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Abstract: In our prior paper [2] we defined a fuzzy function and discussed the theory for optimizing unconstrained fuzzy functions. In that paper we described the need for a measure for comparing different possibility distributions in order to determine the decision makers preference for one distribution over another. In this paper we discuss one such measure on the space of fuzzy numbers and show that this measure defines a normed space of fuzzy numbers if we partition the space into appropriate equivalence classes. We also show that our space is complete if the supports of our fuzzy numbers are contained within a bounded subset of the real line.

Keywords: Fuzzy number, Possibility Distribution, Normed Space

1 Introduction

Suppose we wish to work with an object (for example a number or a function) but can not isolate the object from a set of possible alternatives. Rather than select a single element from the set we can work with the set of alternatives instead. For example, interval analysis is used as the set of possible alternatives when we wish to consider round off error. We can refine our calculations by establishing a weighting over the set of alternatives so that we give greater consideration to alternatives which we decide have greater possibility of being our object. The set of alternatives with the weighting function is called a possibility distribution for the unknown object. The study of possibility distributions is called possibility theory. We remark that the weighting function is not a probability distribution. We are concerned with existence, i.e. is it possible that our object is a particular element in the set not the frequency with which it would turn out to be the particular element. In essence our possibility distribution is a fuzzy set representing the possible alternatives for our object. Thus when we place the word 'fuzzy' before the name of an object we are referring to this fuzzy set of alternatives (we may have additional requirements as will be stated below).

Of particular interest to the author is the case where the unknown object is a function. Various concepts of fuzzy functions have been explored in the literature, from functions with fuzzy parameters (see [1] and [8]) to point to fuzzy set mappings (see [7] and [14]). In our paper [2] we introduced the concept of a fuzzy function as a possibility distribution (satisfying certain properties) over the set of bounded functions mapping a subset of a one vector space into another. We showed that a fuzzy function maps crisp vectors to fuzzy vectors (as defined below). In particular a fuzzy real valued function maps crisp vectors to fuzzy numbers. In [9] and [10] we extended these ideas to practical applications involving constraints.

In this paper we study the space of fuzzy numbers. Our purpose is to examine fuzzy numbers within the context of classical analysis. In particular, we seek sufficient structure on the space of fuzzy numbers to be able to define convergence of fuzzy functions. We will show that there is an equivalence relation on the set of fuzzy numbers that makes the space into a vector space and that on this vector space we can define a norm. We then show that this normed space is isometric to the space of bounded variations on the interval [0,1]. Lastly we use this isometry to show that in most applications Cauchy sequences in our space will converge.

2 Notation and Preliminaries

Let x be an unknown element of set X. We define a **possibility distribution** for x, which we denote $\widetilde{x}$, which is characterized by a function $\mu_{\widetilde{x}} : X \to [0,1]$,where $\mu_{\widetilde{x}}(y) = 1$ means that it is possible that x=y, $\mu_{\widetilde{x}}(y) = 0$ means that it is impossible that x=y and values in between measure degrees of belief between these two. $\mu_{\widetilde{x}}$ is called the **membership function** for the distribution. If, for at least one y$\in X$, $\mu_{\widetilde{x}}(y) = 1$ we say the distribution is **normal**, meaning that at least one element of X is possible. For a given possibility distribution $\widetilde{x}$, we define the $\boldsymbol{\alpha}$ **− cut**, $\tilde{x}_\alpha$, to be the crisp set $\{x \mid \mu_{\widetilde{x}}(x) \geq \alpha \}$ for $\alpha \in(0,1]$ and $\overline{\{x \mid \mu_{\widetilde{x}}(x)>0 \}}$ for α=0 where $\overline{A}$ is the closure of set A. The **strict** α**-cut** for $\tilde{x}$ is defined as the crisp set $\tilde{x}_{\alpha+}=$

$\{x \mid \mu_{\tilde{x}}(x)>\alpha\}$ for $\alpha \in [0,1]$.

Let X be a real finite dimensional vector space with the euclidean norm. We define a **fuzzy vector**, $\tilde{x}$, as a possibility distribution for an unknown vector x such that 1) $\tilde{x}$ is normal and 2) $\forall\alpha \in [0,1]$ the $\alpha-cut$, $\tilde{x}_\alpha$, is compact. We define a **convex fuzzy vector**, $\tilde{x}$, as a fuzzy vector such that $\forall\alpha \in [0,1]$ $\tilde{x}_\alpha$ is convex. We define a **fuzzy number** as a convex fuzzy vector in R, i.e. $\forall\alpha$ $\tilde{x}_\alpha$ is the closed interval $[\tilde{x}_\alpha^-, \tilde{x}_\alpha^+]$.

Let X,Y be real finite dimensional vector spaces with the euclidean norm, $\Omega \subset X$ and $\mathcal{F}=\{f: \Omega \rightarrow Y \mid f \text{ is a bounded function over } \Omega\}$. On $\mathcal{F}$ we define the norm $\|f\|_{\text{sup}}=\sup_{x\in\Omega} \|f(x)\|$. Let f be an unknown element of $\mathcal{F}$ and $\tilde{f}$ a possibility distribution for f. For $x\in \Omega$ we define $\tilde{f}(x)$ to be the possibility distribution in Y with membership function $\mu_{\tilde{f}(x)}(y)=\sup\{\alpha : \mu_{\tilde{f}}(f) = \alpha, f \in \mathcal{F} \text{ and } f(x) = y\}$. A **fuzzy function** over Ω is a fuzzy subset $\tilde{f}$ of $\mathcal{F}$ such that 1) $\tilde{f}$ is normal 2) $\forall\alpha \in [0,1]$ $\tilde{f}_\alpha$ is path connected and compact. If $\forall\alpha \in [0,1]$, $\tilde{f}_\alpha$ is convex we say $\tilde{f}$ is a **convex fuzzy function**.

