Closed-Form Expressions for Other-Cell Interference in Cellular CDMA

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ABSTRACT

We investigate the ratio of other-cell interference to in-cell interference in cellular CDMA. This ratio appears in expressions for the capacity of the system when "mobiles" are distributed as a (spatial) Poisson process (as in [2]), but must be evaluated by numerical integration when the positions of the "base stations" are fixed (e.g. a hexagonal grid). Our model is identical to [2] except that we distribute both the mobiles and base stations as Poisson processes. Each mobile is power controlled by the "best" base station (least signal attenuation) of the N closest base stations. We present closed form expressions when $N = 1, 2$ and ∞ which are very simple when $N = 1$ and ∞. We argue that the performance (system capacity) predicted by our model is pessimistic since a real system has a more regular arrangement of base stations than a typical Poisson process. However, our simple closed-form expressions yield values that do not differ significantly from the (optimistic) values found in [1], where a regular hexagonal arrangement of base stations is assumed.

1. Introduction.

Viterbi and Viterbi [2] show that the capacity of a cellular CDMA system is approximately proportional to $(1+f)^{-1}$ where f is the ratio of the interference at a base station due to users at the other cells to the interference from its own users. In [1], Viterbi, Viterbi and Zehavi calculate the value of f using numerical integration assuming the base stations are arranged on a regular hexagonal grid. There are four integration formulas presented corresponding to the best base station being chosen from among the $N = 1, 2, 3$, and 4 closest base stations. For example, $f \approx 0.55$ assuming $N = 4$, the propagation exponent is $\mu = 4$ and the parameter of the lognormal attenuation factor is $\sigma = 8$. This value is used in [2] to estimate CDMA cellular reverse link capacity with respect to AMPS.

We replace the hexagonal grid of base stations with a two-dimensional spatial Poisson process and derive simple closed form expressions for the "f-factor" in three cases of interest, $N = 1, 2$, and ∞.

Of course, it would be ill-advised to arrange base stations in the pattern of a Poisson process; the location of base stations in a typical cellular system is "more regular" than a typical sample of a two-dimensional Poisson process. However, due to practical considerations a regular hexagonal grid is rarely possible. In real systems there is an element of randomness in the placement of the base stations so it seems safe to say that the performance of a system with base stations arranged as a Poisson process serves as a pessimistic bound on the performance of a real system, and a regular hexagonal grid arrangement serves as an optimistic bound. We obtain $f = 1$ from a Poisson arrangement for the case $N = \infty, \mu = 4$ and $\sigma = 8$ which implies a capacity that differs from the optimistic bound (using $f = 0.55$) by about 25%. Since our expressions for f are simple and do not yield capacity estimates that differ significantly from the numerical results from more optimistic models, they may be useful for "back of the envelope" calculations of system performance.

2. The Model.

Assume that mobiles are located on the plane S according to a spatial two-dimensional Poisson process with density $\rho_M > 0$. Thus the probability of finding a mobile in a small region of area dA is equal to $\rho_M dA$. Assume that the base stations are located according to a Poisson process with density $\rho_B > 0$.

We assume that base stations are numbered by $0, 1, 2, \ldots$ If there is a mobile located at some given point s of the plane S, the attenuation of its signal from the point s to base station j is

$$A_j(s) = r_j(s)^{-\mu} e^{\alpha X_j(s)}$$

where

$r_j(s)$ is the distance from point s to base station j;

$\mu > 0$ is the propagation exponent;

$X_j(s)$ is a standard normal random variable having mean 0 and variance 1.

$\alpha = 10^{-1}(\ln 10)b\sigma$, where σ is the standard deviation (in dB) of the lognormal shadowing component of the signal attenuation, and $0 < b \leq 1$ is a constant.

The random variables $X_j(s)$ corresponding to different base stations, j, and/or different mobiles are independent. We will use the notation $X(s) \equiv \{X_j(s), j = 0, 1, 2, \ldots\}$ for the set of random variables, $X_j(s)$ corresponding to a mobile located at point s. A reasonable value for the constant b (which reflects a base station specific part of the lognormal shadowing) is $b = 1/\sqrt{2}$. (See [1] for a discussion.)

