A Homotopy Based Algorithm for Mixed Complementarity Problems

Stephen C. Billups *

February 13, 1998
(Revised April 3, 1998)

Abstract

This paper develops an algorithm for solving mixed complementarity problems which is based upon probability one homotopy methods. After the complementarity problem is reformulated as a system of nonsmooth equations, a homotopy method is used to solve a sequence of smooth approximations to this system of equations. The global convergence properties of this approach are considerably stronger than other recent algorithms, depending on very weak assumptions about the problem. To improve efficiency, the homotopy algorithm is embedded in a generalized Newton-method.

Keywords: Complementarity problems, homotopy methods, smoothing.

1 Introduction

This paper discusses a robust method for solving mixed complementarity problems, which is based upon the probability one homotopy methods of [13, 31, 33]. The idea is to reformulate the mixed complementarity problem as a system of equations, and then solve smooth approximations of this system with a homotopy method. While extremely robust, the homotopy methods we have considered tend to be slower than Newton-based methods. We therefore propose to embed the homotopy method inside a Newton-based method. A similar approach was successfully applied in the proximal perturbation strategy described in [4, 5, 7]. The idea is to invoke the homotopy technique only when the Newton-based method fails. The homotopy method is used to construct an improved starting point, from which the Newton method can be restarted.

The idea of applying homotopy methods to complementarity problems is not new; Watson [32] proposed such a method to solve the nonlinear complementarity problem. Watson's method involved reformulating the nonlinear complementarity problem as a system of smooth (C^2) equations and applying a homotopy method to solve this system. In the context of Newton-based methods, such smooth reformulations of complementarity problems are inferior to nonsmooth reformulations due to slow local convergence for degenerate solutions. In contrast, nonsmooth reformulations allow much faster (superlinear or quadratic) convergence to degenerate solutions. As such, we are interested in applying the homotopy method in the context of nonsmooth reformulations of the mixed complementarity problem. One such approach was developed by Sellami and Robinson [26, 28, 27] based on the theoretical framework for piecewise smooth continuation methods presented in [1, 2, 3]. This approach was complicated by the fact that a special procedure was needed to make the transition from one smooth segment of the homotopy zero curve to another.

*Department of Mathematics, University of Colorado, Denver, Colorado 80217-3364

In this paper, we consider a different approach; rather than applying the homotopy method to the original nonsmooth equations, we instead apply it to to a smooth approximation of these equations. The solution of this smooth approximation can then be shown to be nearly a zero of the original function. This solution then gives the improved starting point from which to restart Newton's method. The overall strategy is as follows: first apply a nonsmooth Newton method using a linesearch to ensure a reduction of a merit function at each iteration. If the Newton method stalls (for example, at a local minimum of the merit function), then apply the homotopy method to a smooth approximation of the equations. If the smooth approximation is properly chosen, the solution generated by the homotopy method will provide a reduction in the merit function of the original equations. It is then possible to return to the damped Newton method with no risk of returning to the region where the method stalled.

In the remainder of this paper, we describe this approach in more detail. Section 2 provides essential background material, including reformulations of mixed complementarity problems, smoothing functions, and homotopy methods. Section 3 describes the algorithm in general and proves a global convergence result. Section 4 discusses a particular implementation of the approach along with some numerical experimentation. Finally, in Section 5, we give conclusions.

2 Background

Given a rectangular region $\mathbb{B} = \prod_{i=1}^{n}[l_i, u_i]$ (where for each i, $-\infty \leq l_i < u_i \leq \infty$), and a function $F : \mathbb{B} \to \mathbb{R}^n$, the mixed complementarity problem MCP$(F, \mathbb{B})$ is to find $x \in \mathbb{B}$ such that for each $i \in \{1, \ldots, n\}$, either

1. $x_i = l_i$ and $F_i(x) \geq 0$, or
2. $F_i(x) = 0$, or
3. $x_i = u_i$ and $F_i(x) \leq 0$.