We will need the following definitions and results from the theory of functions of bounded variations (see [6] or [12]):

Definition 1 *Let $f:[a,b]\rightarrow R$. f is said to be* ***of bounded variation*** *if $\exists C>0$ such that $\sum_{i=1}^{n} |f(x_i)-f(x_{i-1})| \leq C$ for every partition $a=x_0<x_1<\ldots<x_k=b$ on $[a,b]$. We denote the set of all functions of bounded variation on $[a,b]$ by* ***BV[a,b]****.*

Definition 2 *Let $f:[a,b]\rightarrow R$ be a function of bounded variation. The* ***total, positive and negative variation*** *of f on [a,b] are denoted*
$\boldsymbol{V}_a^b(\boldsymbol{f})=\sup_p \sum_{i=1}^{n} |f(x_i)-f(x_{i-1})|$, $\boldsymbol{P}_a^b(\boldsymbol{f})=\sup_p \sum_{i=1}^{n} \max\{0,f(x_i)-f(x_{i-1})\}$,
$\boldsymbol{N}_a^b(\boldsymbol{f})=\sup_p \sum_{i=1}^{n} \max\{0,f(x_{i-1})-f(x_i)\}$ *where p represents all partitions of [a,b].*

Lemma 3 *Let $f:[a,b]\rightarrow R$ be a function of bounded variation.*
Then $\boldsymbol{V}_a^b(\boldsymbol{f})=\boldsymbol{P}_a^b(\boldsymbol{f})+\boldsymbol{N}_a^b(\boldsymbol{f})$ and $f(b)-f(a)=\boldsymbol{P}_a^b(\boldsymbol{f})-\boldsymbol{N}_a^b(\boldsymbol{f})$.

For a proof see [12] page 99.□

Lemma 4 *Let $f:[a,b]\rightarrow R$ be a function of bounded variation and $a<b<c$. Then $\boldsymbol{V}_a^c(\boldsymbol{f})=\boldsymbol{V}_a^b(\boldsymbol{f})+\boldsymbol{V}_b^c(\boldsymbol{f})$.*

For a proof see [6] page 329.□

3 A Normed Space of Fuzzy Number Equivalence Classes

Let F={$\tilde{x}$ | $\tilde{x}$ is a fuzzy number}. We define an equivalence relationship on FXF by stating that $\tilde{x}\equiv\tilde{y}$ iff $\forall\alpha \in$ [0,1] $\exists\epsilon_\alpha \in$R such that $[\tilde{x}_\alpha^- -\epsilon_\alpha, \tilde{x}_\alpha^+ +\epsilon_\alpha]=[\tilde{y}_\alpha^-, \tilde{y}_\alpha^+]$ where $\tilde{x}_\alpha= [\tilde{x}_\alpha^-, \tilde{x}_\alpha^+]$ and $\tilde{y}_\alpha=[\tilde{y}_\alpha^-, \tilde{y}_\alpha^+]$. It is clear that this is an equivalence relation. Let P be the partition of F into equivalence classes resulting from this relationship. We will denote elements of P using capital letters. In essence we define two fuzzy numbers to be equivalent if the midpoint of each α-cut is the same for the two possibility distributions.

Remark 1 *One way to motivate this relationship is as follows. Suppose our objective is to minimize (in some sense) a fuzzy real valued function whose range at any particular vector is a possibility distribution that is an interval. When we write $\widetilde{f}(x) = [2,3]$ we interpret this to mean that $2 \leq f(x) \leq 3$ where f is the unknown function we are modeling with $\widetilde{f}$ i.e. it is possible that f(x) lies in [2,3] and impossible for f(x) to lie outside of [2,3]. Suppose $\widetilde{f}(x) = [2,3]$ and $\widetilde{f}(y) = [1,5]$. We seek a criteria for selecting between x and y. Suppose we comparing the least and greatest possible outcome for our choice to the least and greatest possible outcome for the alternative choice. We choose the action which more than offsets any increase (decrease) in the greatest possible outcome by a decrease (increase) in the least possible outcome. For example, in comparing action x to y above, if we choose y over x the greatest possible outcome is increased by 2 (from 3 to 5) but we only decrease the least possible outcome by 1 (from 2 to 1). Based on our criteria for minimization we choose action x over action y. We have the following simple result. Given two intervals [a,b], [c,d], which contain the possible outcome from choosing between two actions. Choosing the interval with the smallest midpoint will decrease (increase) the least possible outcome more (less) than it will increase (decrease) the greatest possible outcome. If the midpoints are equal, the increase (decrease) in greatest possible outcome will equal the decrease (increase) in the least possible outcome. This follows since $(a+b)/2\leq(c+d)/2\Rightarrow b-d\leq c-a$. Thus choosing [a,b] over [c,d] we change the least possible outcome from c to a which is a decrease (increase) of c-a while the greatest possible outcome changes from d to b which is an increase (decrease) of b-d. To compare intervals we only need to look at their midpoints. So in a minimization problem a choice between an action whose possibility distribution is the interval of [1,2] is judged superior to one whose possibility* *distribution is the interval [0,4] but we are neutral when we compare it to an action whose possibility distribution is [0,3]. From our definition of P, intervals in the same equivalence class share the same midpoint and so we would be neutral with respect to any action producing possibility distributions in the same equivalence class.*

