FURTHER CONSIDERATIONS ON RESIDUAL-FREE BUBBLES FOR ADVECTIVE-DIFFUSIVE EQUATIONS

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Abstract

We further consider the Galerkin method for advective-diffusive equations in two-dimensions. The finite dimensional space employed is of piecewise polynomials enriched with residual-free bubbles (RFB). We show that, in general, this method does not coincide with the SUPG method, unless the piecewise polynomials are spanned by linear functions. Furthermore a simple stability analysis argument displays the effect of the RFB on the reduced space of piecewise polynomials, which is not equivalent to streamline diffusion for bilinears, in some situations.

1 Introduction

The Galerkin finite element method using piecewise polynomials enriched with *residual-free bubble functions* (RFB) seems to provide a general framework for discretizations [3, 4, 7, 8, 9, 10]. Partitioning our domain into a mesh of elements, the residual-free bubble functions are defined to be as rich as possible within an element. In other words, these functions are assumed to satisfy strongly the partial differential equations in the interior of the element, up to the contribution of the piecewise polynomial functions. In addition, they are also assumed to satisfy a homogeneous Dirichlet condition on the element boundary. The residual-free-bubbles represent the unresolvable part of the solution, whereas the piecewise polynomials are the resolvable part for the given mesh.

This decomposition of the solution into a piecewise polynomial plus residual-free bubbles produces the exact solution of linear differential equations in the one-dimensional case. Furthermore, by inspecting the method after we eliminate the residual-free bubbles, various successful discretization schemes are unveiled, such as upwinding for advective-diffusive equations, mass lumping for a model of the parabolic heat transfer equation, selective reduced integration with adjustment of coefficients for the deflection of a Timoshenko beam, etc [7, 8, 9]. In higher dimensions the computation of the residual-free bubbles becomes a major task, in that only in limited situations (such as rectangular elements) one can employ classical analytical tools to get the exact solution within each element or in limiting cases [4, 6].

In this note we investigate further the effect of the elimination of the residual-free bubbles in the context of the advective-diffusive equation. In particular, we note that in multidimensions, considering the limit of vanishing viscosity, the method that surfaces after the elimination of the bubble is *not* the SUPG method [5] if the basis function is defined on quadrilaterals. For example in the case of bilinear interpolation the method yields different numerical results (Here and in what follows we use, as usual, "bilinear interpolation" and "bilinear element" terms to denote the function which is the isoparametrical image of Q1 defined on a unit square reference domain). Furthermore the interpretation of the additional term is not the addition of a streamline diffusion term in this case.

We organize the paper as follows: in the next Section we first review the residual-free bubble idea and apply it to the advection diffusion equation discretized by either linears or bilinears. Therein the effect of the residual free bubbles is derived in the limit of vanishing viscosity and analyzed with respect to its stability impact on the Galerkin method. In Section 3 we present some numerical results for the bilinear discretization and point out the differences to the results obtained using the SUPG method. We close the paper with conclusions in Section 4.

2 RFB for the Advective-Diffusive Model

Let us first consider the RFB idea in general terms. We wish to approximate the abstract boundary-value problem defined in the domain $\Omega \subset \mathbb{R}^2$ and given by:

$$\begin{cases} Lu = f & \text{in } \Omega\,, \\ u = 0 & \text{on } \Gamma = \partial\Omega\,, \end{cases} \tag{1}$$

where L is a linear differential operator, u is the unknown scalar function and f is a given source function. For the sake of simplicity, we will assume that Ω is a polygonal domain. The standard Galerkin method is formulated in a subspace $V_h \subset V$, where V is the space of functions for which a solution of the continuous problem is sought. The Galerkin method consists of finding $u_h \in V_h$ such that

$$a(u_h, v_h) = (Lu_h, v_h) = (f, v_h) \quad \forall v_h \in V_h\,. \tag{2}$$

Here V_h is defined in a partition of the domain into triangles and/or quadrilaterals in the usual manner (i.e., no overlapping, their union reproduces the domain, no vertex on the edges of a neighboring element, etc). Each member u_h and v_h of the space of functions V_h

is spanned by continuous piecewise linear or bilinear polynomials plus bubble functions to be defined below, i.e.,

$$u_h = u_1 + u_b \,. \tag{3}$$

The bubble part of this space is subject to zero Dirichlet boundary condition on each element K boundary, i.e.,

$$u_b = 0 \quad \text{on}\, \partial K \,. \tag{4}$$

This feature of the bubble functions allows us to employ the classical *static condensation* procedure: first we set $v_h = v_{b,K}$ on K (zero elsewhere) in (2) to obtain

$$a(u_1 + u_b, v_{b,K})_K = (f, v_{b,K})_K \tag{5}$$

where the subscript K indicates that integration is restricted to the element K. This part of the process provides us with the bubble-part of the solution, u_b, at each element as a function of the piecewise linear or bilinear polynomial part of the solution, u_1.

