AN EFFICIENT ITERATIVE METHOD FOR STOKES AND LAMÉ EQUATIONS FOR NEARLY INCOMPRESSIBLE MEDIA WITH HIGHLY DISCONTINUOUS COEFFICIENTS

N. S. BAKHVALOV*, A. V. KNYAZEV† AND R. R. PARASHKEVOV‡

Abstract. We consider an efficient iterative solution technique for the isotropic linear elasticity (Lamé) equations for nearly incompressible media and Stokes equations with highly discontinuous coefficients. The iterative method involves a special choice for an initial guess and a preconditioner based on solving a constant coefficient problem.

For simplicity, we only analyze a periodic boundary value problem. Some other standard boundary value problems can be treated similarly, or can be reduced to the periodic case by using the fictitious domain method.

For the Lamé equations, we also discuss the case of absolutely compressible media.

Key words. linear elasticity, incompressible, Lamé equations, Stokes equations, discontinuous coefficients, iterative methods, divergence-free vector functions, fictitious domain method, Cosserat problem

AMS(MOS) subject classifications. **65F10, 65N12, 73C02, 76D07.**

Dedicated to Olof B. Widlund on the occasion of his 60th birthday

1. Introduction. In [1, 2, 7, 3, 4] efficient preconditioned iterative methods for boundary value problems for stationary differential equations with large jumps in the coefficients have been proposed. For the corresponding discrete problems, analogous methods have been suggested in [12, 5]. For some classes of problems, it has been shown that the increase in the coefficient jump does not cause deterioration in the convergence in a natural coefficient-independent norm. The choice of the preconditioner is determined by the problem. Usually, a problem similar to the original one, but with constant coefficients, needs to be solved on every step of the iterative method. Several very efficient algorithms are known for such problems, and can be used for our preconditioning. We discuss some possibilities in Section 5.

In the present paper, some results from [4] are extended to incompressible and nearly incompressible medium. For simplicity, periodic boundary value problems for the Stokes equations, in Section 2, and the Lamé equations, in Section 3, with piece-wise constant coefficients are considered. Some other standard boundary value problems can be treated similarly, or can be reduced to the periodic case by using the fictitious domain method, see Section 6.

Our theory is based on extension theorems. For the Stokes equations, we use extension of solenoidal, or divergence-free, functions from W_2^1. For the Lamé equa-

* Institute of Numerical Mathematics, Russian Academy of Sciences, Moscow, Russia

† Department of Mathematics, University of Colorado at Denver, P.O. Box 173364, Campus Box 170, Denver, CO 80217-3364. *e-mail*: `aknyazev@math.cudenver.edu` *WWW URL*: `http://www-math.cudenver.edu/~aknyazev`. The second author was supported by NSF grant DMS 9501507.

‡ Department of Mathematics, University of Colorado at Denver, P.O. Box 173364, Campus Box 170, Denver, CO 80217-3364. *e-mail*: `rparashk@carbon.cudenver.edu`

tions, a theorem of extension of functions from W_2^1 preserving the energy norm of the Lamé operator uniformly with respect to the parameter λ as $\lambda \to +\infty$ is proved. We present proofs of the results announced in [3].

2. Stokes Equations. We consider a three-dimensional periodic boundary value problem for Stokes equations in the form

$$
\begin{aligned}
2\frac{\partial\left[\mu(\mathbf{x})\epsilon_{ij}(\mathbf{u})\right]}{\partial x_i} + \frac{\partial p}{\partial x_j} &= \frac{\partial f_{ij}}{\partial x_i}, \\
\operatorname{div}\mathbf{u} &= 0,
\end{aligned} \tag{2.1}
$$

where $\mathbf{u} = (u_1, u_2, u_3)^T$ and $\epsilon_{ij}(\mathbf{u}) \stackrel{\text{def}}{=} \frac{1}{2}\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right]$.

We assume the summation on repeating indices from 1 to 3. For our periodic boundary value problem we assume that all functions are defined on a three-dimensional unit torus $\mathbf{T}$ with Cartesian coordinate system. We define the unit torus to be the unit cube with all pairs of opposite faces identified. For a Lipschitz domain $\mathbf{\Omega} \subseteq \mathbf{T}$ the following norms will be used throughout the paper:

$$
\| \mathbf{w} \|^2_{\left[W_2^1(\mathbf{\Omega})\right]^3} \stackrel{\text{def}}{=} \int_{\mathbf{\Omega}} \frac{\partial w_i}{\partial x_j}\frac{\partial w_i}{\partial x_j} d\mathbf{x}, \quad \mathbf{w} \in [W_2^1(\mathbf{\Omega})]^3/R^3
$$

and

$$
\| \mathbf{w} \|_{\left[W_2^{1/2}(\partial\mathbf{\Omega})\right]^3} \stackrel{\text{def}}{=} \inf_{\substack{\mathbf{v}\in[W_2^1(\mathbf{\Omega})]^3 \\ \mathbf{v}|_{\partial\mathbf{\Omega}}=\mathbf{w}}} \| \mathbf{v} \|_{\left[W_2^1(\mathbf{\Omega})\right]^3}, \quad \mathbf{w} \in \left[W_2^{1/2}(\partial\mathbf{\Omega})\right]^3/R^3.
$$

Let $\mathbf{H}$ be the factor space with regard to the constants from R^3 of the space of *solenoidal* functions from $[W_2^1(\mathbf{T})]^3$, equipped with the norm $\|\cdot\|_{\left[W_2^1(\mathbf{T})\right]^3}$. Given $f_{ij} \in L_2(\mathbf{T})$, a weak solution of problem (2.1) is a function $\mathbf{u} \in \mathbf{H}$, satisfying

$$
\Lambda(\mathbf{u},\mathbf{v}) = \int_{\mathbf{T}} f_{ij}\frac{\partial v_j}{\partial x_i} d\mathbf{x}, \quad \forall\, \mathbf{v} \in \mathbf{H}, \tag{2.2}
$$

where $\Lambda(\mathbf{u},\mathbf{v}) \stackrel{\text{def}}{=} 2\int_{\mathbf{T}} \mu(\mathbf{x})\epsilon_{ij}(\mathbf{u})\epsilon_{ij}(\mathbf{v})d\mathbf{x}$. Let $\mathbf{D} \subset \mathbf{T}$ be a Lipschitz domain and its complement $\mathbf{T}\backslash\overline{\mathbf{D}}$ be also a domain. We assume that the viscosity coefficient $\mu()$ is piece-wise constant, and equals to $\mu > 0$ and $\mu^* > 0$ in $\mathbf{D}$ and $\mathbf{T}\backslash\overline{\mathbf{D}}$ correspondingly. We will need two auxiliary results.

Proposition 2.1. *For any Lipschitz domain $\mathbf{\Omega} \subset \mathbf{T}$ there exists a constant $\kappa_1 \equiv \kappa_1(\mathbf{\Omega}) > 0$, such that for an arbitrary function $\mathbf{v} \in [W_2^1(\mathbf{\Omega})]^3/R^3$ a function $\mathbf{w} \in [W_2^1(\mathbf{\Omega})]^3/R^3$ can be found for which* $\int_{\mathbf{\Omega}} \epsilon_{ij}(\mathbf{w})\epsilon_{ij}(\mathbf{w})d\mathbf{x} \geq \kappa_1 \| \mathbf{w} \|^2_{\left[W_2^{1/2}(\partial\mathbf{\Omega})\right]^3}$

and $\mathbf{w} = \mathbf{v} + C\mathbf{x}$ *in* $\mathbf{\Omega}$ *with some* 3×3 *matrix* $C = -C^T$, *not depending on* $\mathbf{x} \in \mathbf{T}$, *the vector of independent variables on the torus* $\mathbf{T}$.

Proof. The following version of a Korn type inequality is obtained in [4, Section 5]: for a Lipschitz domain $\mathbf{\Omega}$ on the torus $\mathbf{T}$ there can be found a constant $\kappa_1 \equiv \kappa_1(\mathbf{\Omega}) > 0$, for which for any function $\mathbf{v} \in [W_2^1(\mathbf{\Omega})]^3/R^3$ there exist a function $\mathbf{w} \in [W_2^1(\mathbf{\Omega})]^3/R^3$ such that $\int_{\mathbf{\Omega}} \epsilon_{ij}(\mathbf{w})\epsilon_{ij}(\mathbf{w})d\mathbf{x} \geq \kappa_1 \int_{\mathbf{\Omega}} \frac{\partial w_i}{\partial x_j}\frac{\partial w_i}{\partial x_j} d\mathbf{x}$ and $\mathbf{w} = \mathbf{v} + C\mathbf{x}$ in $\mathbf{\Omega}$ with some 3×3 constant matrix $C = -C^T$.

By definition,

$$\begin{aligned} \int_{\mathbf{\Omega}} \frac{\partial w_i}{\partial x_j}\frac{\partial w_i}{\partial x_j} d\mathbf{x} &= \quad \| \mathbf{w} \|^2_{[W_2^1(\mathbf{\Omega})]^3} \\ &\geq \| \mathbf{w} \|^2_{\left[W_2^{1/2}(\partial\mathbf{\Omega})\right]^3}, \end{aligned}$$

and we come to the desired inequality. □

PROPOSITION 2.2. *For any Lipschitz domain* $\mathbf{\Omega} \subset \mathbf{T}$ *there exists a positive constant* $\theta(\mathbf{\Omega}) < \infty$ *such that for an arbitrary function* $\mathbf{v} \in [W_2^{1/2}(\partial\mathbf{\Omega})]^3/R^3$ *under the condition* $\oint(\mathbf{v}, \mathbf{n})d\partial\mathbf{\Omega} = 0$, *there can be found a solenoidal extension* $\mathbf{v} \in [W_2^1(\mathbf{\Omega})]^3/R^3$, *for which* $\| \mathbf{v} \|^2_{[W_2^1(\mathbf{\Omega})]^3} \leq \theta(\mathbf{\Omega}) \| \mathbf{v} \|^2_{[W_2^{1/2}(\partial\mathbf{\Omega})]^3}$.

Proof. In [14], this proposition has been proven for a bounded Lipschitz domain $\mathbf{\Omega} \subset R^3$. The proof is applicable for a Lipschitz domain on a torus as well.

The desired solenoidal extension can also be found explicitly in $\mathbf{\Omega}$ as a weak solution of the following problem:

$$\Delta\mathbf{v} + \text{grad } q = \mathbf{0}$$

$$\text{div } \mathbf{v} = 0,$$

with the trace of $\mathbf{v}$ given on the boundary $\partial\mathbf{\Omega}$. Then, the desired estimate of the lemma is simply the well-known stability estimate for the problem above, e.g., [11]. □

We are now ready to prove our main extension result for solenoidal functions.