A more concise way of stating these conditions is that $\text{mid}(x - l, x - u, F(x)) = 0$, where mid is the componentwise median function.

In the above definition, if $l_i = 0, u_i = \infty$ for all $i = 1, 2, \ldots, n$, then MCP$(F, \mathbb{B})$ reduces to the standard form nonlinear complementarity problem NCP(F), which is to find $x \geq 0$ such that

$$\min(x, F(x)) = 0.$$

In discussing algorithms for solving these problems, it is normal to assume that F is a C^1 function on an open set $\Omega \supset \mathbb{B}$. **For our homotopy approach, we shall make the stronger assumption that F is C^2 on Ω. Furthermore, for simplicity of discussion, we will assume that $\Omega = \mathbb{R}^n$.**

2.1 MCP Reformulations

A common approach to solving the mixed complementarity problem is to define a function $H : \mathbb{R}^n \to \mathbb{R}^n$ such that the zeros of H correspond to solutions of the complementarity problem. To discuss such reformulations, we need to state several definitions, which are equivalent to the NCP function and the BVIP function defined in [24]:

Definition 2.1 *A function $\phi : \mathbb{R}^2 \to \mathbb{R}$ is called an NCP function if $\phi(a, b) = 0$ if and only if* $\min(a, b) = 0$.

Definition 2.2 *A function* $\psi : \mathbb{R}\bigcup\{-\infty\} \times \mathbb{R}\bigcup\{+\infty\} \times \mathbb{R}^2 \to \mathbb{R}$ *is called an* MCP function associated with l and u *if* $\psi(l, u, x, f) = 0$ *if and only if* $\text{mid}(x - l, x - u, f) = 0$.

For simplicity, we will also refer to $\psi_{l,u} := \psi(l, u, \cdot, \cdot)$.

It is useful to further distinguish NCP and MCP functions according to their orientations:

Definition 2.3 *An NCP function* ϕ *is called* positively oriented *if*

$$\text{sign}(\phi(a, b)) = \text{sign}(\min(a, b)).$$

An MCP function ψ *is called* positively oriented *if*

$$\text{sign}(\psi(l, u, x, f)) = \text{sign}(\text{mid}(x - l, x - u, f)).$$

Finally, we can further classify MCP functions with the following definition:

Definition 2.4 *A positively oriented MCP function* ψ *is said to be* median-bounded *if there exists a positive constant* M *such that*

$$|\psi(l, u, x, f)| \leq M|\text{mid}(x - l, x - u, f)|.$$

Two popular NCP functions are the median function and the Fischer-Burmeister [18, 19] defined by

$$\phi_{FB}(a, b) = a + b - \sqrt{a^2 + b^2}. \tag{1}$$

This function is continuously differentiable everywhere except at the origin, and furthermore, it has the nice property that ϕ^2 is continuously differentiable. (Note: this version of the Fischer-Burmeister function is actually the negative of the function presented in [18, 19]. This change of sign makes ϕ_{FB} a positively oriented NCP function.)

Billups [4, 5] showed how both of these NCP functions can be used to construct an MCP function using the formula

$$\psi(l, u, x, f) := \phi(x - l, -\phi(u - x, -f)). \tag{2}$$

In the case where ϕ is the min function, this formula simplifies to $\psi_{l,u}(a, b) = \text{mid}(a - l, a - u, b)$. In the case of the Fischer-Burmeister function, Billups [4][Proposition 3.2.7, Theorem 3.2.8] showed that $\psi_{l,u}$ is a semismooth (see definition 2.6) MCP function.

This approach was generalized by Qi [24], who showed that if ϕ is a regular pseudo-smooth (see [24][definintion 2.1]) NCP function, then ψ defined by (2) is a regular pseudo-smooth MCP function.