The second part of the static condensation procedure consists in setting $v_h = v_1$ in (2), which gives

$$a(u_1 + u_b, v_1) = (f, v_1) \tag{6}$$

or

$$a(u_1, v_1) + a(u_b, v_1) = (f, v_1) \,. \tag{7}$$

If we refer to the standard Galerkin method for piecewise linears (or bilinears), equation (7) shows that eliminating the bubbles yields to a formulation that may be viewed as modifying the variational formulation in the left hand side by the addition of $a(u_b, v_1)$.

We have still not defined the space of bubbles we will use. These are the residual-free bubbles which are bubble functions that satisfy the differential equations strongly in each element K, i.e.,

$$Lu_b = -(Lu_1 - f) \qquad \text{in}\, K \,. \tag{8}$$

Note that this choice of bubbles implies that equation (5) is satisfied automatically, since it is the variational equation for (8).

We now turn to the application of this methodology to the solution of the advection-diffusion equation. For this equation we have

$$L = -\kappa \triangle + \boldsymbol{a} \cdot \nabla \,, \tag{9}$$

where κ is the positive constant viscosity coefficient (or diffusivity) and $\boldsymbol{a}$ is the velocity field. To simplify notation, in what follows we assume $\boldsymbol{a}$ to be constant, although the subsequent analysis and considerations equally apply to velocities which are piecewise constant in the partition of Ω. In addition, for simplicity, we consider a homogeneous Dirichlet boundary condition:

$$u = 0 \qquad \text{on}\, \Gamma = \partial\Omega\,. \tag{10}$$

Substituting (9) into (7), leads to

$$\kappa(\nabla u_1, \nabla v_1) + \kappa(\nabla u_b, \nabla v_1) + (\boldsymbol{a}\cdot\nabla u_1, v_1) + \sum_K (\boldsymbol{a}\cdot\nabla u_b, v_1)_K = (f, v_1)\,. \tag{11}$$

Integrating by parts, this can be rewritten as:

$$\kappa(\nabla u_1, \nabla v_1) + (\boldsymbol{a}\cdot\nabla u_1, v_1) - \sum_K (\boldsymbol{a}\cdot\nabla v_1 + \kappa\triangle v_1, u_b)_K = (f, v_1)\,. \tag{12}$$

The residual-free part of the solution, u_b can be computed from (8) as solution to:

$$-\kappa\triangle u_b + \boldsymbol{a}\cdot\nabla u_b = -\boldsymbol{a}\cdot\nabla u_1 + \kappa\triangle u_1 + f \qquad \text{in}\, K\,, \tag{13}$$

subject to

$$u_b = 0 \qquad \text{on}\, \partial K\,. \tag{14}$$

In the limiting case when $\kappa \to 0$, the problem reduces to finding u_b such that

$$\boldsymbol{a}\cdot\nabla u_b = -\boldsymbol{a}\cdot\nabla u_1 + f \qquad \text{in}\, K\,, \tag{15}$$

subject to

$$u_b = 0 \qquad \text{on}\, \partial K^-\,, \tag{16}$$

where $\partial K^- = \{\boldsymbol{x} \in \partial K : \boldsymbol{a}\cdot\boldsymbol{n}(\boldsymbol{x}) < 0\}$ is the inflow boundary of the element K. (We will also use $\partial K^+ = \{\boldsymbol{x} \in \partial K : \boldsymbol{a}\cdot\boldsymbol{n}(\boldsymbol{x}) \geq 0\}$ as the outflow boundary of the element K.)