LEMMA 2.1. *There exists a constant* $\kappa = \kappa(\mathbf{D}) > 0$, *such that for an arbitrary function* $\mathbf{v} \in \mathbf{H}$ *there exist a function* $\mathbf{w} \in \mathbf{H}$ *such that* $\int_{\mathbf{D}} \epsilon_{ij}(\mathbf{w})\epsilon_{ij}(\mathbf{w})d\mathbf{x} \geq \kappa \int_{\mathbf{T}} \epsilon_{ij}(\mathbf{w})\epsilon_{ij}(\mathbf{w})d\mathbf{x}$ *and* $\mathbf{w} = \mathbf{v} + C\mathbf{x}$ *in* $\mathbf{D}$ *with some* 3×3 *matrix* $C = -C^T$ *independent of* $\mathbf{x}$.

Proof. Let us apply Proposition 2.1 to the function $\mathbf{v}$ in the domain $\mathbf{\Omega} = \mathbf{D}$. There exist a function $\mathbf{w}$ and a constant $\kappa_1 = \kappa_1(\mathbf{D})$, such that

$$\int_{\mathbf{D}} \epsilon_{ij}(\mathbf{w})\epsilon_{ij}(\mathbf{w})d\mathbf{x} \quad \geq \quad \kappa_1 \parallel \mathbf{w} \parallel^2_{[W_2^{1/2}(\partial\mathbf{D})]^3},$$

where $\mathbf{w} = \mathbf{v} + C\mathbf{x}$ in $\mathbf{D}$, for some constant antisymmetric matrix C. By assumption, the function $\mathbf{v}$ is solenoidal on the torus $\mathbf{T}$, therefore it is solenoidal in $\mathbf{D}$. Then $\mathbf{w}$ is solenoidal in $\mathbf{D}$, and its trace on the boundary $\partial\mathbf{D}$ satisfies the following

$$\oint(\mathbf{w}, \mathbf{n})d\partial\mathbf{D} \quad = \quad \int \text{div } \mathbf{w} \, d\mathbf{D} = 0.$$

Next, we use Proposition 2.2 for the domain $\mathbf{\Omega} = \mathbf{T}\backslash\overline{\mathbf{D}}$. For $\mathbf{w} \in [W_2^1(\mathbf{D})]^3/R^3$ there exists a solenoidal extension $\mathbf{w} \in [W_2^1(\mathbf{T}\backslash\overline{\mathbf{D}})]^3/R^3$ such that

$$\parallel \mathbf{w} \parallel^2_{[W_2^1(\mathbf{T}\backslash\overline{\mathbf{D}})]^3} \quad \leq \quad \theta \parallel \mathbf{w} \parallel^2_{[W_2^{1/2}(\partial\mathbf{D})]^3},$$

where $\theta = \theta(\mathbf{T}\backslash\overline{\mathbf{D}})$.

Finally,

$$\parallel \mathbf{w} \parallel^2_{[W_2^1(\mathbf{T}\backslash\overline{\mathbf{D}})]^3} \quad = \quad \int_{\mathbf{T}\backslash\overline{\mathbf{D}}} \frac{\partial w_i}{\partial x_j}\frac{w_i}{\partial x_j} \geq 2\int_{\mathbf{T}\backslash\overline{\mathbf{D}}} \epsilon_{ij}(\mathbf{w})\epsilon_{ij}(\mathbf{w})d\mathbf{x}. \tag{2.3}$$

In this way, the function $\mathbf{w}$ is solenoidal on the whole torus $\mathbf{T}$ and $\mathbf{w} \in [W_2^1(\mathbf{T})]^3/R^3$, i.e. $\mathbf{w} \in \mathbf{H}$; and the following estimate holds

$$\int_{\mathbf{D}} \epsilon_{ij}(\mathbf{w})\epsilon_{ij}(\mathbf{w})d\mathbf{x} \quad \geq \quad 2\frac{\kappa_1}{\theta}\int_{\mathbf{T}\backslash\overline{\mathbf{D}}} \epsilon_{ij}(\mathbf{w})\epsilon_{ij}(\mathbf{w})d\mathbf{x},$$

equivalent to the one we are looking for with

$$\frac{1}{\kappa} = 1 + \frac{\theta}{2\kappa_1}.$$

□

Proof. [Alternative proof of Lemma 2.1] One can prove Lemma 2.1 without using traces, as suggested by E. G. D'yakonov (private communication).

The first step is just the same as in the proof of Proposition 2.1. We use a Korn type inequality obtained in [4, Section 5]: for the given function $\mathbf{v} \in [W_2^1(\mathbf{D})]^3/R^3$ there exist a function $\mathbf{w} \in [W_2^1(\mathbf{D})]^3/R^3$, such that

$$\int_{\mathbf{D}} \epsilon_{ij}(\mathbf{w})\epsilon_{ij}(\mathbf{w})d\mathbf{x} \quad \geq \quad \kappa_1 \parallel \mathbf{w} \parallel^2_{[W_2^1(\mathbf{D})]^3},$$

where $\mathbf{w} = \mathbf{v} + C\mathbf{x}$ in $\mathbf{D}$ with a constant antisymmetric matrix C. By assumption, the function $\mathbf{v}$ is solenoidal on the torus $\mathbf{T}$, thus it is solenoidal in $\mathbf{D}$, and then $\mathbf{w}$ is solenoidal in $\mathbf{D}$ as well.

The second step is different. As shown in [4], for any function $\mathbf{w} \in [W_2^1(\mathbf{D})]^3/R^3$ there exists an extension, not necessarily solenoidal, $\mathbf{r} \in [W_2^1(\mathbf{T})]^3/R^3$, such that $\mathbf{r} = \mathbf{w}$ in $\mathbf{D}$ and $\|\mathbf{r}\|^2_{[W_2^1(\mathbf{T}\backslash\overline{\mathbf{D}})]^3} \le C(\mathbf{D})\,\|\mathbf{w}\|^2_{[W_2^1(\mathbf{D})]^3}$.

Now, we set $\mathbf{w} = \mathbf{r} + \mathbf{s}$ in $\mathbf{T}\backslash\overline{\mathbf{D}}$, where $\mathbf{s}$ is a weak solution to the following homogeneous Dirichlet problem:

$$\Delta(\mathbf{r} + \mathbf{s}) + \text{grad}\, q = \mathbf{0}$$

$$\text{div}\,(\mathbf{r} + \mathbf{s}) = 0,$$

with $\mathbf{s} = \mathbf{0}$ on the boundary $\partial\mathbf{D}$. The boundary condition is chosen in such a way that we can extend $\mathbf{s}$ as zero inside $\mathbf{D}$. We have a stability estimate, e.g., [8, Chapter 2],

$$\|\mathbf{s}\|^2_{[W_2^1(\mathbf{T}\backslash\overline{\mathbf{D}})]^3} \le C_1(\mathbf{D})\,\|\mathbf{r}\|^2_{[W_2^1(\mathbf{T}\backslash\overline{\mathbf{D}})]^3},$$

which shows that

$$\|\mathbf{w}\|^2_{[W_2^1(\mathbf{T}\backslash\overline{\mathbf{D}})]^3} = \|\mathbf{s}+\mathbf{r}\|^2_{[W_2^1(\mathbf{T}\backslash\overline{\mathbf{D}})]^3} \le 2\left(\|\mathbf{s}\|^2_{[W_2^1(\mathbf{T}\backslash\overline{\mathbf{D}})]^3} + \|\mathbf{r}\|^2_{[W_2^1(\mathbf{T}\backslash\overline{\mathbf{D}})]^3}\right) \le 2(1+C_1)\|\mathbf{r}\|^2_{[W_2^1(\mathbf{T}\backslash\overline{\mathbf{D}})]^3}.$$

Combining with (2.3) and the previous estimates, we derive the desired inequality with $\kappa^{-1} = 1 + \kappa_1^{-1}C(1 + C_1)$. □

Let us denote

$$\Lambda^*(\mathbf{v}, \mathbf{w}) \overset{\text{def}}{=} 2\mu^* \int_{\mathbf{T}} \epsilon_{ij}(\mathbf{v})\epsilon_{ij}(\mathbf{w})d\mathbf{x}, \qquad \forall\, \mathbf{v}, \mathbf{w} \in \mathbf{H}, \tag{2.4}$$

and notice that the bilinear form $\frac{1}{\mu^*}\Lambda^*(\cdot,\cdot)$ describes a new scalar product on $\mathbf{H}$ and the corresponding norm, which is equivalent to the original norm $\|\cdot\|^2_{[W_2^1(\mathbf{T})]^3}$.

We consider the following iterative method for problem (2.2):

$$\Lambda^*(\frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\tau}, \mathbf{v}) + \Lambda(\mathbf{u}^n, \mathbf{v}) = \int_{\mathbf{T}} f_{ij}\frac{\partial v_j}{\partial x_i}d\mathbf{x}, \quad \forall\, \mathbf{v} \in \mathbf{H}, \;\; n = 0, 1, \cdots, \tag{2.5}$$

where the initial guess $\mathbf{u}^0 \in \mathbf{H}$ is determined by the equation

$$\Lambda^*(\mathbf{u}^0, \mathbf{v}) = \int_{\mathbf{T}} g_{ij}\frac{\partial v_j}{\partial x_i}d\mathbf{x}, \quad \forall\, \mathbf{v} \in \mathbf{H}, \tag{2.6}$$

with

$$g_{ij} = \begin{cases} \text{an arbitrary function of } L_2(\mathbf{D}) \text{ in } \mathbf{D}, \\ f_{ij} \text{ in } \mathbf{T}\backslash\overline{\mathbf{D}}. \end{cases} \tag{2.7}$$

Using Lemma 2.1 and arguments analogous to those in [4], we prove the following

LEMMA 2.2. *We define the subspace* $\mathbf{N} \subset \mathbf{H}$ *of functions* $\mathbf{w} \in \mathbf{H}$ *such that* $\mathbf{w} = C\mathbf{x}$ *in the domain* $\mathbf{D}$ *where* $\mathbf{x} \in \mathbf{T}$ *is the vector of independent variables on the torus* $\mathbf{T}$ *and* $C = -C^T$ *is any* 3×3 *matrix, independent of* $\mathbf{x}$. *Let the subspace* $\mathbf{R} \subset \mathbf{H}$ *be defined as the orthogonal, with respect to the bilinear form* $\Lambda^*(\cdot,\cdot)$, *complement of* $\mathbf{N}$, *i.e.*

$$\Lambda^*(\mathbf{v}, \mathbf{w}) = 0, \quad \forall\, \mathbf{v} \in \mathbf{R}, \mathbf{w} \in \mathbf{N}.$$

Then:

(a) for the initial guess $\mathbf{u}^0$, *we have* $\mathbf{u}^0 - \mathbf{u} \in \mathbf{R}$ *where* $\mathbf{u}$ *is the true solution of our problem (2.2);*

(b) $\mathbf{R}$ *is an invariant subspace for the error propagation operator from* $\boldsymbol{\epsilon}^n = \mathbf{u}^n - \mathbf{u}$ *to* $\boldsymbol{\epsilon}^{n+1} = \mathbf{u}^{n+1} - \mathbf{u}$ *acting according to the rule:*

$$\Lambda^*(\frac{\boldsymbol{\epsilon}^{n+1} - \boldsymbol{\epsilon}^n}{\tau}, \mathbf{v}) + \Lambda(\boldsymbol{\epsilon}^n, \mathbf{v}) \quad = \quad 0, \quad \forall\, \mathbf{v} \in \mathbf{H},\ n = 0, 1, \cdots; \tag{2.8}$$

(c) on this invariant subspace, we have

$$\kappa\, \frac{\mu}{\mu^*}\, \Lambda^*(\mathbf{v}, \mathbf{v}) \leq \Lambda(\mathbf{v}, \mathbf{v}) \quad \leq \quad \max\{\frac{\mu}{\mu^*}, 1\}\, \Lambda^*(\mathbf{v}, \mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{R}$$

PROPOSITION 2.3. *The following equality holds:*

$$\Lambda^*(\mathbf{v}, \mathbf{w}) \quad = \quad \Lambda(\mathbf{v}, \mathbf{w}), \quad \forall \mathbf{v} \in \mathbf{H}, \mathbf{w} \in \mathbf{N}.$$

Proof. We note that $\epsilon_{ij}(\mathbf{w}) = 0$ in the domain $\mathbf{D}$ when $\mathbf{w} \in \mathbf{N}$. Therefore,

$$\begin{aligned} \Lambda^*(\mathbf{v}, \mathbf{w}) = 2\mu^* \int_{\mathbf{T}} \epsilon_{ij}(\mathbf{v})\epsilon_{ij}(\mathbf{w}) d\mathbf{x} \quad &= \quad 2\mu^* \int_{\mathbf{T}\setminus\overline{\mathbf{D}}} \epsilon_{ij}(\mathbf{v})\epsilon_{ij}(\mathbf{w})\, d\mathbf{x} = \\ 2\int_{\mathbf{T}\setminus\overline{\mathbf{D}}} \mu(\mathbf{x})\epsilon_{ij}(\mathbf{v})\epsilon_{ij}(\mathbf{w})\, d\mathbf{x} \quad &= \quad 2\int_{\mathbf{T}} \mu(\mathbf{x})\epsilon_{ij}(\mathbf{v})\epsilon_{ij}(\mathbf{w}) d\mathbf{x} = \Lambda(\mathbf{v}, \mathbf{w}). \end{aligned}$$

□

Proof. [Lemma 2.2]

(a) According to the definition of the space $\mathbf{R}$, we need to verify the equality $\Lambda^*(\mathbf{u}^0 - \mathbf{u}, \mathbf{w}) = 0, \mathbf{w} \in \mathbf{N}$, which is equivalent to the equality $\Lambda^*(\mathbf{u}, \mathbf{w}) = \Lambda(\mathbf{u}, \mathbf{w})$. The last equality is true by Proposition 2.3.

(b) Let us consider the bounded linear operator $A : \mathbf{H} \to \mathbf{H}$ defined by

$$\Lambda^*(A\mathbf{v}, \mathbf{w}) = \Lambda(\mathbf{v}, \mathbf{w}), \quad \forall \mathbf{v}, \mathbf{w} \in \mathbf{H}.$$

The error propagation operator from $\boldsymbol{\epsilon}^n$ to $\boldsymbol{\epsilon}^{n+1}$ is $I - \tau A$, where I is the identity operator, i.e. $\boldsymbol{\epsilon}^{n+1} = (I - \tau A)\boldsymbol{\epsilon}^n$. The statement needed to be verified is equivalent to the statement that $\mathbf{R}$ is an invariant subspace for the operator A. If $\Lambda^*(\mathbf{v}, \mathbf{w}) =$

0, $\mathbf{v} \in \mathbf{H}, \mathbf{w} \in \mathbf{N}$, then $\Lambda(\mathbf{v}, \mathbf{w}) = 0$ by Proposition 2.3 and, therefore, $\Lambda^*(A\mathbf{v}, \mathbf{w}) = 0$ by the definition of the operator A.

(c) The right inequality is true for any function $\mathbf{v} \in \mathbf{H}$ and can be proven directly. Let us verify the left inequality. For any $\mathbf{v} \in \mathbf{H}$, we have

$$\begin{aligned} \Lambda(\mathbf{v}, \mathbf{v}) &= 2\mu \int_{\mathbf{D}} \epsilon_{ij}(\mathbf{v})\epsilon_{ij}(\mathbf{v}) d\mathbf{x} + 2\mu^* \int_{\mathbf{T}\setminus\overline{\mathbf{D}}} \epsilon_{ij}(\mathbf{v})\epsilon_{ij}(\mathbf{v}) d\mathbf{x} \\ &\geq 2\mu \int_{\mathbf{D}} \epsilon_{ij}(\mathbf{v})\epsilon_{ij}(\mathbf{v}) d\mathbf{x}. \end{aligned}$$

Also, by Lemma 2.1, we can find a function $\mathbf{w} \in \mathbf{H}$ such that $\mathbf{v} - \mathbf{w} \in \mathbf{N}$ and

$$\begin{aligned} \int_{\mathbf{D}} \epsilon_{ij}(\mathbf{v})\epsilon_{ij}(\mathbf{v}) d\mathbf{x} &= \int_{\mathbf{D}} \epsilon_{ij}(\mathbf{w})\epsilon_{ij}(\mathbf{w}) d\mathbf{x} \\ &\geq \kappa \int_{\mathbf{T}} \epsilon_{ij}(\mathbf{w})\epsilon_{ij}(\mathbf{w}) d\mathbf{x} = \frac{\kappa}{2\mu^*}\Lambda^*(\mathbf{w}, \mathbf{w}) \ . \end{aligned}$$

Taking into account that $\mathbf{v} \in \mathbf{R}$, and using the definition of the subspace $\mathbf{R}$, we can derive $\Lambda^*(\mathbf{v}-\mathbf{w}, \mathbf{w}) = 0$, and therefore, $\Lambda^*(\mathbf{w}, \mathbf{w}) = \Lambda^*(\mathbf{v}, \mathbf{v}) + \Lambda^*(\mathbf{v}-\mathbf{w}, \mathbf{v}-\mathbf{w}) \geq \Lambda^*(\mathbf{v}, \mathbf{v})$. □

As a direct consequence of Lemma 2.2 (cf. [4]), we obtain the following

THEOREM 2.1. *Let $\mu \geq \mu^* > 0$ and $\tau = \dfrac{\mu^*}{\mu}$. Then the sequence of approximations $\{\mathbf{u}^n\}$ given by the method (2.5) with the initial guess computed from (2.6) satisfies the following convergence rate estimate:*

$$\Lambda^*(\mathbf{u}^n - \mathbf{u}, \mathbf{u}^n - \mathbf{u}) \quad \leq \quad q^{2n}\Lambda^*(\mathbf{u}^0 - \mathbf{u}, \mathbf{u}^0 - \mathbf{u}), \quad 0 \leq q = 1 - \kappa < 1.$$

Proof. Using notations of Lemma 2.2, we write the identity $\boldsymbol{\epsilon}^{n+1} = (I - \tau A)\boldsymbol{\epsilon}^n$, where $\boldsymbol{\epsilon}^{n+1}, \boldsymbol{\epsilon}^n \in \mathbf{R}$. The operator $I - \tau A : \mathbf{R} \to \mathbf{R}$ is selfadjoint and is a contraction in the scalar product $\Lambda^*(\cdot, \cdot)$ which follows from statement (c) from Lemma 2.2, which in operator form reads $\kappa\mu/\mu^* I \leq A \leq \mu/\mu^* I$. For example, when $\tau = \mu/\mu^*$ we obtain $0 \leq I - \tau A \leq qI$, which had to be proven. □

The theorem shows uniform, with respect to $\mu^*, 0 < \mu^* \leq \mu$, convergence of the method. The next theorem demonstrates that the initial error in our iterative method is uniformly bounded under some natural assumptions required to have a possibility of taking the limit $\mu^* \to 0$, cf. [4].

THEOREM 2.2. *Let $\mu \geq \mu^* > 0$, where μ is fixed. Let also*

$$tr\{(F_{\mathbf{D}} - G_{\mathbf{D}})C_{\mathbf{D}}\} = 0, \tag{2.9}$$

where $F_{\mathbf{D}}$ and $G_{\mathbf{D}}$ are averages in $\mathbf{D}$ of the matrices $f = (f_{ij})$ and $g = (g_{ij})$, and $C_{\mathbf{D}}$ is an arbitrary 3×3 antisymmetric matrix, $C_{\mathbf{D}} = -C_{\mathbf{D}}^T$, such that there exists a continuous single-value branch of the multi-place function $C_{\mathbf{D}}\mathbf{x}\,|_{\mathbf{D}}$ restricted on $\mathbf{D}$, see [4].

Then, the initial guess computed from (2.6) satisfies the following estimate:

$$\frac{1}{\mu^*}\Lambda^*(\mathbf{u}^0-\mathbf{u},\mathbf{u}^0-\mathbf{u}) \quad \leq \quad const\left(\left\|\frac{g}{\mu^*}\right\|^2_{[L_2(\mathbf{T})]^9}+\|f\|^2_{[L_2(\mathbf{T})]^9}\right)$$

where here and below const denotes a generic constant independent of μ^*.

Proof. We write

$$\Lambda(\mathbf{u}^0-\mathbf{u},\mathbf{v})=\Lambda(\mathbf{u}^0,\mathbf{v})-\int_{\mathbf{T}} f_{ij}\frac{\partial v_j}{\partial x_i}d\mathbf{x}, \quad \forall \mathbf{v}\in\mathbf{H} \tag{2.10}$$

and write the vector $\mathbf{v}\in\mathbf{H}$ as an orthogonal sum with respect to $\Lambda^*(\cdot,\cdot)$,

$$\mathbf{v}=\mathbf{v}_N+\mathbf{v}_R,\ \ \mathbf{v}_N\in\mathbf{N},\ \mathbf{v}_R\in\mathbf{R}.$$

We will now show that all terms containing $\mathbf{v}_N$ in (2.10) vanish.

For the left hand side, $\Lambda(\mathbf{u}^0-\mathbf{u},\mathbf{v}_N)=0$, because of Proposition 2.3, as $\mathbf{u}^0-\mathbf{u}\in\mathbf{R}$ by Lemma 2.2 and $\mathbf{v}_N\in\mathbf{N}$.