We now consider the problem of comparing two actions whose results are two general fuzzy numbers where our objective is to minimize the outcome. Suppose we wish to choose between fuzzy number $\widetilde{x}$ and fuzzy number $\widetilde{y}$. If we consider only possibilities that are at least α-possible, then we can compare the interval $\widetilde{x}_\alpha$ to $\widetilde{y}_\alpha$ by comparing their midpoints as discussed above. However we would like to consider all possibility levels. We propose to use the possibility weighted average midpoint as our basis for selection. For example, let $\widetilde{x}$ be the triangular fuzzy number (1,2,3) so that $\widetilde{x}_\alpha = [1+\alpha, 3-\alpha]$ and $\widetilde{y}$ the trapezoidal fuzzy number (.5,2,2.5,3) so that $\widetilde{y}_\alpha = [.5 + 1.5\alpha, 3 - .5\alpha]$. Then the possibility weighted average midpoint for $\widetilde{x}$ is $\int_0^1 \alpha(1+\alpha+3-\alpha)d\alpha = 2$ and the possibility weighted average midpoint for $\widetilde{y}$ is $\int_0^1 \alpha(.5+1.5\alpha+3-.5\alpha)d\alpha = 2.0833$ so we consider the action resulting in $\widetilde{x}$ superior to that resulting in $\widetilde{y}$.

We can interpret this calculation as follows. For each α-level we calculate the increase (decrease) in greatest possible outcome $(\widetilde{x}_\alpha^+ - \widetilde{y}_\alpha^+)$ and the decrease (increase) in least possible outcome $(\widetilde{y}_\alpha^- - \widetilde{x}_\alpha^-)$ for a choice of $\widetilde{x}_\alpha$ over $\widetilde{y}_\alpha$. We then calculate the possibility weighted average of these two amounts and choose $\widetilde{x}$ over $\widetilde{y}$ if the possibility weighted average increase (decrease) in greatest possible outcome is less than the possibility weighted average decrease (increase) in least possible outcome, i.e. $\int_0^1 \alpha(\widetilde{x}_\alpha^+ - \widetilde{y}_\alpha^+)d\alpha \leq \int_0^1 \alpha(\widetilde{y}_\alpha^- - \widetilde{x}_\alpha^-)d\alpha$. Given two fuzzy numbers $\widetilde{x}$ and $\widetilde{y}$, choosing the one with the smallest possibility weighted average midpoint is equivalent to choosing the one such that the possibility weighted average increase (decrease) in the greatest possible outcome will be less than the possibility weighted average decrease (increase) in the least possible outcome. This follows since $\int_0^1 \alpha(\widetilde{x}_\alpha^+ + \widetilde{x}_\alpha^-)d\alpha \leq \int_0^1 \alpha(\widetilde{y}_\alpha^+ + \widetilde{y}_\alpha^-)d\alpha$ implies $\int_0^1 \alpha(\widetilde{x}_\alpha^+ - \widetilde{y}_\alpha^+)d\alpha \leq \int_0^1 \alpha(\widetilde{y}_\alpha^- - \widetilde{x}_\alpha^-)d\alpha$. From our definition of P, two fuzzy numbers in the same equivalence class share the same midpoint at each $\alpha - cut$ so their possibility weighted average midpoints will be equal i.e. we would be neutral with respect to two actions whose possibility distributions are in the same equivalence class.

Remark 2 *Another interpretation for our equivalence relation is that we compare intervals by looking at the crisp point which minimizes the maximum* *possible error when compared to the possible outcome within the interval. For example, if we choose 1.5 from the possibility distribution [1,2] then at most our error is .5 (2-1.5 or 1.5-1). For general fuzzy numbers we compare the possibility weighted midpoint.*

Lemma 5 *Let $\tilde{X} \in P$, then $\exists \tilde{x} \in \tilde{X}$* ***of minimal possibility****, i.e. if $\tilde{x}_\alpha = [\tilde{x}_\alpha^-, \tilde{x}_\alpha^+]$ and $\tilde{y} \in \tilde{X}$ then $\forall \alpha \in [0,1]$ $\tilde{y}_\alpha^- \leq \tilde{x}_\alpha^-$ and $\tilde{x}_\alpha^+ \leq \tilde{y}_\alpha^+$.*

Proof:

Let $\tilde{x}_\alpha = \cap_{\tilde{y} \in \tilde{X}} [\tilde{y}_\alpha^-, \tilde{y}_\alpha^+]$, then $\tilde{x}_\alpha$ is closed and convex, thus $\tilde{x}$ is a fuzzy number.□

Theorem 6 *P with addition and scalar multiplication defined in the usual way (minmax convolution, see [3]) forms a vector space over R.*

Proof:

Let $\tilde{X}$, $\tilde{Y}$ and $\tilde{Z} \in P$ and $r,s \in R$.

(1) Vector Addition - see [3] for commutative and associative properties. For the zero element and inverses let $\tilde{x} \in \tilde{X}$. Consider $-\tilde{x}$ defined by $-\tilde{x}_\alpha = [-x_\alpha^+, -x_\alpha^-]$. Then $[x_\alpha^-, x_\alpha^+] + [-x_\alpha^+, -x_\alpha^-] = [x_\alpha^- + -x_\alpha^+, x_\alpha^+ - x_\alpha^-] \equiv 0$. Next, let $\tilde{y} \equiv 0$, then $\tilde{y}_\alpha = [-a,a]$ for some $a \geq 0$. Then $[x_\alpha^-, x_\alpha^+] + [-a,a] = [x_\alpha^- - a, x_\alpha^+ + a]$ so $\tilde{x} + \tilde{y} \equiv \tilde{x}$.