Proposition 2.5 *The function* ψ *defined by (2) is a median-bounded MCP function.*

Proof Let ϕ be the Fischer-Burmeister function defined by (1). It is easy to show that

$$\min(a, b) \geq 0 \implies 0 \leq \phi(a, b) \leq \min(a, b) \tag{3}$$

$$\text{and} \quad \min(a, b) < 0 \implies 0 > \phi(a, b) \geq M \min(a, b), \tag{4}$$

where $M = 2 + \sqrt{2}$. Let $c := -\phi(u - x, -f)$. We now consider several cases:

If $c < 0$, then $\phi(u-x,-f) > 0$, and by positive orientation, $\min(u-x,-f) > 0$. It follows by (3) that $0 \leq \phi(u-x,-f) \leq \min(u-x,-f)$ or, equivalently, $-\min(u-x,-f) \leq c < 0$. Since $\min(x-l,c) < 0$, then

$$\begin{aligned} 0 &> \phi(x-l,c) \\ &\geq M\min(x-l,c) \\ &\geq M\min(x-l,-\min(u-x,-f)) \\ &= M\,\mathrm{mid}(x-l,x-u,f). \end{aligned}$$

If $c \geq 0$, then $0 \leq c \leq -M\min(u-x,-f)$. Now, if $x-l \geq 0$, then $\min(x-l,c) \geq 0$, so

$$\begin{aligned} 0 &\leq \phi(x-l,c) \\ &\leq \min(x-l,c) \\ &\leq \min(x-l,-M\min(u-x,-f)) \\ &\leq M\min(x-l,-\min(u-x,-f)) \\ &= M\,\mathrm{mid}(x-l,x-u,f). \end{aligned}$$

If instead $x-l<0$, then $\min(x-l,c) = x-l < 0$, so $0 \geq \phi(x-l,c) \geq M(x-l)$. But, in this case, since $x-u < x-l < 0$ and $c \geq 0$, we must have that $f \geq 0$. Thus, $\mathrm{mid}(x-l,x-u,f) = x-l$. Thus, $0 \geq \phi(x-l,c) \geq M\,\mathrm{mid}(x-l,x-u,f)$.

In every case, we get $|\psi(l,u,x,f)| = |\phi(x-l,c)| \leq M|\,\mathrm{mid}(x-l,x-u,f)|$. □

It follows from the definitions that if we define $H : \mathbb{R}^n \to \mathbb{R}^n$ by

$$H_i(x) := \psi(l_i,u_i,x_i,F_i(x)) \tag{5}$$

then a point x is a solution of MCP$(F,\mathbb{B})$ if and only if $H(x)=0$. Thus, the problem of solving the MCP reduces to finding a zero of the function H. Given such a function H, it is usual to define the *natural* merit function

$$\theta(\cdot) := \frac{1}{2}\|H(\cdot)\|^2,$$

which is useful for linesearch strategies.

2.2 Generalized Newton Algorithms

Since the function H is not smooth, Newton's method cannot be applied directly; however, a generalization can be stated using the notion of the B-subdifferential.

By Rademacher's theorem, if $F : \mathbb{R}^n \to \mathbb{R}^m$ is locally Lipschitzian, it is differentiable almost everywhere. Let D_F be the set where F is differentiable. We define the B-subdifferential by

$$\partial_B F(x) := \left\{ V \,\middle|\, \exists\{x^k\} \to x, x^k \in D_F, \quad \text{with} \quad V = \lim_{k\to\infty} \nabla F(x_k) \right\}.$$

The Clarke subdifferential $\partial F(x)$ is the convex hull of $\partial_B F(x)$.

Definition 2.6 *We say that F is* semismooth *at x if*

$$\lim_{\substack{V \in \partial F(x+th') \\ h' \to h, t\downarrow 0}} \{Vh'\}$$

exists for any $h \in \mathbb{R}^n$.