For the right hand side, using Proposition 2.3, we rewrite the first term

$$\Lambda(\mathbf{u}^0,\mathbf{v}_N)=\Lambda^*(\mathbf{u}^0,\mathbf{v}_N)=\int_{\mathbf{T}} g_{ij}\frac{\partial(v_N)_j}{\partial x_i}d\mathbf{x},$$

using the definition (2.6) of $\mathbf{u}^0$ for the last equality. Making that substitution, we obtain the right hand side of (2.10) in the form

$$\int_{\mathbf{T}}(g_{ij}-f_{ij})\frac{\partial(v_N)_j}{\partial x_i}d\mathbf{x}=\int_{\mathbf{D}}(g_{ij}-f_{ij})\frac{\partial(v_N)_j}{\partial x_i}d\mathbf{x}$$

as $\mathbf{v}_N\in\mathbf{N}$, and this value is zero owing to condition (2.9).

Thus, we just proved that it is sufficient to take $\mathbf{v}\in\mathbf{R}$ in (2.10). Lemma 2.2 states that $\mathbf{u}^0-\mathbf{u}\in\mathbf{R}$ and the symmetric bilinear form $\Lambda(\cdot,\cdot)$ is coercive and bounded on the subspace $\mathbf{R}$ with respect to the form $\frac{\mu}{\mu^*}\Lambda^*(\cdot,\cdot)$ with constants κ and 1 respectively.

We now estimate the norm, with respect to the $\frac{1}{\mu^*}\Lambda^*(\cdot,\cdot)$ scalar product, of the linear functional of the right hand side of (2.10). We will not use the fact that $\mathbf{v}\in\mathbf{R}$ in our arguments below.

For the first term,

$$|\Lambda(\mathbf{u}^0,\mathbf{v})|^2\leq\Lambda(\mathbf{u}^0,\mathbf{u}^0)\Lambda(\mathbf{v},\mathbf{v})\leq\left(\frac{\mu}{\mu^*}\right)^2\Lambda^*(\mathbf{u}^0,\mathbf{u}^0)\Lambda^*(\mathbf{v},\mathbf{v}),$$

where

$$\frac{1}{\mu^*}\Lambda^*(\mathbf{u}^0,\mathbf{u}^0)\leq const\left\|\frac{g}{\mu^*}\right\|^2_{[L_2(\mathbf{T})]^9}. \tag{2.11}$$

For the second term,

$$\left|\int_{\mathbf{T}} f_{ij}\frac{\partial v_j}{\partial x_i}d\mathbf{x}\right|^2\leq\|f\|^2_{[L_2(\mathbf{T})]^9}\|\mathbf{v}\|^2_{[W_2^1(\mathbf{T})]^3}\leq\|f\|^2_{[L_2(\mathbf{T})]^9}\frac{const}{\mu^*}\Lambda^*(\mathbf{v},\mathbf{v}),\ \ \mathbf{v}\in\mathbf{H},$$

with a constant *const* independent of μ^*.

We now combine the two estimates,

$$\left|\Lambda(\mathbf{u}^0,\mathbf{v}) - \int_{\mathbf{T}} f_{ij}\frac{\partial v_j}{\partial x_i}d\mathbf{x}\right|^2 \leq const\left(\left\|\frac{g}{\mu^*}\right\|^2_{[L_2(\mathbf{T})]^9} + \|f\|^2_{[L_2(\mathbf{T})]^9}\right)\frac{1}{\mu^*}\Lambda^*(\mathbf{v},\mathbf{v}).$$

Therefore, problem (2.10) is well-posed in the subspace $\mathbf{R}$ of $\mathbf{H}$, and the stability estimate holds. □

REMARK 2.1. *The coefficient $\mu(\mathbf{x})$ can be variable in $\mathbf{D}$. Let $\kappa\chi\mu \leq \mu(\mathbf{x}) \leq \mu$, for almost all $\mathbf{x} \in \mathbf{D}$, and χ, μ be positive constants. Then Theorem 2.1 holds with $q = 1 - \kappa\chi < 1$. Actually, only statement (c) of Lemma 2.2 changes and the corresponding inequalities can be replaced by*

$$\kappa\chi\frac{\mu}{\mu^*}\Lambda^*(\mathbf{v},\mathbf{v}) \leq \Lambda(\mathbf{v},\mathbf{v}) \leq \max\{\frac{\mu}{\mu^*},1\}\Lambda^*(\mathbf{v},\mathbf{v}), \quad \mathbf{v}\in\mathbf{R}.$$

3. Lamé Equations. Following [2, 4], we consider a three-dimensional periodic boundary value problem for the isotropic linear elasticity (Lamé) equations:

$$2\frac{\partial\left[\mu(\mathbf{x})\epsilon_{ij}(\mathbf{u})\right]}{\partial x_i} + \frac{\partial[\lambda(\mathbf{x})\mathrm{div}\,\mathbf{u}]}{\partial x_j} = \frac{\partial f_{ij}}{\partial x_i}, \tag{3.1}$$

where $\mathbf{u} = (u_1, u_2, u_3)^T$ and $\epsilon_{ij}(\mathbf{u}) \stackrel{\mathrm{def}}{=} \frac{1}{2}\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right]$.

We re-define the space $\mathbf{H} \stackrel{\mathrm{def}}{=} [W_2^1(\mathbf{T})]^3/R^3$, i.e. we no longer require vectors in $\mathbf{H}$ to be solenoidal. Then a weak solution to problem (3.1), given $f_{ij} \in L_2(\mathbf{T})$, is a function $\mathbf{u}\in\mathbf{H}$ such that

$$\Lambda(\mathbf{u},\mathbf{v}) = \int_{\mathbf{T}} f_{ij}\,\frac{\partial v_j}{\partial x_i}d\mathbf{x}, \quad \forall\,\mathbf{v}\in\mathbf{H}, \tag{3.2}$$

where $\Lambda(\mathbf{u},\mathbf{v}) \stackrel{\mathrm{def}}{=} \int_{\mathbf{T}}[2\mu(\mathbf{x})\epsilon_{ij}(\mathbf{u})\epsilon_{ij}(\mathbf{v}) + \lambda(\mathbf{x})\mathrm{div}\,\mathbf{u}\ \mathrm{div}\,\mathbf{v}]d\mathbf{x}$. We assume that Lamé coefficients $\lambda(\mathbf{x})$ and $\mu(\mathbf{x})$ are piece-wise constant and equal to (λ,μ) and (λ^*,μ^*) in $\mathbf{D}$ and $\mathbf{T}\backslash\overline{\mathbf{D}}$ correspondingly. Also, we assume

$$\mu,\mu^* > 0, K \equiv \lambda + \frac{2}{3}\mu > 0, K^* \equiv \lambda^* + \frac{2}{3}\mu^* > 0, 0 < \eta \leq \frac{\mu}{\mu^*}\frac{K^*}{K} \leq \eta^{-1}, \tag{3.3}$$

for some constant $\eta \leq 1$.

LEMMA 3.1. *Under the assumptions (3.3), we have the following estimates:*

$$0 < \alpha \leq \frac{2\mu\sum_{i,j}|a^i_j|^2 + \lambda|\sum_i a^i_i|^2}{2\mu^*\sum_{i,j}|a^i_j|^2 + \lambda^*|\sum_i a^i_i|^2} \leq \beta,$$

$$\frac{\alpha}{\beta} \geq \eta > 0,$$

where $\alpha \equiv \min\{\frac{\mu}{\mu^*}, \frac{K}{K^*}\}$, $\beta \equiv \max\{\frac{\mu}{\mu^*}, \frac{K}{K^*}\}$, *and* a^i_j, $i, j = 1, 2, 3$ *are arbitrary real numbers, not all equal to zero.*

Proof. First, we re-write the fraction in the form $\frac{\mu x + K y}{\mu^* x + K^* y}$, where

$$x = 2\sum_{i,j} |a^i_j - \frac{1}{3}\delta_{ij} \sum_k a^k_k|^2, \quad \text{and} \quad y = |\sum_i a^i_i|^2.$$

Then, we use the well-known inequalities

$$\min\{\frac{\mu}{\mu^*}, \frac{K}{K^*}\} \leq \frac{\mu x + K y}{\mu^* x + K^* y} \leq \max\{\frac{\mu}{\mu^*}, \frac{K}{K^*}\},$$

which are valid for $\mu, \mu^* > 0, x, y \geq 0$, and $K, K^* > 0$ to obtain the result. □

COROLLARY 3.1. *The constants* α *and* β *are actually bounds for the ratio of the squares of the following energy norms:*

$$0 < \alpha \leq \frac{\int_{\mathbf{D}} [2\mu\epsilon_{ij}(\mathbf{w})\epsilon_{ij}(\mathbf{w}) + \lambda|\text{div } \mathbf{w}|^2] d\mathbf{x}}{\int_{\mathbf{D}} [2\mu^*\epsilon_{ij}(\mathbf{w})\epsilon_{ij}(\mathbf{w}) + \lambda^*|\text{div } \mathbf{w}|^2] d\mathbf{x}} \leq \beta, \quad \forall \mathbf{w} \in [W^1_2(\mathbf{D})]^3/R^3.$$

Proof. Noting that $\epsilon_{ij}(\mathbf{w}) \in L_2(\mathbf{D})$ and applying Lemma 3.1 with $a^i_j = \epsilon_{ij}(\mathbf{w}(\mathbf{x}))$, $\mathbf{x} \in \mathbf{D}$ almost everywhere, we obtain the estimates by integrating the numerator and the denominator over $\mathbf{D}$. □

Let us define on $\mathbf{H}$ the following bilinear form:

$$\Lambda^*(\mathbf{u}, \mathbf{v}) \stackrel{\text{def}}{=} 2\mu^* \int_{\mathbf{T}} \epsilon_{ij}(\mathbf{u})\epsilon_{ij}(\mathbf{v}) + \lambda^* \int_{\mathbf{T}} \text{div}\, \mathbf{u} \,\text{div}\, \mathbf{v} \; d\mathbf{x},$$

and let the forms $\Lambda^*_{\mathbf{D}}(\mathbf{u}, \mathbf{v})$ and $\Lambda^*_{\mathbf{T}\setminus\overline{\mathbf{D}}}(\mathbf{u}, \mathbf{v})$ be defined in the same way, except for the domain of integration.

THEOREM 3.1. *Let* $\frac{K^*}{\mu^*} \geq \delta > 0$. *Then, there exist a positive constant* $\kappa = \kappa(\delta, \mathbf{D})$, *such that for every* $\mathbf{v} \in \mathbf{H}$ *there exist a function* $\mathbf{w} \in \mathbf{H}$, *such that* $\Lambda^*_{\mathbf{D}}(\mathbf{w}, \mathbf{w}) \geq \kappa\Lambda^*(\mathbf{w}, \mathbf{w})$ *and* $\mathbf{w} = \mathbf{v} + C\mathbf{x}$ *in* $\mathbf{D}$, *where* $C = -C^T$ *is some* 3×3 *matrix independent of* $\mathbf{x}$.