(2) Scalar Multiplication - For $r \geq 0$, $r[x_\alpha^-, x_\alpha^+] = [rx_\alpha^-, rx_\alpha^+]$,

for $r<0$, $r[x_\alpha^-, x_\alpha^+] = [rx_\alpha^+, rx_\alpha^-]$.

Case 1- $r,s>0$ $r(s[x_\alpha^-, x_\alpha^+]) = r[sx_\alpha^-, sx_\alpha^+] = [rsx_\alpha^-, rsx_\alpha^+] = (rs)[x_\alpha^-, x_\alpha^+]$

Case 2 -$r>0, s<0$ $r(s[x_\alpha^-, x_\alpha^+]) = r[sx_\alpha^+, sx_\alpha^-] = [rsx_\alpha^+, rsx_\alpha^-] = (rs)[x_\alpha^-, x_\alpha^+]$

Case 3 -$r<0, s>0$ $r(s[x_\alpha^-, x_\alpha^+]) = r[sx_\alpha^-, sx_\alpha^+] = [rsx_\alpha^+, rsx_\alpha^-] = (rs)[x_\alpha^-, x_\alpha^+]$

$1[x_\alpha^-, x_\alpha^+] = [x_\alpha^-, x_\alpha^+]$ so the multiplicative identity is the number 1.

(3) Distributive Laws

Case 1 - $r \geq 0$ $r([x_\alpha^-, x_\alpha^+] + [y_\alpha^-, y_\alpha^+]) = r[x_\alpha^- + y_\alpha^-, x_\alpha^+ + y_\alpha^+] = [rx_\alpha^- + ry_\alpha^-, rx_\alpha^+ + ry_\alpha^+] = r[x_\alpha^-, x_\alpha^+] + r[y_\alpha^-, y_\alpha^+]$

Case 2 - $r<0$ $r([x_\alpha^-, x_\alpha^+] + [y_\alpha^-, y_\alpha^+]) = r[x_\alpha^- + y_\alpha^-, x_\alpha^+ + y_\alpha^+] = [rx_\alpha^+ + ry_\alpha^+, rx_\alpha^- + ry_\alpha^-] = r[x_\alpha^-, x_\alpha^+] + r[y_\alpha^-, y_\alpha^+]$

Case 1 - $r,s \geq 0$ $(r+s)[x_\alpha^-, x_\alpha^+] = [(r+s)x_\alpha^-, (r+s)x_\alpha^+] = r[x_\alpha^-, x_\alpha^+] + s[x_\alpha^-, x_\alpha^+]$

Case 2 - $r,s<0$ $(r+s)[x_\alpha^-, x_\alpha^+] = [(r+s)x_\alpha^+, (r+s)x_\alpha^-] = r[x_\alpha^-, x_\alpha^+] + s[x_\alpha^-, x_\alpha^+]$

Case 3 - $r \geq 0, s<0, r+s<0$ $(r+s)[x_\alpha^-, x_\alpha^+] = [(r+s)x_\alpha^+, (r+s)x_\alpha^-] =$
$[rx_\alpha^+ + sx_\alpha^+, rx_\alpha^- + sx_\alpha^-] \equiv [rx_\alpha^+ + sx_\alpha^+ - r\delta, rx_\alpha^- + sx_\alpha^- + r\delta] = [rx_\alpha^- + sx_\alpha^+, rx_\alpha^+ + sx_\alpha^-] =$
$[rx_\alpha^-, rx_\alpha^+] + [sx_\alpha^+, sx_\alpha^-] = r[x_\alpha^-, x_\alpha^+] + s[x_\alpha^-, x_\alpha^+]$ where $r\delta = (rx_\alpha^+ - rx_\alpha^-)$

Case 4 - $r \geq 0, s<0, r+s \geq 0$ $(r+s)[x_\alpha^-, x_\alpha^+] = [(r+s)x_\alpha^-, (r+s)x_\alpha^+] =$
$[rx_\alpha^- + sx_\alpha^-, rx_\alpha^+ + sx_\alpha^+] \equiv [rx_\alpha^- + sx_\alpha^- - s\delta, rx_\alpha^+ + sx_\alpha^+ + s\delta] = [rx_\alpha^- + sx_\alpha^+, rx_\alpha^+ + sx_\alpha^-] =$
$[rx_\alpha^-, rx_\alpha^+] + [sx_\alpha^+, sx_\alpha^-] = r[x_\alpha^-, x_\alpha^+] + s[x_\alpha^-, x_\alpha^+]$ where $s\delta = (sx_\alpha^- - sx_\alpha^+)$□

Definition 7 *On the vector space P, define the function* $\left\|\tilde{X}\right\|_{pwa} = \int_0^1 \alpha \left|\tilde{x}_\alpha^- + \tilde{x}_\alpha^+\right| d\alpha$ *where* $\tilde{x} \in \tilde{X}$.