This problem can be solved by noting that $u_b = u_b^0 + u_b^f$ where u_b^0 and u_b^f are solutions of:

$$\begin{cases} \boldsymbol{a}\cdot\nabla u_b^0 = -\boldsymbol{a}\cdot\nabla u_1 & \text{in } K\,, \\ u_b^0 = 0 & \text{on } \partial K^-\,, \end{cases} \tag{17}$$

and

$$\begin{cases} \boldsymbol{a}\cdot\nabla u_b^f = f & \text{in } K\,, \\ u_b^f = 0 & \text{on } \partial K^-\,. \end{cases} \tag{18}$$

The solution for (17) is simply:

$$u_b^0(\boldsymbol{x}) = -u_1(\boldsymbol{x}) + u_1(\boldsymbol{x}_P) \tag{19}$$

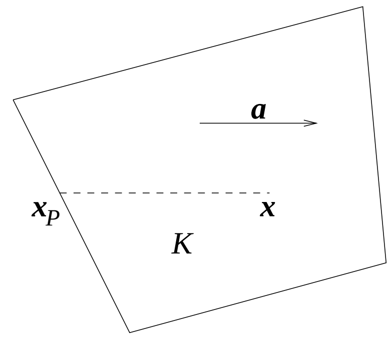


Figure 1: Obtaining $\boldsymbol{x}_P$ from any $\boldsymbol{x} \in K$

where $\boldsymbol{x}_P = \boldsymbol{x}_P(\boldsymbol{x})$ is the intersection of a line parallel to $\boldsymbol{a}$ passing through the point $\boldsymbol{x}$ in equation (19) with the inflow boundary ∂K^- (see Figure 1).

Substituting (19) into (12) and taking into account that for $\kappa \ll h|\boldsymbol{a}|$ the term $(\kappa \triangle v_1, u_b)_K$ is negligible with respect to the other terms, yields:

$$\begin{aligned}\kappa(\nabla u_1, \nabla v_1) + (\boldsymbol{a} \cdot \nabla u_1, v_1) \quad &+(\boldsymbol{a} \cdot \nabla v_1, u_1) - \sum_K (\boldsymbol{a} \cdot \nabla v_1, u_1(\boldsymbol{x}_P))_K \\ &= (f, v_1) + \sum_K (\boldsymbol{a} \cdot \nabla v_1, u_b^f)_K \,. \end{aligned} \tag{20}$$

Up to the third, fourth and last term this is the Galerkin method for the advective diffusive equation using either linears or bilinears. These additional terms represent the effect of the residual-free bubble contribution on the reduced space of linears or bilinears. The formulation can be readily implemented in this form, up to the last term. The solution to (18) can be easily obtained as

$$u_b^f(\boldsymbol{x}) = \frac{1}{|\boldsymbol{a}|} \int_{\boldsymbol{x}_P}^{\boldsymbol{x}} f \, d\, s \tag{21}$$

In order to further investigate what these additional terms represent, we look at a standard stability argument. We will need to define $\boldsymbol{x}_Q$ as the coordinate of the point on the outflow boundary of the element K, aligned with the point $\boldsymbol{x}_P$ in the streamline direction (see Figure 2).

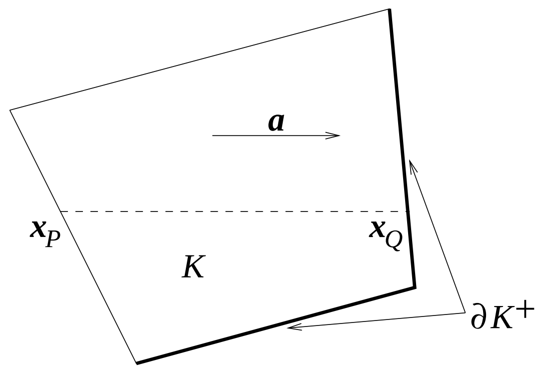


Figure 2: From $\boldsymbol{x}_P$ to $\boldsymbol{x}_Q$ on ∂K^+ along with $\boldsymbol{a}$

Replacing $v_1 = u_1$ and examining the extra terms of (20), we obtain:

$$
\begin{aligned}
(\boldsymbol{a}\cdot\nabla u_1, u_1) - \sum_K (\boldsymbol{a}\cdot\nabla u_1, u_1(\boldsymbol{x}_P))_K &= \sum_K [(\boldsymbol{a}\cdot\nabla u_1, u_1)_K - (\boldsymbol{a}\cdot\nabla u_1, u_1(\boldsymbol{x}_P))_K] \\
&= \sum_K (\boldsymbol{a}\cdot\nabla u_1, u_1 - u_1(\boldsymbol{x}_P))_K \\
&= \sum_K \int_K \boldsymbol{a}\cdot\nabla u_1 [u_1 - u_1(\boldsymbol{x}_P)] d\,\Omega \\
&= \sum_K \int_K \boldsymbol{a}\cdot\nabla u_1 \frac{1}{|\boldsymbol{a}|}\left[\int_{\boldsymbol{x}_P}^{\boldsymbol{x}} \boldsymbol{a}\cdot\nabla u_1 d\,s\right] d\,\Omega \\
&= \sum_K \frac{1}{2|\boldsymbol{a}|^2}\int_K \boldsymbol{a}\cdot\nabla\left[\int_{\boldsymbol{x}_P}^{\boldsymbol{x}} \boldsymbol{a}\cdot\nabla u_1 d\,s\right]^2 d\,\Omega \\
&= \sum_K \frac{1}{|\boldsymbol{a}|^2}\int_{\partial K}\left[\int_{\boldsymbol{x}_P}^{\boldsymbol{x}_Q} \boldsymbol{a}\cdot\nabla u_1 d\,s\right]^2 \boldsymbol{a}\cdot\boldsymbol{n}\, d\,\Gamma \\
&= \sum_K \frac{1}{2|\boldsymbol{a}|}\int_{\partial K^+} [u_1(\boldsymbol{x}_Q) - u_1(\boldsymbol{x}_P)]^2 \boldsymbol{a}\cdot\boldsymbol{n}\, d\,\Gamma
\end{aligned}
$$

Clearly the final expression is positive, since at the outflow boundary $\boldsymbol{a}\cdot\boldsymbol{n} \geq 0$.

From this development, it is apparent that the effect of the residual-free bubbles on the formulation is to add stability through these "jump" terms measuring the differences in the streamline direction of the bilinear (or linear) values of the solution. It is a method which does not coincide with the SUPG method for bilinear u_1. (For linear u_1 (and only for *linear* u_1), it is straightforward to show that the additional term is adding streamline diffusion as in SUPG).

However, this final expression can be shown to be equivalent to the L_2-norm of the streamline diffusion term as in the SUPG method, except for quadrilaterals having two opposite vertices aligned with the streamline direction. We prove this by demonstrating that this expression can only be zero if and only if the streamline derivative is also zero.

There is no loss in generality in stating the following result for the streamline direction aligned with the x-direction, i.e., for constant velocity field $\boldsymbol{a} = (1, 0)$ in element K. Then we close this section by showing the following:

Theorem 1 *Assume there are no opposite vertices of a quadrilateral aligned with the x-direction. For a bilinear u, if $u(\boldsymbol{x}_Q) = u(\boldsymbol{x}_P)$ for every $\boldsymbol{x}_Q \in \partial K^+$, then $u_x = 0$ everywhere in K.*

To prove it, let us use the following Lemmas:

Lemma 1 *Under the assumptions of the theorem, for a bilinear element there exist at least two different vertices not aligned with x where $u_x = 0$.*

Proof: The simple proof of this fact is left to the reader.

Lemma 2 *If u is bilinear and $u_x = 0$ at two different vertices, then there exists a $p \in P_1(K)$ such that $p = u$ at the four corners. Moreover, $\partial p / \partial x = 0$.*

Proof: Let V_1 and V_2 be the two vertices where $u_x = 0$. Let p_1 be the linear interpolant which coincides with u at V_1 and at its two adjacent vertices. Let p_2 be the linear interpolant which coincides with u at V_2 and at its two adjacent vertices. Then $\nabla p_1 = \nabla u$ at V_1 and $\nabla p_2 = \nabla u$ at V_2, which imply that:

$$\frac{\partial p_1}{\partial x} = \frac{\partial p_2}{\partial x} = 0\,. \tag{22}$$

If V_1 and V_2 are consecutive vertices then $p_1 = p_2$ on the edge connecting V_1 and V_2, and, since V_1 and V_2 are not aligned with x, equation (22) implies that $p_1 = p_2$ everywhere.

If V_1 and V_2 are opposite vertices then $p_1 = p_2$ on the straight line connecting the other two opposite vertices. As these vertices are not aligned with x by assumption, we conclude again that $p_1 = p_2$ everywhere.

Lemma 3 *Under the assumptions of the theorem, and if there exists a $p \in P_1(K)$ such that $p = u$ at the four corners of an element K, then $p = u$ everywhere.*

Proof: To start with, $u = p$ on each edge of ∂K, since both functions are linear and they coincide at the vertices. Then we remark that u will also be linear on each straight line that is the image of a line parallel to the axes on the parent (reference) domain. With the same argument as before, we then have $u = p$ on each of such lines, using now that u and p coincide at the two endpoints on ∂K. Since those lines fill K, this ends the proof.