Proof. Let us choose a fixed function $\mathbf{w}_f \in \mathbf{H}$, such that $\oint (\mathbf{w}_f, \mathbf{n})\, d\partial\mathbf{D} = 1$. For example, if the domain $\mathbf{D}$ does not wrap around the torus $\mathbf{T}$ in the direction of x_1, one can choose $\mathbf{w}_f = (x_1, x_1, x_1)$. Then, by Proposition 2.1, there exist a constant $\kappa_1 = \kappa_1(\mathbf{D})$, such that for $\forall \mathbf{v} \in \mathbf{H}$ there exist a function $\mathbf{w} \in [W^1_2(\mathbf{D})]^3/R^3$, such that

$$\int_{\mathbf{D}} \epsilon_{ij}(\mathbf{w})\epsilon_{ij}(\mathbf{w}) \geq \kappa_1 \|\mathbf{w}\|^2_{[W^{1/2}_2(\partial\mathbf{D})]^3},$$

and $\mathbf{w} = \mathbf{v} + C\mathbf{x}$ in $\mathbf{D}$, where $C = -C^T$ is some 3×3 matrix independent of $\mathbf{x}$. Setting $\mathbf{w}_s = \mathbf{w} - \mathbf{w}_f \oint (\mathbf{w}, \mathbf{n})\, d\partial\mathbf{D}$ in $\mathbf{D}$, we have $\mathbf{w}_s \in [W_2^1(\mathbf{D})]^3/R^3$ and $\oint (\mathbf{w}_s, \mathbf{n})\, d\partial\mathbf{D} = 0$. Then, by Proposition 2.2, there exists a solenoidal extension $\mathbf{w}_s \in [W_2^1(\mathbf{T}\backslash\overline{\mathbf{D}})]^3/R^3$ satisfying

$$\|\mathbf{w}_s\|^2_{[W_2^1(\mathbf{T}\backslash\overline{\mathbf{D}})]^3} \leq \theta \|\mathbf{w}_s\|^2_{[W_2^{1/2}(\partial\mathbf{D})]^3},$$

with some positive constant $\theta = \theta(\mathbf{T}\backslash\overline{\mathbf{D}})$. Now we can extend $\mathbf{w}$ from $\mathbf{D}$ into $\mathbf{T}\backslash\overline{\mathbf{D}}$ by $\mathbf{w} = \mathbf{w}_s + \mathbf{w}_f \oint (\mathbf{w}, \mathbf{n})\, d\partial\mathbf{D}$, and then $\mathbf{w} \in \mathbf{H}$ with

$$\Lambda^*_{\mathbf{T}\backslash\overline{\mathbf{D}}}(\mathbf{w}, \mathbf{w}) \leq 2 \left(\Lambda^*_{\mathbf{T}\backslash\overline{\mathbf{D}}}(\mathbf{w}_s, \mathbf{w}_s) + \Lambda^*_{\mathbf{T}\backslash\overline{\mathbf{D}}}(\mathbf{w}_f, \mathbf{w}_f)\Phi(\mathbf{w}) \right),$$

where $\Phi(\mathbf{w}) = |\oint (\mathbf{w}, \mathbf{n})\, d\partial\mathbf{D}|^2$. Let us estimate each term on the right-hand side separately. First,

$$\Lambda^*_{\mathbf{T}\backslash\overline{\mathbf{D}}}(\mathbf{w}_s, \mathbf{w}_s) = 2\mu^* \int_{\mathbf{T}\backslash\overline{\mathbf{D}}} \epsilon_{ij}(\mathbf{w}_s)\epsilon_{ij}(\mathbf{w}_s) d\mathbf{x} \leq \mu^* \|\mathbf{w}_s\|^2_{[W_2^1(\mathbf{T}\backslash\overline{\mathbf{D}})]^3} \leq$$

$$\theta\mu^* \|\mathbf{w}_s\|^2_{[W_2^{1/2}(\partial\mathbf{D})]^3} \leq 2\theta\mu^* \{ \|\mathbf{w}\|^2_{[W_2^{1/2}(\partial\mathbf{D})]^3} + c_1 \Phi(\mathbf{w}) \} \leq$$

$$2\theta\mu^* \{ \frac{1}{\kappa_1} \int_{\mathbf{D}} \epsilon_{ij}(\mathbf{w})\epsilon_{ij}(\mathbf{w}) d\mathbf{x} + c_1 \Phi(\mathbf{w}) \},$$

with $c_1 = \|\mathbf{w}_f\|^2_{[W_2^{1/2}(\mathbf{T}\backslash\overline{\mathbf{D}})]^3}$. In the above inequalities we have used that div $\mathbf{w}_s = 0$ in $\mathbf{T}\backslash\overline{\mathbf{D}}$, and $\mathbf{w}_s = \mathbf{w} - \mathbf{w}_f \oint (\mathbf{w}, \mathbf{n})\, d\partial\mathbf{D}$ on $\partial\mathbf{D}$.

Second, using $\lambda^* \leq K^*$, we have

$$\Lambda^*_{\mathbf{T}\backslash\overline{\mathbf{D}}}(\mathbf{w}_f, \mathbf{w}_f) = 2\mu^* c_2 + \lambda^* c_3 \leq 2\mu^* c_2 + K^* c_3,$$

with $c_2 = \int_{\mathbf{T}\backslash\overline{\mathbf{D}}} \epsilon_{ij}(\mathbf{w}_f)\epsilon_{ij}(\mathbf{w}_f) d\mathbf{x}$ and $c_3 = \int_{\mathbf{T}\backslash\overline{\mathbf{D}}} |\text{div } \mathbf{w}_f|^2\, d\mathbf{x}$.

Finally, using the Cauchy inequality, we get

$$\Phi(\mathbf{w}) = |\oint (\mathbf{w}, \mathbf{n})\, d\partial\mathbf{D}|^2 = |\int_{\mathbf{D}} \text{div } \mathbf{w}\, d\mathbf{x}|^2 \leq mes(\mathbf{D}) \int_{\mathbf{D}} |\text{div } \mathbf{w}|^2\, d\mathbf{x}.$$

Combining the above estimates, we obtain

$$\Lambda^*_{\mathbf{T}\backslash\overline{\mathbf{D}}}(\mathbf{w}, \mathbf{w}) \leq 2\mu^\$ \int_{\mathbf{D}} \epsilon_{ij}(\mathbf{w})\epsilon_{ij}(\mathbf{w}) d\mathbf{x} + \lambda^\$ \int_{\mathbf{D}} |\text{div } \mathbf{w}|^2\, d\mathbf{x},$$

where $\mu^\$ = \dfrac{2\mu^*\theta}{\kappa_1}$ and $\lambda^\$ = (4\mu^*(\theta c_1 + c_2) + 2K^* c_3) mes(\mathbf{D})$.

Now, by setting $\mu = \mu^\$$ and $\lambda = \lambda^\$$ in the assumptions of Corollary 3.1 and taking into account $\dfrac{K^*}{\mu^*} \geq \delta > 0$, we get

$$\Lambda^*_{\mathbf{T}\setminus\overline{\mathbf{D}}}(\mathbf{w},\mathbf{w}) \leq \beta^\$ \Lambda^*_{\mathbf{D}}(\mathbf{w},\mathbf{w}),$$

where $\beta^\$ = 2\max\left\{\dfrac{\theta}{\kappa_1}, \dfrac{1}{\delta} mes(\mathbf{D})\left[2\theta\left\{\dfrac{1}{3\kappa_1 mes(\mathbf{D})} + c_1\right\} + 2c_2 + \delta c_3\right]\right\}$, which is equivalent to the desired inequality with $\kappa = \dfrac{1}{1+\beta^\$}$. □

Remark 3.1. *A weaker version of Theorem 3.1 has been proven in [4], where the constant $\kappa \to 0$ in the incompressible limit, i.e. when $\dfrac{K^*}{\mu^*} \to +\infty$. In contrast, the constant κ in Theorem 3.1 depends only on the lower bound on $\dfrac{K^*}{\mu^*}$ and thus does not deteriorate in the incompressible limit.*

Next, we show that the result of Theorem 3.1 is sharp in the sense that in general, $\kappa \to 0$ when $\dfrac{K^*}{\mu^*} \to 0$.

Lemma 3.2. *If the domain $\mathbf{D}$ does not wrap around the torus $\mathbf{T}$, then there exist a nontrivial function $\mathbf{v} \in \mathbf{H}$, such that for any function $\mathbf{w} \in \mathbf{H}$, such that $\mathbf{w} = \mathbf{v} + C\mathbf{x}$ in $\mathbf{D}$, for some 3×3 matrix $C = -C^T$ the ratio $\dfrac{\Lambda^*_{\mathbf{D}}(\mathbf{w},\mathbf{w})}{\Lambda^*(\mathbf{w},\mathbf{w})} \to 0$ as $\dfrac{K^*}{\mu^*} \to 0$. Namely, an example of such vector function is $\mathbf{v}(\mathbf{x}) = \mathbf{x}$ in $\mathbf{D}$ and extended as a function in $[W^1_2(\mathbf{T})]^3$ outside $\mathbf{D}$.*

Proof. First of all, the fact that function $\mathbf{v}(\mathbf{x}) = \mathbf{x}$ in $\mathbf{D}$ can be extended to the whole torus in $[W^1_2(\mathbf{T})]^3$ follows from extension arguments in [4]. For this function, $\epsilon_{ij}(\mathbf{v})$ is a spherical tensor in $\mathbf{D}$, i.e. $\epsilon_{ij}(\mathbf{v}) = \Psi(\mathbf{x})\delta_{ij}$ in $\mathbf{D}$, namely, with $\Psi(\mathbf{x}) \equiv 1$. For any $\mathbf{w} \in \mathbf{H}$, such that $\mathbf{w} = \mathbf{v} + C\mathbf{x}$ in $\mathbf{D}$, we have $\epsilon_{ij}(\mathbf{w}) = \epsilon_{ij}(\mathbf{v}) = \Psi(\mathbf{x})\delta_{ij}$ in $\mathbf{D}$. Then, on one hand

$$\Lambda^*_{\mathbf{D}}(\mathbf{w},\mathbf{w}) = 2\mu^*\int_{\mathbf{D}} \epsilon_{ij}(\mathbf{v})\epsilon_{ij}(\mathbf{v})\,d\mathbf{x} + \lambda^*\int_{\mathbf{D}} |\text{div }\mathbf{v}|^2\,d\mathbf{x} = 9K^*\int_{\mathbf{D}} \Psi^2(\mathbf{x})d\mathbf{x}.$$