Example 8 *For the triangular fuzzy number* $\tilde{x}$*=(-2,-1,1) with* $\tilde{x}_\alpha = [-2 + \alpha, 1 - 2\alpha]$ $\left\|\tilde{X}\right\|_{pwa} = \int_0^1 \alpha \left|-2 + \alpha + 1 - 2\alpha\right| d\alpha = .83333$

This is well defined since $\tilde{x}_\alpha^-$ and $\tilde{x}_\alpha^+$ integrable on [0,1] (since each is monotonic, see 6.9 in [6]) implies $\tilde{x}_\alpha^- + \tilde{x}_\alpha^+$ integrable implies $|\tilde{x}_\alpha^- + \tilde{x}_\alpha^+|$integrable. This function measures the possibility weighted average of the absolute value of the midpoint of each possibility level set. The subscript pwa is an abbreviation for *possibility weighted average.* For a crisp number this function reduces to the absolute value of the number. We will show that this function defines a norm on P. To do this we first make the following observation.

Let $\tilde{x} \in F$. Consider $\tilde{x}_\alpha^-$ and $\tilde{x}_\alpha^+$ as functions of α. We observe that these functions are monotonic on [0,1] and so are continuous except at countably many points in [0,1] and all discontinuities are of the first kind (see [13] page 96).

Lemma 9 *As functions of* α, $\tilde{x}_\alpha^-$ *and* $\tilde{x}_\alpha^+$ *are continuous from the left.*

Proof

This lemma is proven in Theorem 2.13 (c) of [5] but we reproduce it here in our setting. We will prove the lemma for $\tilde{x}_\alpha^-$ the proof for $\tilde{x}_\alpha^+$ is identical. Let $\alpha \in (0,1]$ and $\beta = \sup_{0<\gamma<\alpha} \tilde{x}_\gamma^-$. We know β exists and $\beta \leq \tilde{x}_\alpha^-$ since $\tilde{x}_\alpha^-$ is monotone increasing (see 4.29 in [12]). By definition, $\forall \epsilon > 0$ $\mu_{\tilde{x}}(\beta) \geq \mu_{\tilde{x}}(\tilde{x}_{\alpha-\epsilon}^-) \geq \alpha - \epsilon$ so that $\mu_{\tilde{x}}(\beta) \geq \alpha \Rightarrow \beta \geq \tilde{x}_\alpha^-$ so $\beta = \tilde{x}_\alpha^-$.□

Theorem 10 *The function* $\left\|\tilde{X}\right\|_{pwa}$ *defines a norm on P.*

Proof

(N1) $\left\|\tilde{X}\right\|_{pwa} \geq 0$ since $\alpha \left|\tilde{x}_\alpha^- + \tilde{x}_\alpha^+\right| \geq 0$ $\forall \alpha \in [0,1]$

(N2) $\left\|\tilde{X}\right\|_{pwa} = 0 \Rightarrow \tilde{X} = 0$..

Case(1) Assume for some $\gamma > 0$ we have $\left|\tilde{x}_\gamma^- + \tilde{x}_\gamma^+\right| = r > 0$. Since $\tilde{x}_\beta^- \to \tilde{x}_\gamma^-$ as $\beta \to \gamma$ from below and similarly for $\tilde{x}_\gamma^+$, $\exists \delta > 0$ such that $\left|\tilde{x}_\beta^- - \tilde{x}_\gamma^-\right| < r/3$ and $\left|\tilde{x}_\beta^+ - \tilde{x}_\gamma^+\right| < r/3$ and thus $\left|\tilde{x}_\beta^- + \tilde{x}_\beta^+\right| > 0$ $\forall \beta < \gamma$ with $\gamma - \beta < \delta$.

Then $\int_{\gamma-\delta}^{\gamma} \alpha\, |\tilde{x}_\alpha^- + \tilde{x}_\alpha^+|\, d\alpha > 0 \Rightarrow \int_0^1 \alpha\, |\tilde{x}_\alpha^- + \tilde{x}_\alpha^+|\, d\alpha > 0$, a contradiction. Thus $\forall\alpha>0$, $|\tilde{x}_\alpha^- + \tilde{x}_\alpha^+| = 0 \Rightarrow \tilde{x}_\alpha^- = -\tilde{x}_\alpha^+$ i.e. $\tilde{x} \equiv 0$.

Case(2) Let $|\tilde{x}_0^- + \tilde{x}_0^+| = r > 0$. Then for some γ arbitrarily close to 0 and $r > s > 0$

$|\tilde{x}_\gamma^- + \tilde{x}_\gamma^+| = s > 0$ and case(1) applies.

(N3) $\left\|r\tilde{X}\right\|_{pwa} = |r| \left\|\tilde{X}\right\|_{pwa}$. Clear since absolute value is a norm.

(N4) $\left\|\tilde{X}+\tilde{Y}\right\|_{pwa} \leq \left\|\tilde{X}\right\|_{pwa} + \left\|\tilde{Y}\right\|_{pwa}$. $\left\|\tilde{X}+\tilde{Y}\right\|_{pwa} = \int_0^1 \alpha\, |\tilde{x}_\alpha^- + \tilde{y}_\alpha^- + \tilde{x}_\alpha^+ + \tilde{y}_\alpha^+|\, d\alpha$ $\leq \int_0^1 \alpha\, \{|\tilde{x}_\alpha^- + \tilde{x}_\alpha^+| + |\tilde{y}_\alpha^- + \tilde{y}_\alpha^+|\}\, d\alpha = \left\|\tilde{X}\right\|_{pwa} + \left\|\tilde{Y}\right\|_{pwa}$ since absolute value is a norm and by properties of the Lebesgue integral.□

4 An Isomorphism between P and Functions of Bounded Variation

We now show that each equivalence class of P can be represented by an equivalence class of functions from BV[0,1], the space of bounded variations on the interval [0,1]. Using this representation, we will show that P is isometric to this space under the pwa-norm above and that this space can be viewed as a subspace of $L^1[0,1]$, the space of Lebesque integrable functions on [0,1] (partitioned into functions which are equivalent up to a set of measure zero).