Definition 2.7 *We say that a semismooth function F is* BD-regular *at x if all elements in $\partial_B F(x)$ are nonsingular.*

Definition 2.8 *Suppose that $F : \mathbb{R}^n \to \mathbb{R}^m$ is B-differentiable in a neighborhood of x. We say that the directional derivative $F'(\cdot;\cdot)$ is* semicontinuous *at x if, for every $\epsilon > 0$, there exists a neighborhood N of x such that, for all $x + h \in N$,*

$$\|F'(x+h;h) - F'(x;h)\| \leq \epsilon \|h\| .$$

We say that $F'(\cdot;\cdot)$ is semicontinuous of degree 2 *at x if there exist a constant L and a neighborhood N of x such that, for all $x + h \in N$,*

$$\|F'(x+h;h) - F'(x;h)\| \leq L \left\|h^2\right\| .$$

A nonsmooth version of a damped Newton method, which is discussed in [5] is shown in Figure 1.

Figure 1: Generalized Damped Newton Method

Step 1 [Initialization] Select linesearch parameters $\alpha, \sigma \in (0, 1)$, a positive integer m_{max}, a starting point $x^0 \in \mathbb{R}^n$, and a stopping tolerance *tol*. Set $k = 0$.

Step 2 [Direction generation] Choose $V^k \in \partial_B H(x^k)$. If V^k is singular, stop, returning the point x^k along with a failure message. Otherwise choose the direction

$$d^k = -(V^k)^{-1} H(x^k). \tag{6}$$

Step 3 [Steplength determination] Let m_k be the smallest nonnegative integer $m \leq m_{max}$ such that

$$\theta(x^k + \alpha^m d^k) - \theta(x^k) \leq -2\sigma\alpha^m \theta(x^k). \tag{7}$$

If no such m_k exists, stop, returning the point x^k along with a failure message. Otherwise set $x^{k+1} = x^k + \alpha^{m_k} d^k$.

Step 4 [Termination check] If $\theta(x^{k+1}) < tol$ stop, returning the point x^{k+1}. Otherwise, return to Step 2, with k replaced by $k + 1$.

The algorithm has three features that make it attractive for use in our context:

1. The calculation of the search direction at each iteration is very cheap; it only requires solving the single linear equation (6).

2. The algorithm either fails in a finite number of steps, or produces a sequence of iterates $\{x^k\}$ such that the corresponding merit function values $\{\theta(x^k)\}$ are strictly decreasing. This property, which is an obvious consequence of the upper bound m_{max} placed on m_k for the steplength determination step, is essential for our purposes. When the algorithm fails, we

intend to employ a homotopy method to construct an improved starting point $\tilde{x}$ for which $\theta(\tilde{x})$ is smaller than any merit function values evaluated thus far. It will then be possible to restart the Newton method from $\tilde{x}$ with the guarantee that the iterates will not return to the region where the algorithm failed previously.

3. The algorithm has fast local convergence behavior, which is summarized in the following theorem from Qi [23].

Theorem 2.9 *Suppose that x^* is a solution of $H(x) = 0$, and that H is semismooth and BD-regular at x^*. Then the iteration method defined by $x^{k+1} = x^k + d^k$, where d^k is given by (6) is well defined and convergent to x^* superlinearly in a neighborhood of x^*. In addition, if $H(x^k) \neq 0$ for all k, then*

$$\lim_{k\to\infty} \frac{\left\| H(x^{k+1}) \right\|}{\left\| H(x^k) \right\|} = 0.$$

If, in addition, H is directionally differentiable at a neighborhood of x^* and $H'(\cdot;\cdot)$ is semicontinuous of degree 2 at x^*, then the convergence of the iteration method is quadratic.

One consequence of this local convergence theorem is that within a neighborhood of a BD-regular solution x^*, the linesearch criteria (7) will be satisfied by $m_k = 0$. Thus, the inner algorithm will take full Newton steps and achieve the fast local convergence rates specified by the theorem.