On the other hand, we have

$$\Lambda^*(\mathbf{w},\mathbf{w}) = 2\mu^*\int_{\mathbf{T}} \epsilon_{ij}(\mathbf{w})\epsilon_{ij}(\mathbf{w})\,d\mathbf{x} + \lambda^*\int_{\mathbf{T}} |\text{div }\mathbf{w}|^2\,d\mathbf{x} =$$

$$\mu^*\int_{\mathbf{T}} \frac{\partial w_i}{\partial x_j}\frac{\partial w_i}{\partial x_j}d\mathbf{x} + (\lambda^*+\mu^*)\int_{\mathbf{T}} |\text{div }\mathbf{w}|^2\,d\mathbf{x} > (\lambda^*+\mu^*)\int_{\mathbf{T}} |\text{div }\mathbf{w}|^2\,d\mathbf{x},$$

after integrating by parts. Now we use the assumption $K^* \geq 0$ to replace $\lambda^* + \mu^* \geq \dfrac{1}{3}\mu^*$, and integrate over the smaller domain $\mathbf{D}$ to obtain

$$\Lambda^*(\mathbf{w},\mathbf{w}) > \frac{1}{3}\mu^*\int_{\mathbf{D}} |\text{div }\mathbf{w}|^2\,d\mathbf{x} = 3\mu^*\int_{\mathbf{D}} \Psi^2(\mathbf{x})d\mathbf{x},$$

where we have used the identities div $\mathbf{w}$ = div $\mathbf{v} = 3\Psi(\mathbf{x})$ in $\mathbf{D}$. Therefore,

$$\frac{\Lambda^*_{\mathbf{D}}(\mathbf{w},\mathbf{w})}{\Lambda^*(\mathbf{w},\mathbf{w})} < 3\frac{K^*}{\mu^*}.$$

□

In Lemma 3.2, we give only an example of a function $\mathbf{v}$. In the next section, we find all such functions.

REMARK 3.2. *If* $\mathbf{D}$ *wraps around the torus* $\mathbf{T}$ *at least in one direction, it seems possible to prove Theorem 3.1 even if* $\frac{K^*}{\mu^*} \to 0$. *One approach is to try to estimate* $\Lambda^*_{\mathbf{D}}(\mathbf{w},\mathbf{w})$ *from below directly using arguments similar to those applied to other types of boundary value problems in [6]. Other possibilities are using results of [16] on Cosserat eigenproblem, or adopting arguments from [15]. We discuss it in some more details at the end of the next section.*

The rest of this section is very similar to the last part of the previous section.

LEMMA 3.3. *Define the space* $\mathbf{N} \subset \mathbf{H}$ *of functions* $\mathbf{v} \in \mathbf{H}$ *of the type* $\mathbf{v} = C\mathbf{x}$ *in the domain* $\mathbf{D}$ *where* $\mathbf{x} \in \mathbf{T}$ *is the vector of independent variables on the torus* $\mathbf{T}$, *and* $C = -C^T$ *is some* 3×3 *matrix, independent of* $\mathbf{x}$. *Let the subspace* $\mathbf{R} \subset \mathbf{H}$ *be defined by the formula* $\Lambda^*(\mathbf{v},\mathbf{w}) = 0, \forall \mathbf{v} \in \mathbf{R}, \mathbf{w} \in \mathbf{N}$.

Then:

a) for the initial guess $\mathbf{u}^0$, *we have* $\mathbf{u}^0 - \mathbf{u} \in \mathbf{R}$;

(b) $\mathbf{R}$ *is an invariant subspace for the error propagation operator from* $\boldsymbol{\epsilon}^n = \mathbf{u}^n - \mathbf{u}$ *to* $\boldsymbol{\epsilon}^{n+1} = \mathbf{u}^{n+1} - \mathbf{u}$ *acting by formula (2.5) according to the rule:*

$$\Lambda^*(\frac{\boldsymbol{\epsilon}^{n+1} - \boldsymbol{\epsilon}^n}{\tau}, \mathbf{v}) + \Lambda(\boldsymbol{\epsilon}^n, \mathbf{v}) \quad = \quad 0 \quad \forall\, \mathbf{v} \in \mathbf{H},\ n = 0, 1, \cdots; \tag{3.4}$$

(c) if $\frac{K^*}{\mu^*} \geq \delta > 0$, *then on this invariant subspace we have*

$$\kappa\ \alpha\ \Lambda^*(\mathbf{v},\mathbf{v}) \leq \Lambda(\mathbf{v},\mathbf{v}) \quad \leq \quad \max\{\beta, 1\}\ \Lambda^*(\mathbf{v},\mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{R},$$

where $\alpha = \min\{\frac{\mu}{\mu^*}, \frac{K}{K^*}\}$, $\beta = \max\{\frac{\mu}{\mu^*}, \frac{K}{K^*}\}$, *and* κ *is determined in Theorem 3.1.*

PROPOSITION 3.1. *The following equality holds:*

$$\Lambda^*(\mathbf{v},\mathbf{w}) \quad = \quad \Lambda(\mathbf{v},\mathbf{w}), \quad \forall \mathbf{v} \in \mathbf{H}, \mathbf{w} \in \mathbf{N}.$$

Proof. We note that $\epsilon_{ij}(\mathbf{w}) = 0$ and div $\mathbf{w} = 0$ in the domain $\mathbf{D}$ when $\mathbf{w} \in \mathbf{N}$, and therefore,

$$\begin{aligned}
\Lambda^*(\mathbf{v},\mathbf{w}) \quad = \quad & 2\mu^* \int_{\mathbf{T}} \epsilon_{ij}(\mathbf{v})\epsilon_{ij}(\mathbf{w})\, d\mathbf{x} + \lambda^* \int_{\mathbf{T}} \mathrm{div}\ \mathbf{v}\ \mathrm{div}\ \mathbf{w}\, d\mathbf{x} \quad = \\
& 2\mu^* \int_{\mathbf{T}\setminus\overline{\mathbf{D}}} \epsilon_{ij}(\mathbf{v})\epsilon_{ij}(\mathbf{w})\, d\mathbf{x} + \lambda^* \int_{\mathbf{T}\setminus\overline{\mathbf{D}}} \mathrm{div}\ \mathbf{v}\ \mathrm{div}\ \mathbf{w}\, d\mathbf{x} \quad = \\
& \int_{\mathbf{T}\setminus\overline{\mathbf{D}}} \{2\mu(\mathbf{x})\epsilon_{ij}(\mathbf{v})\epsilon_{ij}(\mathbf{w}) + \lambda(\mathbf{x})\mathrm{div}\ \mathbf{v}\ \mathrm{div}\ \mathbf{w}\}\, d\mathbf{x} \quad = \\
& \int_{\mathbf{T}} \{2\mu(\mathbf{x})\epsilon_{ij}(\mathbf{v})\epsilon_{ij}(\mathbf{w}) + \lambda(\mathbf{x})\mathrm{div}\ \mathbf{v}\ \mathrm{div}\ \mathbf{w}\}\, d\mathbf{x} \quad = \quad \Lambda(\mathbf{v},\mathbf{w}).
\end{aligned}$$

□

Proof. [Lemma 3.3]

(a) and (b) can be proven exactly like in Lemma 2.2, except we need to use Proposition 3.1 now.

(c) The right inequality is true for all functions $\mathbf{v} \in \mathbf{H}$ and can be proven directly by Corollary 3.1. The left inequality can be shown as follows. For any $\mathbf{v} \in \mathbf{H}$ we have

$$\begin{aligned}
\Lambda(\mathbf{v},\mathbf{v}) &= \int_{\mathbf{T}} \{2\mu(\mathbf{x})\epsilon_{ij}(\mathbf{v})\epsilon_{ij}(\mathbf{v}) + \lambda(\mathbf{x})|\mathrm{div}\ \mathbf{v}|^2\}\, d\mathbf{x} &\geq \\
& \int_{\mathbf{D}} \{2\mu\epsilon_{ij}(\mathbf{v})\epsilon_{ij}(\mathbf{v}) + \lambda|\mathrm{div}\ \mathbf{v}|^2\} d\mathbf{x} &\geq \\
& \alpha \int_{\mathbf{D}} \{2\mu^*\epsilon_{ij}(\mathbf{v})\epsilon_{ij}(\mathbf{v}) + \lambda^*|\mathrm{div}\ \mathbf{v}|^2\} d\mathbf{x} &= \alpha\Lambda^*_{\mathbf{D}}(\mathbf{v},\mathbf{v}),
\end{aligned}$$

by Corollary 3.1. Next, by Theorem 3.1 there exist a function $\mathbf{w} \in \mathbf{H}$, such that $\mathbf{v}-\mathbf{w} \in \mathbf{N}$ and $\Lambda^*_{\mathbf{D}}(\mathbf{v},\mathbf{v}) = \Lambda^*_{\mathbf{D}}(\mathbf{w},\mathbf{w}) \geq \kappa\Lambda^*(\mathbf{w},\mathbf{w})$. Noting that $\mathbf{v} \in \mathbf{R}$ and using the definition of the space $\mathbf{R}$, we get $\Lambda^*(\mathbf{v}-\mathbf{w},\mathbf{w}) = 0$, and therefore, $\Lambda^*(\mathbf{w},\mathbf{w}) = \Lambda^*(\mathbf{v},\mathbf{v}) + \Lambda^*(\mathbf{v}-\mathbf{w},\mathbf{v}-\mathbf{w}) \geq \Lambda^*(\mathbf{v},\mathbf{v})$. □

Lemma 3.3 implies the following

THEOREM 3.2. *Let* $\frac{K^*}{\mu^*} \geq \delta > 0$, $\beta = \max\{\frac{\mu}{\mu^*}, \frac{K}{K^*}\} \geq 1$, *and* $\tau = \frac{1}{\beta}$. *Then the sequence of approximations* $\{\mathbf{u}^n\}$ *given by the method (2.5) with the initial guess computed from (2.6) satisfies the following convergence rate estimate:*

$$\Lambda^*(\mathbf{u}^n-\mathbf{u},\mathbf{u}^n-\mathbf{u}) \quad\leq\quad q^{2n}\Lambda^*(\mathbf{u}^0-\mathbf{u},\mathbf{u}^0-\mathbf{u}), \quad 0 \leq q = 1-\kappa\eta < 1.$$

Proof. The proof is the same as in Theorem 2.1, except that we use Lemma 3.3 now. □

We can also prove an estimate for the initial error, analogous to that derived in the previous section for the Stokes equations.