Definition 11 *A family $(A_\alpha)_{\alpha\in(0,1]}$ of subsets of R is called a* ***set representation*** *of fuzzy number $\tilde{x}$ if 1) $\alpha<\beta \Rightarrow A_\beta \subseteq A_\alpha$ and 2) $\mu_{\tilde{x}}(x) = sup\{\alpha \mid x \in A_\alpha\}$ where $sup\emptyset \equiv 0$.*

Remark 3 *For any such set representation, we can define $A_0 = cls(\cup_{\gamma\in(0,1]} A_\gamma)$. When we refer to a set representation $(A_\alpha)_{\alpha\in[0,1]}$, we will mean this extended family of sets.*

Lemma 12 *Let $(A_\alpha)_{\alpha\in[0,1]}$ be a set representation of fuzzy number $\tilde{x}$. Then $\forall\alpha \in (0,1]$ $\tilde{x}_\alpha = \cap_{\gamma<\alpha} A_\gamma$ and $\forall\alpha \in [0,1)$ $\tilde{x}_{\alpha+} = \cup_{\gamma>\alpha} A_\gamma$.*

Proof:

We will first show that $\tilde{x}_\alpha = \cap_{\gamma<\alpha} A_\gamma$

$\subseteq$ - Let $x \in \tilde{x}_\alpha$ and assume $x \notin A\gamma$ for some $\gamma<\alpha$. Then $x \notin A_\beta$ $\forall\beta \geq \gamma \Rightarrow$ $.\mu_{\tilde{x}}(x) \leq = \gamma < \alpha$ which is a contradiction.

$\supseteq$ - Let x$\in \cap_{\gamma<\alpha}$A$_\gamma$. Then $\mu_{\tilde{x}}$(x)=sup{α | x $\in$ A$_\alpha$} $\geq \alpha \Rightarrow$x$\in\tilde{x}_\alpha$.
We next show that $\tilde{x}_{\alpha+}=\cup_{\gamma>\alpha}A_\gamma$.
$\subseteq$ - Let x$\in \cup_{\gamma>\alpha}$A$_\gamma$. Then x$\in$A$_\gamma$ for some $\gamma>\alpha$ so $\mu_{\tilde{x}}$(x)$\geq \gamma>\alpha$.
$\supseteq$ - Let x$\in\tilde{x}_{\alpha+}$□

Lemma 13 *Let $(A_\alpha)_{\alpha\in[0,1]}$ be a family of non-empty closed nested intervals with $\alpha<\beta \Rightarrow A_\beta \subseteq A_\alpha$ and A_0=cls$(\cup_{\gamma\in(0,1]}A_\gamma)$. Define $\mu_{\tilde{x}}(x)$=sup$\{\alpha \mid x\in A_\alpha\}$ with sup$\emptyset\equiv$0. Then $\tilde{x}$ is a fuzzy number.*

Proof:

Since each Aγ is closed and non-empty $\cap_{\gamma<\alpha}$A$_\gamma$ is closed and non-empty (see Theorem 2.24(b) and Theorem 2.36 in [13]) and, since it is bounded, it is compact. It is also convex (see Prop 1 pg 464 of [11]), so it is a closed interval. From the prior lemma we know that $\tilde{x}_\alpha=\cap_{\gamma<\alpha}A_\gamma$. Then each level set is a closed interval (satisfying the compact and convex condition) and non-empty (satisfying the normality condition).□

Theorem 14 *The family of sets $(A_\alpha)_{\alpha\in(0,1)}$ is a set representation of fuzzy number $\tilde{x}$ if and only if it holds that $\tilde{x}_{\alpha+}\subseteq A_\alpha \subseteq\tilde{x}_\alpha$ $\forall\alpha\in(0,1)$.*

See Theorem 2.34 of [5] for a proof or apply the prior lemma and use the nested relation of the sets.□

Definition 15 *Let $\tilde{x}$ be a fuzzy number and let $(A_\alpha)_{\alpha\in[0,1]}$ be a set representation of fuzzy number $\tilde{x}$ consisting of non-empty closed intervals with A_0=cls$(\cup_{\gamma\in(0,1)}A_\gamma)$. Define $f_{\tilde{x}}$:[0,1]$\to$R, called a* ***functional representative of $\tilde{x}$****, by $f_{\tilde{x}}(\alpha)=(a_\alpha^-+a_\alpha^+)/2$ where $A_\alpha=[a_\alpha^-,a_\alpha^+]$.*

Lemma 16 *Let $f_{\tilde{x}}$ be a functional representative of $\tilde{x}$, then $f_{\tilde{x}}\in BV[0,1]$ and $f_{\tilde{x}}(\alpha-)=(\beta^-+\beta^+)/2$ where $[\beta^-,\beta^+]=\cap_{\gamma<\alpha}A_\gamma=\tilde{x}_\alpha$ and $f_{\tilde{x}}(\alpha+)=(\beta^-+\beta^+)/2$ where $[\beta^-,\beta^+]$=cls$(\cup_{\gamma>\alpha}A_\gamma)$.*