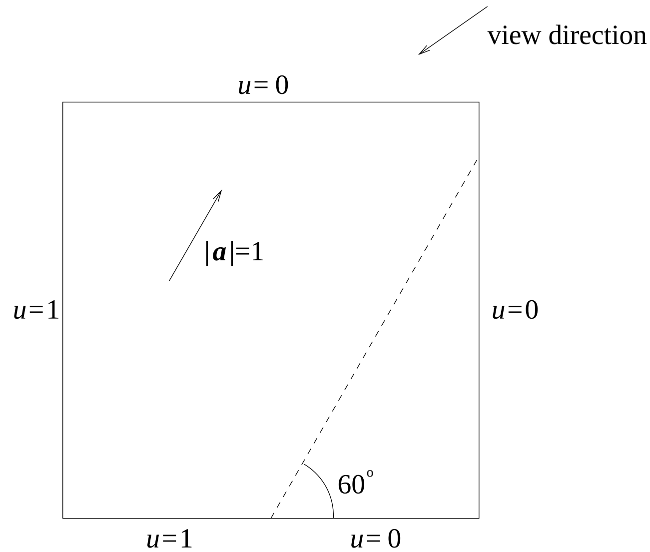


Figure 3: Problem statement

3 Some Numerical Results

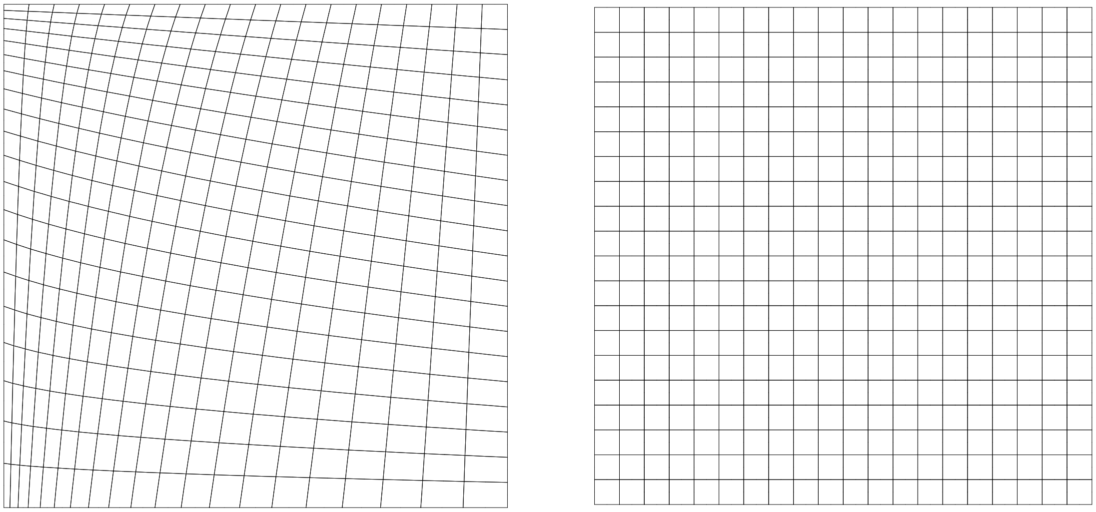
Let us consider the propagation of a discontinuity in the boundary data with a uniform velocity field of size one forming a 60^o with the horizontal axis, on a unit square, with fluid viscosity equal to $\kappa = 10^{-6}$ (see Figure 3 for a problem statement). We first consider this problem on a uniform mesh of 20×20 Q1 elements shown in Fig. 4, and contrast the solution with the SUPG method versus the solution obtained with the RFB method given by equation (20). In Figs. 5 and 6 elevation and contour plots are displayed for both methods. We rerun the experiment on the nonuniform mesh shown in Fig. 4 and plot the results in Figs. 7 and 8. In all the results obtained both methods present comparable graphs, and present only slight differences among them.

Remark We have been able to prove the equivalence of the SUPG and the residual-free bubbles methods under the assumptions of Theorem 1, which exclude the case of a mesh with quadrilaterals whose vertices are aligned with the advection field. However, this seems to be only a technical problem. As a matter of fact, we have run our code on a uniform mesh with a constant field at 45 degrees and our method performs very well, as shown in Figs. 9 and 10.