4. Null Space Corresponding to the Absolutely Compressible Media. In this section, we characterize the null space of the quadratic form $\Lambda^*_{\mathbf{D}}(\mathbf{v},\mathbf{v})$ with $K^* = 0$. To that end, we need to introduce some notation. Let

$$\nabla \times \mathbf{u} = \begin{pmatrix} 0 & -\partial_{x_3} & \partial_{x_2} \\ \partial_{x_3} & 0 & -\partial_{x_1} \\ -\partial_{x_2} & \partial_{x_1} & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

be the curl operator in three dimensions. We use $\nabla \times \tau \times \nabla$ to denote the following second order differential operator acting on a given second order tensor τ:

$$\begin{pmatrix} 0 & -\partial_{x_3} & \partial_{x_2} \\ \partial_{x_3} & 0 & -\partial_{x_1} \\ -\partial_{x_2} & \partial_{x_1} & 0 \end{pmatrix} \begin{pmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{pmatrix} \begin{pmatrix} 0 & -\partial_{x_3} & \partial_{x_2} \\ \partial_{x_3} & 0 & -\partial_{x_1} \\ -\partial_{x_2} & \partial_{x_1} & 0 \end{pmatrix}^T,$$

such that $\nabla \times \tau \times \nabla \in [L_2(\mathbf{D})]^9$.

Let us recall the well-known necessary condition for a symmetric tensor to be the symmetric part of the gradient of some vector in $[W_2^1]^3$:

PROPOSITION 4.1. *Let* $\mathbf{v} \in [W_2^1(\mathbf{D})]^3$. *Then* $\nabla \times \epsilon(\mathbf{v}) \times \nabla = 0$ *in* $[L_2(\mathbf{D})]^9$.

Proof. The statement is known to be true for smooth vectors functions. Noting that the subspace of all symmetric tensors τ in $[L_2(\mathbf{D})]^9$ such that $\nabla \times \tau \times \nabla = 0$ is closed with respect to the topology induced by the L_2 norm, we complete the proof by a standard density argument. □

LEMMA 4.1. *Let* $\mathbf{D}$ *does not wrap around the torus* $\mathbf{T}$. *Then the kernel of the quadratic form* $\Lambda^*_{\mathbf{D}}(\mathbf{v},\mathbf{v})$ *with* $K^* = 0$, *in addition to the standard rigid body motions (translations and rotations), consists of the following 4-dimensional space:*

$$\text{span}\left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}; \begin{pmatrix} \frac{1}{2}(x_1^2 - x_2^2 - x_3^2) \\ x_1 x_2 \\ x_1 x_3 \end{pmatrix}; \begin{pmatrix} x_1 x_2 \\ \frac{1}{2}(x_2^2 - x_1^2 - x_3^2) \\ x_2 x_3 \end{pmatrix}; \begin{pmatrix} x_1 x_3 \\ x_2 x_3 \\ \frac{1}{2}(x_3^2 - x_1^2 - x_2^2) \end{pmatrix} \right\}.$$

Proof. First, we note that the following decomposition

$$\epsilon_{ij} = \frac{1}{3}\epsilon_{ii}\delta_{ij} + \epsilon_{ij}^d,$$

where ϵ^d is the deviatoric part of the strain tensor, is orthogonal with respect to the energy inner product, i.e.

$$\Lambda^*_{\mathbf{D}}(\mathbf{v},\mathbf{v}) = K^* \int_{\mathbf{D}} [\epsilon_{ii}(\mathbf{v})]^2 d\mathbf{x} + 2\mu^* \int_{\mathbf{D}} \epsilon_{ij}^d(\mathbf{v})\epsilon_{ij}^d(\mathbf{v}) d\mathbf{x}, \quad \forall\, \mathbf{v} \in \mathbf{H}.$$

When $K^* = 0$, the kernel is characterized by $\epsilon_{ij}^d(\mathbf{v}) = 0$ in $L_2(\mathbf{D})$, which means that $\epsilon_{ij}(\mathbf{v})$ is a spherical tensor. For any spherical tensor, $\epsilon_{ij} = \Psi(\mathbf{x})\delta_{ij}$, the necessary condition of Proposition 4.1 simplifies to $\dfrac{\partial^2 \Psi}{\partial x_i \partial x_j} = 0$, $i,j = 1,2,3$. Solving for Ψ gives $\Psi(\mathbf{x}) \in \text{span}\{1, x_1, x_2, x_3\} \subset L_2(\mathbf{D})$. Now, the corresponding solutions $\mathbf{v}$ of the equation $\epsilon_{ij}(\mathbf{v}) = \Psi(\mathbf{x})\delta_{ij}$ belong to the 4-dimensional space stated in the Lemma. Finally, we note that for a given scalar function Ψ a solution $\mathbf{v}$ is unique up to a rigid body motion, as if $\mathbf{w}$ is another solution, then $\epsilon_{ij}(\mathbf{v} - \mathbf{w}) = 0$ and the standard arguments apply. □

REMARK 4.1. *The four vector functions in Lemma 4.1 are the four eigenfunctions of the Cosserat eigenproblem with Neumann boundary conditions corresponding to the eigenvalue* K^**=0, see [16]. For a sphere, those functions were found in 1901 by Eugène and François Cosserat [9, 10].*

In the next lemma, we show that if we fix the function $\mathbf{v}$ in Theorem 3.1 and allow the constant κ depend on $\mathbf{v}$, then the functions from the null-space, and only those, can cause trouble.

LEMMA 4.2. *Let* $\mathbf{v} \in \mathbf{H}$ *be fixed. Then there exist a constant* $\kappa(\mathbf{v}) > 0$, *such that*

$$\frac{\Lambda^*_{\mathbf{D}}(\mathbf{v},\mathbf{v})}{\Lambda^*(\mathbf{v},\mathbf{v})} \geq \kappa, \quad \text{as} \quad \frac{K^*}{\mu^*} \to 0,$$

if and only if $\mathbf{v}$ *does not belong to the null space of* $\Lambda^*_{\mathbf{D}}(\cdot,\cdot)$ *with* $K^* = 0$.

Proof. If $\mathbf{v}$ is in the kernel, then $\epsilon_{ij}(\mathbf{v})$ is a spherical tensor and all the arguments of Lemma 3.2 apply.

If $\mathbf{v}$ is not in the kernel, then the deviatoric part of the strain tensor $\epsilon^d_{ij}(\mathbf{v}) \neq 0$; and we have

$$\text{(4.1)} \qquad \frac{\Lambda^*_{\mathbf{D}}(\mathbf{v},\mathbf{v})}{\Lambda^*(\mathbf{v},\mathbf{v})} \geq \frac{\int_{\mathbf{D}} 2\mu^* \epsilon^d_{ij}(\mathbf{v})\epsilon^d_{ij}(\mathbf{v})\, d\mathbf{x}}{\Lambda^*(\mathbf{v},\mathbf{v})} \geq \frac{\int_{\mathbf{D}} 2\mu^* \epsilon^d_{ij}(\mathbf{v})\epsilon^d_{ij}(\mathbf{v})\, d\mathbf{x}}{(3K^* + 2\mu^*)\|\mathbf{v}\|^2_{[W^1_2(\mathbf{T})]^3}}.$$

To obtain the last inequality, we use

$$\Lambda^*(\mathbf{v},\mathbf{v}) = \mu^* \int_{\mathbf{T}} \frac{\partial v_i}{\partial x_j}\frac{\partial v_i}{\partial x_j} d\mathbf{x} + (\lambda^* + \mu^*) \int_{\mathbf{T}} |\text{div } \mathbf{v}|^2\, d\mathbf{x} \leq$$

$$\mu^* \int_{\mathbf{T}} \frac{\partial v_i}{\partial x_j}\frac{\partial v_i}{\partial x_j} d\mathbf{x} + 3(\lambda^* + \mu^*) \int_{\mathbf{T}} \frac{\partial v_i}{\partial x_j}\frac{\partial v_i}{\partial x_j} d\mathbf{x} = (3K^* + 2\mu^*)\|\mathbf{v}\|^2_{[W^1_2(\mathbf{T})]^3}.$$

We complete the proof by taking the limit $\dfrac{K^*}{\mu^*} \to 0$ in (4.1). $\square$

REMARK 4.2. *Let now the domain* $\mathbf{D}$ *wrap around the torus* $\mathbf{T}$ *at least in one direction. Then none of the functions of Lemma 4.1 can be realized, and the null-space of* $\Lambda^*_{\mathbf{D}}(\mathbf{v},\mathbf{v})$ *with* $K^* = 0$ *consists only of some standard rigid body motions (translations and rotations). This gives us hope that for such domains* $\mathbf{D}$ *the statement of Theorem 3.1 holds even if* $\dfrac{K^*}{\mu^*} \to 0$. *The proof is outside of the scope of the paper, however, we want to highlight some arguments.*

Without loss of generality, we set $\mu^* = 1$. *Let*

$$1 \geq K^* \geq 0.$$

We notice that the term $\Lambda^*(\mathbf{w},\mathbf{w})$ *is uniformly (in* K^**) equivalent to* $\int_{\mathbf{T}} \epsilon_{ij}(\mathbf{w})\epsilon_{ij}(\mathbf{w})\, d\mathbf{x}$. *Thus, if we could prove that*

$$\Lambda^*_{\mathbf{D}}(\mathbf{w},\mathbf{w}) \geq C \int_{\mathbf{D}} \epsilon_{ij}(\mathbf{w})\epsilon_{ij}(\mathbf{w})\, d\mathbf{x}$$

uniformly in K^*, *then we would be able to use Theorem 3.1 in its current form to prove that it holds when* $K^* \to 0$ *as well, provided that* $\mathbf{D}$ *wraps around the torus.*

In the worst case, when $K^* = 0$, *the estimate above takes the form*

$$\int_{\mathbf{D}} \epsilon^d_{ij}(\mathbf{w})\epsilon^d_{ij}(\mathbf{w})\, d\mathbf{x} \geq C \int_{\mathbf{D}} \epsilon_{ij}(\mathbf{w})\epsilon_{ij}(\mathbf{w})\, d\mathbf{x}, \ \forall \mathbf{w} \in [W^1_2(\mathbf{D})]^3.$$

We can ignore rigid body motions here as they are in the null-space of both quadratic forms.

One can try to establish the inequality directly using arguments similar to those applied to other types of boundary value problems in [6]. A simpler possibility is adopting arguments used in [15], pp. 28-34, to prove the Korn-type inequality.

Yet another choice is applying results of [16], where it is proved that the value $K^* = 0$ corresponds to the extreme eigenvalue of the Cosserat eigenproblem with second-type boundary conditions, and this eigenvalue is isolated and that of finite multiplicity, see the previous remark. We cut our domain to get a new domain that does not wrap around the torus so results of [16] can be immediately applied. Now, functions on our original domain form a subspace in the space of functions on the new domain. This subspace has only a trivial intersection with the finite-dimensional eigenspace found in Lemma 4.1. Then, the desired estimate can be obtained using known theory of the Rayleigh–Ritz method, e.g., [13].