Proof

Note that $f_{\tilde{x}}(\alpha)=(a_\alpha^-+a_\alpha^+)/2=(f_{\tilde{x}}^- -(-f_{\tilde{x}}^+))/2$. Where $f_{\tilde{x}}^-(\alpha)=a_\alpha^-$ and $f_{\tilde{x}}^+(\alpha)=-a_\alpha^+$. Since $(A_\alpha)_{\alpha\in[0,1]}$ is nested, $f_{\tilde{x}}^-$ and $-f_{\tilde{x}}^+$ are monotone increasing real-valued functions so $f_{\tilde{x}}$ is a bounded variation (see Theorem 4. on page 100 of [12]). Since $f_{\tilde{x}}$ is a bounded variation, $f_{\tilde{x}}(\alpha-)$ and $f_{\tilde{x}}(\alpha+)$ exist (see problem 4.a on page 100 of [12]). Let $[\beta^-,\beta^+]=\cap_{\gamma<\alpha}A_\gamma$. We know that $f_{\tilde{x}}^-(\alpha-)=\beta^-$ and $-f_{\tilde{x}}^+(\alpha-)=\beta^+$ since each is monotone increasing so clearly $f_{\tilde{x}}(\alpha-)=(f_{\tilde{x}}^-(\alpha-)-(-f_{\tilde{x}}^+(\alpha-)))/2=(\beta^-+\beta^+)/2$. Similarly for the limit from the right.

Definition 17 *Let f,g∈BV[0,1]. Let f∼g mean that f(x)=g(x) almost everywhere.*

This defines an equivalence relation over BV[0,1].

Lemma 18 *Let $(A_\alpha)_{\alpha\in[0,1]}$ and $(B_\alpha)_{\alpha\in[0,1]}$ be set representations of fuzzy number $\tilde{x}$ consisting of non-empty closed intervals. Let f and g be the corresponding functional representatives. Then f∼g.*

Proof:

Since $\tilde{x}_\alpha=\cap_{\gamma<\alpha}A_\gamma=\cap_{\gamma<\alpha}B_\gamma$ $\forall\alpha\in(0,1]$ we have $f(\alpha-)=g(\alpha-)$. But this implies that f∼g. This follows since the points at which a function of bounded variation is discontinuous are countable and the measure of a countable set is zero. At all points of continuity we have $f(\alpha)=f(\alpha-)=g(\alpha-)=g(\alpha)$.□

Lemma 19 *Let $(A_\alpha)_{\alpha\in[0,1]}$ and $(B_\alpha)_{\alpha\in[0,1]}$ be set representations of fuzzy numbers $\tilde{x}$ and $\tilde{y}$ consisting of non-empty closed intervals and assume $\tilde{x}\sim\tilde{y}$, then if f and g are functional representations for these two set representations respectively, f∼g.*

Proof:

$\forall\alpha\in[0,1]$, $(a_\alpha^-+a_\alpha^+)/2=(b_\alpha^-+b_\alpha^+)/2$ where $\tilde{x}_\alpha=[a_\alpha^-,a_\alpha^+]$ and $\tilde{y}_\alpha=[b_\alpha^-,b_\alpha^+]$ by definition of our equivalence relation. Thus $f(\alpha-)=g(\alpha-)$ and the same argument as applied in the previous lemma holds.□

We have shown that there is a single functional representation (up to a set of measure zero) for each equivalence class of fuzzy numbers. Thus the map of fuzzy number equivalence classes to the space of equivalence classes of functions of bounded variation is well defined. We now proceed to show that it is not only well defined but establishes an isomorphism. The next lemma establishes that the map is one-to-one and the following one establishes that it is onto.

Lemma 20 *Let $\tilde{X}\neq\tilde{Y}$ be two fuzzy number equivalence classes. Then their functional representations are not equal.*

Proof:

Let f and g be the functional representations for $\tilde{X}$ and $\tilde{Y}$ respectively. Since $\tilde{X}\neq\tilde{Y}$ $\exists\alpha\in(0,1]$ such that $(\tilde{x}_\alpha^-+\tilde{x}_\alpha^+)/2\neq(\tilde{y}_\alpha^-+\tilde{y}_\alpha^+)/2$. Then for this α, $f(\alpha-)\neq g(\alpha-)$. Then ∃ disjoint neighborhoods of $f(\alpha-)$ and $g(\alpha-)$ with positive separation so in a neighborhood near α (a set with measure greater than zero), f≠g.□

Lemma 21 *Let f∈BV[0,1] then ∃$\tilde{X}$∈P such that f is the functional representation for $\tilde{X}$ and f is the functional representation for $\tilde{X}$.*

Proof:
For $\alpha \in(0,1]$, let $[\tilde{x}_\alpha^-, \tilde{x}_\alpha^+]$=[f(1)-2$\mathbf{P}_a^1$(**f**),f(1)+2$\mathbf{N}_a^1$(**f**)] and
$[\tilde{x}_0^-, \tilde{x}_0^+]$=cls($\cup_{\alpha>0}[\tilde{x}_\alpha^-, \tilde{x}_\alpha^+]$).

We claim that the intervals $[\tilde{x}_\alpha^-, \tilde{x}_\alpha^+]$ are a set representation for a fuzzy number $\tilde{x}$. Since each interval is closed and non-empty, by our prior lemma we only need to show that $\alpha<\beta \Rightarrow [\tilde{x}_\beta^-, \tilde{x}_\beta^+]\subset [\tilde{x}_\alpha^-, \tilde{x}_\alpha^+]$, but this is immediate from the definition of $\mathbf{P}_a^1$(**f**) and $\mathbf{N}_a^1$(**f**).