4 Conclusions

If computed exactly, residual-free bubbles yield the exact solution of linear one dimensional problems. For one-dimensional problems we can *derive* in this way several numerical tricks such as upwinding, mass lumping and selective reduced integration.

For multi-dimensional problems the method is approximate, since, in general, the ex-



Non uniform Uniform

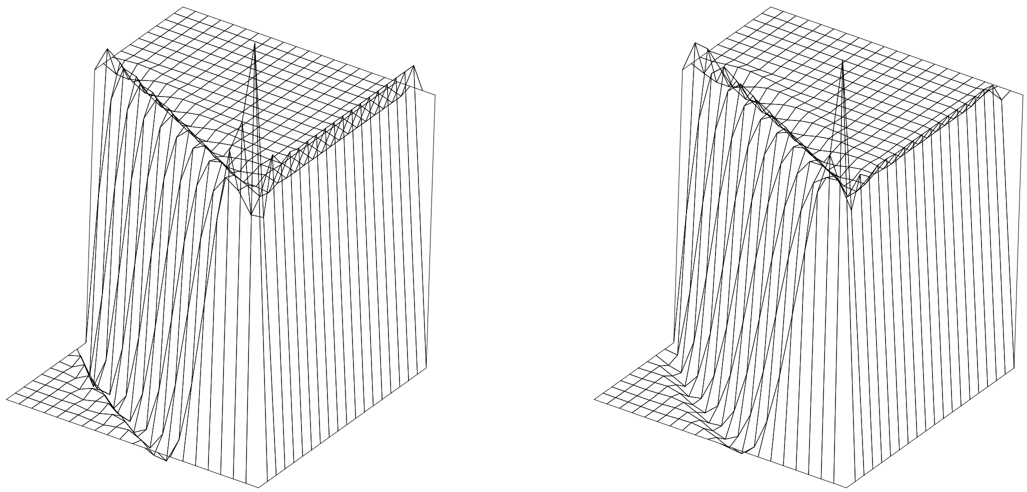
Figure 4: Meshes used in the experiments

act solution may not be linear on element boundaries, as required by the decomposition of the solution (see equation (3)). However, preliminary computations indicate improvement in the accuracy of the results when we compute these functions using analytical techniques such as separation-of-variables or in limiting cases, such as the one discussed in this paper.

From this paper we conclude that if we use bilinear interpolation as the reduced space in a Galerkin method and enrich it with residual-free bubbles, we obtain a stabilization which is not equal to the streamline diffusion one. Only if the reduced space is constituted with linears we recover the SUPG with optimal parameters. We would like to point out that this result does not contradict previously published works on bubbles. In [2] the Galerkin method enriched with bubbles is shown to be equivalent to the SUPG method if the reduced space is of linears. In [1] it is shown that virtual bubbles can be constructed to recover the SUPG or another stabilized method for any order of interpolation. Herein we fix the choice of bubbles to be the residual-free bubble space and further investigate what the numerical scheme is that surfaces when we eliminate them.

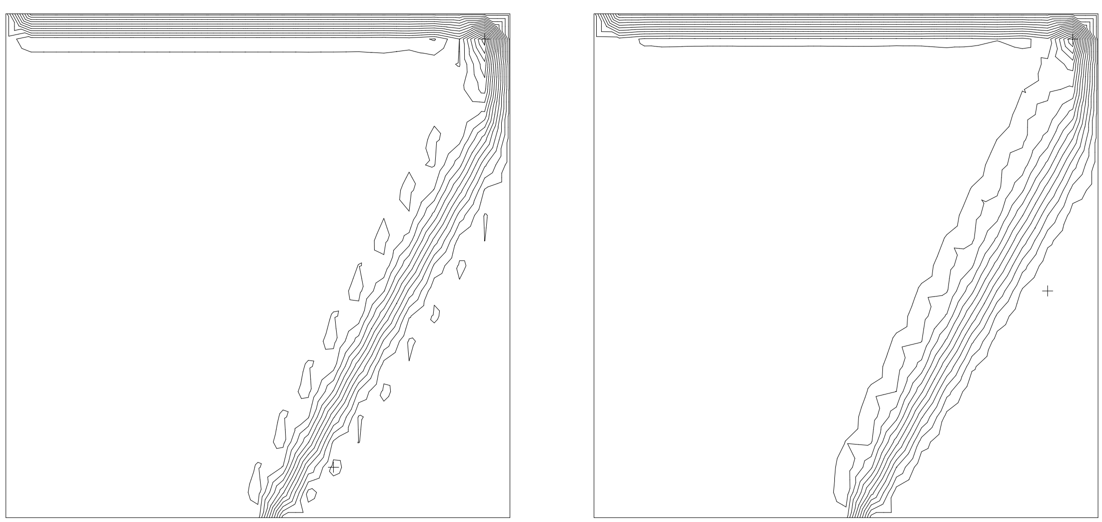
Also from our analysis, when we take $v_1 = u_1$, the additional terms, due to the residual-free bubbles, produce a sum of squares of the jumps that u_1 makes going from ∂K^- to ∂K^+ on the streamline direction. These terms may be identified with the streamline diffusion stabilization, except for some situations as we described in Theorem 1.