The ratio of the signal attenuations from the point s to two base stations j and k is then

$$R_{jk}(s) = \frac{r_j(s)^{-\mu} e^{\alpha X_j(s)}}{r_k(s)^{-\mu} e^{\alpha X_k(s)}}$$

If a mobile is located at point s, it is *controlled* by base station $j_c(s)$. The value of $j_c(s)$ depends on the particular rule used for choosing which base station controls each mobile. We consider rules where $j_c(s)$ is the base station among the N closest to the point s that has the greatest value of $A_j(s)$ (i.e. the lowest attenuation). Therefore, $j_c(s)$ depends only on the base station locations and the set of random variables $X(s)$. In Section 4 we will consider $N = 1, 2$, and ∞.

We assume that the signal power received from a mobile located at point s by the base station $j_c(s)$ controlling it is equal to $Y(s)$, where $Y(s)$ is a realization of a random variable Y. Field trials suggest that in a power-controlled CDMA system, the power Y received from an (in-cell) mobile is proportional to $10^{Q/10}$, where Q is a normal random variable with mean 7 and standard deviation 2.4 (see [2]). However, the distribution of Y has no significance to our analysis. All we need is that Y has a finite mean:

$$E(Y) < \infty$$

We also assume that power levels $Y(s)$ received from different mobiles are independent. Therefore, the power received by base station k from a mobile located at point s is

$$W_k(s) = Y(s)R_{k,j_c(s)}(s) \tag{1}$$

Expression (1) is valid for any base station k including the base station $k = j_c(s)$ controlling the mobile, because

$$R_{j,j}(s) = 1$$

The following observation follows from the assumptions we have made above.

The set of ratios $\{R_{k,j}(s), \quad j,k = 0,1,\ldots\}$ *corresponding to a given mobile located at point* s *(and therefore the choice of the base station controlling the mobile) depends only on the location* s *of the mobile, the locations of all base stations, and the set of random variables* $X(s)$ *corresponding to the mobile, and does not depend on the other mobiles' locations and their corresponding sets* $X(\cdot)$.

3. A general expression for the other-cell interference factor f.

The other-cell interference factor is the ratio

$$f = \frac{E(U_{out})}{E(U_{in})}$$

where $E(U_{in})$ is the average "in-cell" interference - the average total power (interference) received by an *arbitrarily picked* base station from all mobiles controlled by *that base station*; $E(U_{out})$ is the average "other-cell" interference - the average total power received by an *arbitrarily picked* base station from all mobiles controlled by *other base stations.*

Due to symmetry we know that the average number of mobiles controlled by an arbitrarily picked base station is ρ_M/ρ_B. Therefore,

$$E(U_{in}) = E(Y)(\rho_M/\rho_B)$$

Without loss of generality, we can assume that the randomly picked base station is base station 0, and (by translating the axes) it is located at the origin of the plane S. The following expression for $E(U_{out})$ follows:

$$E(U_{out}) = E(Y)\int_S E(R_{0,j_c(s)}(s)I\{j_c(s) \neq 0\})\rho_M dA$$

where $\rho_M dA$ is the probability that there is a mobile located in a small neighborhood (of area dA) of a point s; $I\{C\}$ is an indicator function, equal 1 or 0 if C is respectively true or false.

Therefore, we obtain the following expression for the f-factor:

$$f = \rho_B \int_S E(R_{0,j_c(s)}(s) I\{j_c(s) \neq 0\}) dA \tag{2}$$

The following scaling argument proves that f *depends on* ρ_M *and* ρ_B *only through their ratio.* Consider our model with fixed ρ_M and ρ_B. Consider also another model *constructed on the same probability space* with locations of both base stations and mobiles *scaled by a factor* $a > 0$. This means that if in the original model there is a base station (or mobile) located at a point s, then there is a base station (or mobile) located at a point as in the scaled model. There is an inherent one-to-one correspondence between base stations and mobiles in both models. According to the construction, each pair of (corresponding to each other) mobiles in the two models is assigned the same values of $Y(\cdot)$ and the sequence $X(\cdot)$. Therefore, each pair of corresponding mobiles has the same set $\{R_{jk}(\cdot)\}$, and therefore the same assignment of mobiles to base stations. ($R_{jk}(\cdot)$ depends on the ratio $r_j(\cdot)/r_k(\cdot)$ which is the same in both models.) Therefore the powers received from corresponding mobiles by corresponding base stations are the same in both models. Thus, both models have the same f-factor. But, the scaled model is a model with mobile and base stations location densities equal to ρ_M/a^2 and ρ_B/a^2 respectively. Since $a > 0$ can be chosen arbitrarily, any two models having equal ratio ρ_M/ρ_B have the same f-factor.