To show that this is a functional representation for $\tilde{X}$ note that
g(α)=($\tilde{x}_\alpha^-$+$\tilde{x}_\alpha^+$)/2=f(1)+(2$\mathbf{N}_a^1$(**f**)-2$\mathbf{P}_a^1$(f))/2=f(1)+(f(α)-f(1))=f(α)□

We have established a one-to-one correspondence between P and BV[0,1]. This mapping clearly preserves addition and multiplication by scalars.. Thus the mapping is an isomorphism.

Theorem 22 *The space of BV[0,1] is isometric to P.*

Proof
We have already established an isomorphism between P and BV[0,1], we need only to establish an equivalent norm. We note that if $\tilde{X}$∈P and $f_{\tilde{x}}$ is the functional representation for $\tilde{X}$, then $\left\|\tilde{X}\right\|_{pwa}=2\int_0^1 \alpha\,|f_{\tilde{x}}(\alpha)|\,d\alpha$. Since all functional representations for an equivalence class are equal almost everywhere, the integral is equal.□

5 Completeness

We have established that our equivalence classes of fuzzy numbers are isometric to the space BV[0,1]. With the pwa norm, BV[0,1] is equivalent to $\{f \mid f \in BV[0,1]$ and f(0)=0$\}$ as a subspace of L^1[0,1] which is complete. Thus every Cauchy sequence of fuzzy numbers in the space P will converge but not necessarily to a fuzzy number. However, we have the following result.

Theorem 23 *Suppose $\left\{\widetilde{X}_n\right\}$ is Cauchy in P under the pwa norm and suppose that ∃ M such that ∀ n, $V_0^1(f_n) < M$, where f_n is the functional representation for $\widetilde{X}_n$. Then ∃ $\widetilde{X} \in P$ such that $\widetilde{X}_n \to \widetilde{X}$ in the pwa norm.*

Proof:

By assumption $\{\alpha f_n(\alpha)\}$ is Cauchy in $L^1[0,1]$. Thus $\exists$ g$\in$ $L^1[0,1]$ such that $\alpha f_n(\alpha) \to g(\alpha)$ in the mean since $L^1[0,1]$ is complete. Let f(α) $= \frac{g(\alpha)}{\alpha}$ for $\alpha \in (0,1]$ and f(0)=0. Then $\alpha f_n(\alpha) \to \alpha f(\alpha)$ in the mean. We know that $\exists$ a subsequence $\alpha f_{n_k}(\alpha) \to \alpha f(\alpha)$ almost everywhere (see [6] page 388 problem 7c). Note that $f_n \to f$.where ever $\alpha f_{n_k}(\alpha) \to \alpha f(\alpha)$. Let S equal the subset of [0,1] where we have pointwise convergence. Assume that f is not of bounded variation on S. Since f is not of bounded variation $\exists$ $\{\alpha_i \in S \mid i = 1, m\}$ such that $\sum |f(\alpha_i) - f(\alpha_{i+1)}| > 2M$. We can chose N large enough such that $\forall i$, $|f(\alpha_i) - f_N(\alpha_i)| < \epsilon$ for ϵ arbitrarily small. Then $\sum |f_N(\alpha_i) - f_N(\alpha_{i+1)}| > \sum |f(\alpha_i) - f(\alpha_{i+1)}| - m\epsilon$. But then $V_0^1(f_N) > M$ which is a contradiction. Therefore f is of bounded variation on S. We need to show that there is a function of bounded variation on [0,1] which is equal to f almost everywhere. For every $\alpha \in [0,1] - S$. Let $\{x_n\}$ be a sequence from S such that $x_n \to \alpha$. Such a sequence exists since [0,1]-S has measure zero. Define f(α) $= \limsup f(x_n)$. This limit exists and is finite since f is a bounded variation on S and, therefore, bounded on S. But then f as just defined is a bounded variation on [0,1] since $\forall \alpha \in [0,1] - S$ we can find $x \in S$ arbitrarily close to α such that $|f(x) - f(\alpha)| < \epsilon$ for arbitrarily small ϵ.□

Corollary 24 *Let $\{\widetilde{x}_n\}$ be a sequence of fuzzy numbers with the property that $\forall n$ $\widetilde{x}_0 \subset B$ where B is a bounded subset of R. If $\left\{\widetilde{X}_n\right\}$, the sequence of equivalence classes in P, is Cauchy then it converges.*

Proof:

Recall that $f_{\tilde{x}}$ defined by $f_{\tilde{x}}(\alpha)=(a_\alpha^- + a_\alpha^+)/2$ where $\widetilde{x}_\alpha=[a_\alpha^-, a_\alpha^+]$ is a functional representative of $\tilde{x}$. But $V_0^1(f_{\tilde{x}}) = |a_\alpha^+ - a_\alpha^-| < M$, where M is the bound on B.□

6 Conclusion

We have shown that when the image space of a fuzzy function consists of fuzzy numbers we can partition the space into equivalence classes which, with the appropriate norm, forms a normed vector space. We have shown that this space is isometric to the space of functions of bounded variation with the same norm. We have used this isometric relationship to show that for most applications this space is complete. We hope to use this space to find convergent algorithms to the optimum in applications of optimization. We also hope to study fuzzy functions further as a branch of analysis. Future research will examine additional properties of this normed space and the implications to applying the theory.

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