Finally, the numerical results do not indicate a major qualitative difference between this and the SUPG method.



RFB SUPG

Figure 5: Elevation plots for RFB versus SUPG: Uniform mesh



RFB SUPG

Figure 6: Contour plots for RFB versus SUPG: Uniform mesh

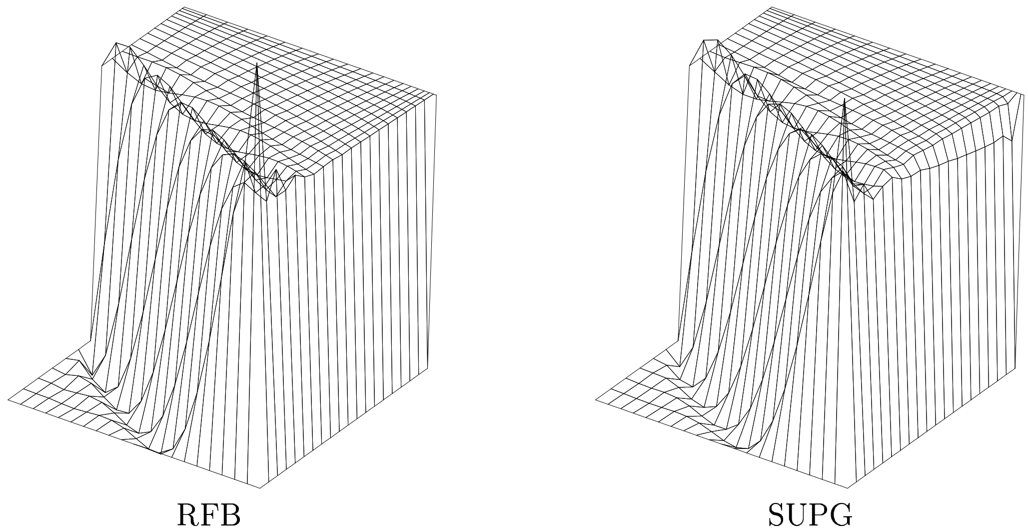


Figure 7: Elevation plots for RFB versus SUPG: Non uniform mesh

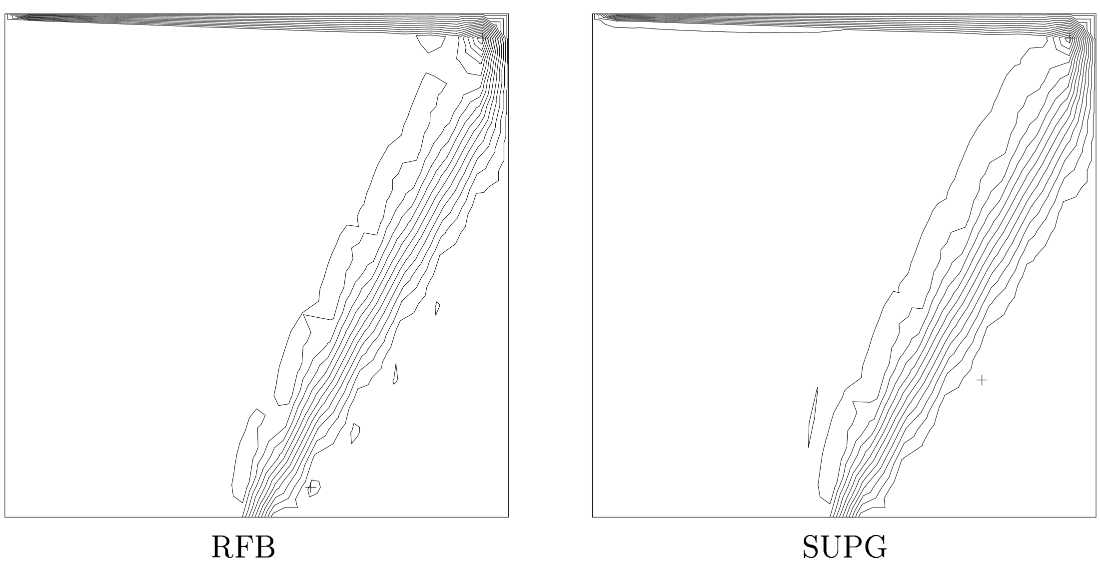
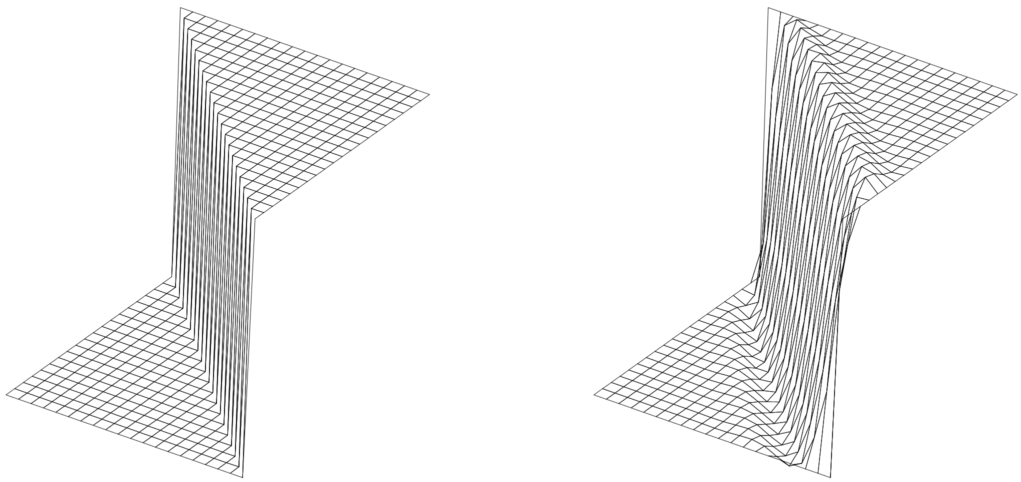