From (2) we see that f does not depend on ρ_M and the scaling argument shows that it may depend only on the ratio ρ_M/ρ_B. Therefore, the integral in (2) must (implicitly) contain ρ_B^{-1}. In the next section, we assume (without loss of generality) that $\rho_B = 1$ in the evaluation of (2).

4. Closed form expressions for the f-factor.

In this section we derive closed form expressions for the factor f when $N = 1$ and $N = \infty$. The case $N = 2$ is considered in the appendix.

4.1. $N = 1$: Each mobile is controlled by the closest base station.

The distribution of the distance x from a mobile to the closest base station has the following density,

$$g(x) = 2\pi x e^{-\pi x^2}, \; x \geq 0$$

Consider an arbitrarily picked base station. (By our assumption it is located at the origin.) Taking the integral over the distances y from that base station to all possible mobile locations on the entire plane, we get from (2):

$$f = \int_0^\infty 2\pi y dy \int_0^y g(x)dx \int_{-\infty}^\infty \phi(z_1)dz_1 \int_{-\infty}^\infty \phi(z_2)dz_2 \frac{y^{-\mu}e^{\alpha z_1}}{x^{-\mu}e^{\alpha z_2}}$$

Here, $\phi(z)$ is the density of the standard normal distribution,

$$\phi(z) = (2\pi)^{-1/2}e^{-z^2/2}$$

From this multiple integral we arrive at the following simple expression for f.

$$f = \frac{2}{\mu - 2}e^{\alpha^2} \tag{3}$$

4.2. $N = \infty$: Each mobile is controlled by the best base station on the entire plane.

First, for a fixed mobile let us calculate the distribution of the random variable

$$A^* = \max_{j \in J} A_j$$

which is the maximum of attenuation from the mobile to all base stations on the plane. (A^* does not depend on s.) Since $P(A^* \leq t) = \prod_{j \in J} P(A_j(s) \leq t)$ we have

$$\begin{aligned}
\ln P(A^* \leq e^u) &= \int_0^\infty \ln[(1 - 2\pi x dx) + 2\pi x dx P(x^{-\mu}e^{\alpha X} \leq e^u)] \\
&= \int_0^\infty \ln[1 - 2\pi x dx P(x^{-\mu}e^{\alpha X} > e^u)] \\
&= -2\pi \int_0^\infty x dx \int_{(u+\mu \ln x)/\alpha}^\infty \phi(z)dz \\
&= -\pi e^{-\frac{2u}{\mu} + \frac{2\alpha^2}{\mu^2}}
\end{aligned}$$

For convenience let

$$F(u) = \pi e^{-\frac{2u}{\mu} + \frac{2\alpha^2}{\mu^2}}$$

Considering an arbitrarily picked base station and taking the integral over all possible mobile locations, we can write

$$
\begin{aligned}
f &= \int_0^\infty 2\pi y dy \int_{-\infty}^\infty \phi(z) dz \int_{-\infty}^\infty d(e^{-F(u)}) I\{y^{-\mu}e^{\alpha z} < e^u\} y^{-\mu} e^{\alpha z} e^{-u} \\
&= 2\pi \int_{-\infty}^\infty d(e^{-F(u)}) \int_{-\infty}^\infty \phi(z) dz e^{\alpha z - u} \int_{e^{(\alpha z - u)/\mu}}^\infty y^{-\mu+1} dy \\
&= \frac{2}{\mu - 2} \int_{-\infty}^\infty d(e^{-F(u)}) F(u) \\
&= \frac{2}{\mu - 2}
\end{aligned}
\tag{4}
$$

As we see, the f factor in this case is invariant with respect to α, and therefore σ.