5. Solving The Preconditioner Problem. In order to compute the initial guess (2.6), and at any iteration step of the process (2.5), we need to find a solution $\mathbf{u} \in [W_2^1(\mathbf{T})]^3/R^3$ to Stokes or Lamé equations

$$
\begin{array}{rcll}
\mu\Delta\mathbf{u} + \mathrm{grad}\, p & = & \mathbf{g} & \\
\mathrm{div}\ \mathbf{u} & = & 0 & \text{(Stokes)} \\
\mathrm{div}\ \mathbf{u} & = & \dfrac{p}{\lambda+\mu} & \text{(Lamé)}
\end{array} \tag{5.1}
$$

THEOREM 5.1. *The solution of problem (5.1) is given by*

$$\mu\mathbf{u} = \Delta^{-1}\mathbf{g} - \Delta^{-1}\mathrm{grad}\ \mathrm{div}\ \Delta^{-1}\mathbf{g}$$

for Stokes equations, and by

$$\mu\mathbf{u} = \Delta^{-1}\mathbf{g} - \Delta^{-1}\mathrm{grad}\ \frac{\lambda+\mu}{\lambda+2\mu}\ \mathrm{div}\ \Delta^{-1}\mathbf{g}$$

for Lamé equations.

Proof. After applying the div operator on both sides of the first equation in (5.1), and using the identity $\Delta\,\mathrm{div} = \mathrm{div}\,\Delta$ for our periodic case, we get a Poisson equation for p, or $\dfrac{\lambda+2\mu}{\lambda+\mu}p$, correspondingly. Then we plug the solution to the Poisson equation back into the first equation. □

REMARK 5.1. *In the above derivations we never used that λ is a constant, and therefore, the theorem holds for variable λ in the interval $-\frac{2}{3}\mu \leq \lambda(\mathbf{x}) \leq +\infty$.*

Once problem (5.1) is reduced to six scalar periodic boundary value problems for the Laplace operator, one can apply a variety of efficient solvers, like separation of variables, or multilevel methods. Also, if we interchange the places of Δ^{-1} and grad, the number of scalar equations can be reduced to four.

One can directly apply the Fourier method to problem (5.1): find the unknown scalars $c^j_{n_1,n_2,n_3}$ and c_{n_1,n_2,n_3}, such that

$$u_j(\mathbf{x}) \approx \sum_{n_1,n_2,n_3} c^j_{n_1,n_2,n_3} \exp\left(2\pi i(n_1x_1 + n_2x_2 + n_3x_3)\right),$$

$$p(\mathbf{x}) \approx \sum_{n_1,n_2,n_3} c_{n_1,n_2,n_3} \exp\left(2\pi i(n_1x_1+n_2x_2+n_3x_3)\right),$$

where $i=\sqrt{-1}$ and $j=1,2,3$.

There is also another version of the above method. We find, by solving a system of ordinary differential equation, unknown functions $c^j_{n_1,n_2}(x_3)$ and $c_{n_1,n_2}(x_3)$, such that

$$u_j(\mathbf{x}) \approx \sum_{n_1,n_2} c^j_{n_1,n_2}(x_3) \exp\left(2\pi i(n_1x_1+n_2x_2)\right),$$

$$p(\mathbf{x}) \approx \sum_{n_1,n_2} c_{n_1,n_2}(x_3) \exp\left(2\pi i(n_1x_1+n_2x_2)\right),$$

where $i=\sqrt{-1}$ and $j=1,2,3$. The last method is applicable not only on a torus, but also on a cylinder with homogeneous boundary conditions specified at the top and the bottom.

6. Method of Fictitious Domains. One can obtain an iterative method for solving problem (2.1) with $\mu(\mathbf{x})=0$ in $\mathbf{T}\backslash\overline{\mathbf{D}}$ (and for problem (3.1) with $\mu(\mathbf{x})=\lambda(\mathbf{x})=0$ in $\mathbf{T}\backslash\overline{\mathbf{D}}$) by letting $\mu^*\to 0$ in (2.5)–(2.6) (and $\lambda^*\to 0$ with $\dfrac{\mu^*}{\lambda^*}=\dfrac{\mu}{\lambda}$ in the case of problem (3.1)).

Similarly, by letting $\mu\to+\infty$ (and $\lambda\to+\infty$ with $\dfrac{\mu^*}{\lambda^*}=\dfrac{\mu}{\lambda}$ in the case of problem (3.1)), we obtain an iterative method for solving problem (2.1) with $\dfrac{1}{\mu}=0$ in $\mathbf{D}$ (and in case of problem (3.1) with $\dfrac{1}{\mu}=\dfrac{1}{\lambda}=0$ in $\mathbf{D}$). These types of problems arise (cf. [4]) when the method of fictitious domains is applied to boundary value problems of the first and second kind.

Mixed boundary value problems can be reduced to boundary value problems on a cylinder by applying the fictitious domains method. For example, let us consider Lamé equations with coefficients λ and μ in $\mathbf{D}$, where $\mathbf{D}$ is a brick. Let homogeneous Dirichlet boundary conditions be specified on the horizontal faces of $\mathbf{D}$ and Neumann boundary conditions everywhere else. Then we consider a fictitious brick $\mathbf{\Pi}$ containing $\mathbf{D}$, and having the same height, but larger length and depth. On $\mathbf{\Pi}$ we consider Lamé equations with homogeneous Dirichlet boundary conditions specified on the horizontal faces and periodic boundary conditions at other pairs of opposite faces. We choose some small coefficient λ^*, while keeping $\dfrac{\mu^*}{\lambda^*}=\dfrac{\mu}{\lambda}$. Specifying periodic boundary conditions is equivalent to identifying the corresponding faces of $\mathbf{\Pi}$, i.e. we have re-formulated a problem on a brick to one on a cylinder. Now we can apply all results from Section 3 to the problem on a cylinder. In this way, by letting in (2.5)–(2.6) $\mu^*\to 0$ and $\lambda^*\to 0$, we obtain an efficient iterative technique for the original mixed boundary value problem with convergence rate uniform in λ as $\lambda\to+\infty$.

REMARK 6.1. *If the the problem in considerations possesses (odd or even) symmetry with respect to one or more coordinate planes, then the solution will have the same symmetry. In that case, all approximations in (2.5)–(2.6) will have the symmetry property. This allows us to solve the problem only on a part of the torus by imposing on the planes of symmetry boundary conditions of the third or fourth kind. The above statements also apply to a problem on a cylinder.*

REMARK 6.2. *The coefficients λ and μ can be variable in $\mathbf{D}$. Let for some positive constants χ, μ we have $\chi\mu \leq \mu(\mathbf{x}) \leq \mu$ and $\chi\lambda \leq \lambda(\mathbf{x}) \leq \lambda$ for almost all $\mathbf{x} \in \mathbf{D}$. Then Theorem 2.1 holds with $q = 1 - \chi\kappa < 1$ and Theorem 3.2 holds with $q = 1 - \chi\kappa\eta < 1$.*

REFERENCES

[1] N. S. BAKHVALOV AND A. V. KNYAZEV, *A new iterative algorithm for solving problems of the fictitious flow method for elliptic equations*, Soviet Math. Doklady, 41 (1990), pp. 481–485.

[2] ——, *Efficient computation of averaged characteristics of composites of a periodic structure of essentially different materials*, Soviet Math. Doklady, 42 (1991), pp. 57–62.

[3] ——, *An efficient iterative method for solving the Lamé equations for almost incompressible media and Stokes' equations*, Soviet Math. Doklady, 44 (1992), pp. 4–9.

[4] ——, *Fictitious domain methods and computation of homogenized properties of composites with a periodic structure of essentially different components*, in Numerical Methods and Applications, G. I. Marchuk, ed., CRC Press, Boca Raton, 1994, pp. 221–276.

[5] ——, *Preconditioned iterative methods in a subspace for linear algebraic equations with large jumps in the coefficients*, in Domain Decomposition Methods in Science and Engineering, D. Keyes and J. Xu, eds., vol. 180 of Contemporary Mathematics, American Mathematical Society, Providence, 1994, pp. 157–162. Proceedings of the Seventh International Conference on Domain Decomposition, October 27-30, 1993, held at the Pennsylvania State University.

[6] N. S. BAKHVALOV, A. V. KNYAZEV, AND M. E. EGLIT, *An inequality of the Korn type and orthogonal decompositions in spaces of matrices*, Russian Acad. Sci. Math. Doklady, 48 (1994), pp. 127–129.

[7] N. S. BAKHVALOV, A. V. KNYAZEV, AND G. M. KOBEL'KOV, *Iterative methods for solving equations with highly varying coefficients*, in Fourth International Symposium on Domain Decomposition Methods for Partial Differential Equations, R. Glowinski, Y. A. Kuznetsov, G. A. Meurant, J. Périaux, and O. Widlund, eds., Philadelphia, PA, 1991, SIAM, pp. 197–205.

[8] F. BREZZI AND M. FORTIN, *Mixed and Hybrid Finite Element Methods*, Springer–Verlag, New York, 1991.

[9] E. COSSERAT AND F. COSSERAT, *Sur une application des fonctions potentielles de la théorie de l'élasticité*, C. R. Acad. Sci. (Paris), 133 (1901), pp. 210–213.

[10] ——, *Sur une point critique particulier de la solution des équations de l'élasticité dans le cas où les efforts sur la frontière sont données*, C. R. Acad. Sci. (Paris), 133 (1901), pp. 382–384.

[11] V. GIRAULT AND P.-A. RAVIART, *Finite element methods for Navier–Stokes equations*, vol. 5 of Springer Series in Computational Mathematics, Springer-Verlag, Berlin, 1986. Theory and algorithms.

[12] A. V. KNYAZEV, *Iterative solution of PDE with strongly varying coefficients: algebraic version*, in Iterative Methods in Linear Algebra, R. Beauwens and P. de Groen, eds., Amsterdam, 1992, Elsevier, pp. 85–89. Proceedings IMACS Symp. Iterative Methods in Linear Algebra, Brussels, 1991.

[13] A. V. KNYAZEV, *New estimates for Ritz vectors*, Math. Comp., 66 (1997), pp. 985–995.

[14] O. A. Ladyzhenskaya, *The mathematical theory of viscous incompressible flow*, Gordon and Breach, New York, 1969.
[15] V. G. Litvinov, *Motion of a nonlinearly viscous fluid*, "Nauka", Moscow, 1982. (In Russian).
[16] S. G. Mikhlin, *The spectrum of a family of operators in the theory of elasticity*, Russian Math Surveys, 28 (1973), pp. 45–88.