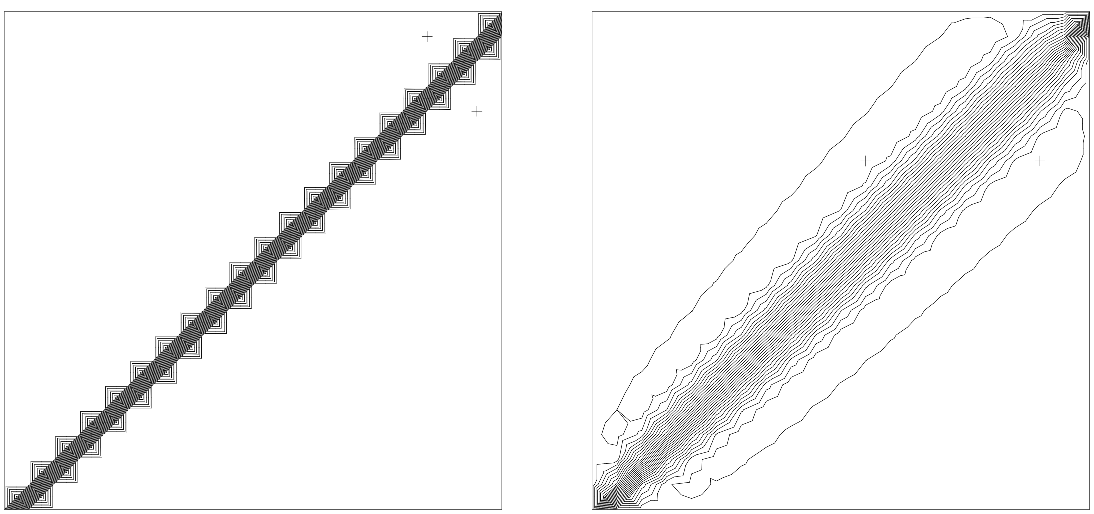


Figure 8: Contour plots for RFB versus SUPG: Non uniform mesh



RFB SUPG

Figure 9: Elevation plots for RFB versus SUPG: Uniform mesh with vertices aligned with the convection field



RFB SUPG

Figure 10: Contour plots for RFB versus SUPG: Uniform mesh with vertices aligned with the convection field

5 Acknowledgments

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