5. Conclusions.

A major conclusion of this work is that the other-cell interference factor f *is* sensitive to the pattern of base station locations. The more "irregular" the placement of base stations the greater f-factor. It is natural to expect, that a placement of base stations following a sample of a Poisson process is "more irregular" than any placement of base stations likely to occur in practice. Thus, we believe that the simple formulas for the f-factor we derived can serve as "worst case" (upper) bounds for the f-factor in a real system.

Another important observation is that the soft handoff mechanism, i.e., the possibility for a mobile to choose the "best" base station, mitigates in some sense the negative impact of the lognormal shadowing on the system capacity. Indeed, in our model the other-cell interference factor in the system where mobiles choose the best base station ($N = \infty$) *does not depend* on the standard deviation σ of the lognormal shadowing, and has the same value $2/(\mu - 2)$ as in the system with no lognormal shadowing, where the best base station is always the closest one.

APPENDIX

We consider here the case $N = 2$: each mobile is controlled by the better of the two closest base stations.

In the expression for f we will use the following notation:

$$
q(\xi, z_\xi; \eta, z_\eta) \equiv \frac{\xi^{-\mu} e^{\alpha z_\xi}}{\eta^{-\mu} e^{\alpha z_\eta}}
$$

Also recall that $I\{C\}$ is an indicator function, equal 1 or 0 if C is respectively true or false, and $\phi(z)$ is the density of the standard normal distribution.

We can write the following expression for f

$$f = \int_0^\infty 2\pi y dy \int_0^\infty 2\pi x dx \exp(-\pi x^2) \int_x^\infty 2\pi h dh \exp(-\pi(h^2 - x^2))$$

$$\cdot \int_{-\infty}^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty \phi(z_y) dz_y \phi(z_x) dz_x \phi(z_h) dz_h$$

$$\cdot \{I\{y < h\} I\{q(y, z_y; x, z_x) < 1\} q(y, z_y; x, z_x)$$

$$+ I\{h \le y\} \left[I\{q(x, z_x; h, z_h) < 1\} q(y, z_y; h, z_h) + I\{q(x, z_x; h, z_h) \ge 1\} q(y, z_y; x, z_x) \right] \}$$

We omit the evaluation of the integral (which is not complicated, but rather routine and long) and present here the result only:

$$f = (2\pi)^2 e^{\alpha^2} (d_1 + d_2) \tag{5}$$

where

$$d_1 = B(1 - \mu, 2) + \frac{2\pi}{\mu - 2} B(3 - \mu, 1)$$

$$d_2 = C(1 + \mu, 2) + \frac{2\pi}{\mu - 2} B(3 + \mu, 1)$$

$$B(\delta, \beta) = \frac{1}{2(\mu + 2)} \pi^{-(\delta + \mu + 3)/2} ((\delta + \mu + 1)/2)! \times$$

$$\left[\exp(-(\mu + 2)\beta\alpha^2/\mu + (\mu + 2)^2 \alpha^2/\mu^2) \bar{\Phi}(-\beta\alpha/\sqrt{2} + \sqrt{2}(\mu + 2)\alpha/\mu) + \Phi(-\beta\alpha/\sqrt{2}) \right]$$

$$C(\delta, \beta) = \frac{1}{2(2 - \mu)} \pi^{-(\delta - \mu + 3)/2} ((\delta - \mu + 1)/2)! \times$$

$$\left[-\exp((2 - \mu)\beta\alpha^2/\mu + (2 - \mu)^2 \alpha^2/\mu^2) \bar{\Phi}(\beta\alpha/\sqrt{2} + \sqrt{2}(2 - \mu)\alpha/\mu) + \bar{\Phi}(\beta\alpha/\sqrt{2}) \right]$$

$$\Phi(x) = \int_{-\infty}^x \phi(z) dz$$

$$\bar{\Phi}(x) = 1 - \Phi(x)$$

References.

[1] A.J.Viterbi, A.M.Viterbi, and E.Zehavi, "Other-Cell Interference in Cellular Power-Controlled CDMA", *IEEE Trans. Commun.*, Vol. 42, (1994), pp. 1501-1504, No. 2/3/4.

[2] A.J.Viterbi, and A.M.Viterbi, "Erlang Capacity of a Power Controlled CDMA System", *IEEE Journal on Selected Areas in Communications*, Vol. 11, (1993), pp. 892-899, No. 6.