2.3 Homotopy Methods

The probability one homotopy methods we consider in this paper are based on the following proposition from [13, 30, 31]:

Proposition 2.10 *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a C^2 function and suppose there exists a C^2 map*

$$\rho : \mathbb{R}^m \times [0,1) \times \mathbb{R}^n \to \mathbb{R}^n$$

such that

1. *the $n \times (m+1+n)$ Jacobian matrix $\nabla\rho(a,\lambda,x)$ has rank n on the set*

$$\rho^{-1}(0) = \{(a,\lambda,x) \mid a \in \mathbb{R}^m, 0 \le \lambda < 1, x \in \mathbb{R}^n, \rho(a,\lambda,x) = 0\},$$

   and for any fixed $a \in \mathbb{R}^m$,

2. *$\rho_a(0,x) := \rho(a,0,x) = 0$ has a unique solution x_0,*

3. *$\rho_a(1,x) = F(x)$,*

4. *$\rho_a^{-1}(0)$ is bounded,*

then for almost all $a \in \mathbb{R}^m$ (in the sense of Lebesgue measure) there exists a zero curve γ of ρ_a along which the Jacobian matrix $\nabla\rho_a$ has rank n, emanating from $(0, x_0)$ and reaching a zero $\bar{x}$ of F at $\lambda = 1$. Moreover, γ does not intersect itself and is disjoint from any other zeros of ρ_a.

The expression "reaching a zero" requires some clarification. This expression means that there exists a sequence of points $\{(\lambda_k, x^k)\}$ in γ converging to $\bar{x}$.

The full rank conclusion of $\nabla\rho_a$ on $\rho^{-1}(0)$ allows us to parameterize γ by arc length. Thus, we denote by $\gamma(s)$ the point on γ of arclength s along γ from $(0, x_0)$.

Given such a homotopy mapping ρ, a globally convergent algorithm can be constructed which picks $a \in \mathbb{R}^m$ (which uniquely determines x_0), and then tracks the homotopy zero curve γ. Perhaps the simplest choice of homotopy mapping is given by $\rho : \mathbb{R}^n \times [0, 1) \times \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$\rho(a, \lambda, x) := \lambda F(x) + (1 - \lambda)(x - a). \tag{8}$$

When F is a C^2 map, this choice of ρ satisfies properties 1)–3), but not necessarily 4). However, there are fairly general sufficient conditions on $F(x)$ so that it does satisfy property 4). One such sufficient condition is particularly relevant in our context and gives us the following theorem from [32].

Theorem 2.11 *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a C^2 map such that*

$$\text{for some } r > 0, x^\top F(x) \geq 0 \text{ whenever } \|x\| = r. \tag{9}$$

Then

1. *F has a zero in the ball $\{x \in \mathbb{R}^n \mid \|x\| \leq r\}$, and for almost all a in the interior of this ball, there is a zero curve γ of*

$$\rho_a(\lambda, x) := \lambda F(x) + (1 - \lambda)(x - a),$$

along which the Jacobian matrix $\nabla\rho_a(\lambda, x)$ has full rank, emanating from $(0, a)$ and reaching a zero $\bar{x}$ of F at $\lambda = 1$. Furthermore, γ has finite arc length if $\nabla F(\bar{x})$ is nonsingular.

Conceptually, the homotopy method is very simple: construct the homotopy mapping ρ_a and follow the zero curve γ from the point x^0 to the solution. However, implementing this idea into an efficient computer algorithm is very difficult. Clearly, it is impractical to trace the zero curve exactly. Instead we must generate a sequence of points $\{\lambda^k, x^k\}$ which loosely follow the zero curve (within some prescribed tolerances) and which make reliable progress along its arclength. These points should not be too close together, since this requires more function evaluations than are really necessary. However, if these points are spaced too loosely, one can end up tracing a different component of the zero set, or reversing direction on the zero curve γ, thereby never reaching the desired solution.

Obviously it is not possible to ensure "perfect" curve tracking; however, much research has been devoted to this problem and reliable codes have been developed. One such code, which we use in our implementation is HOMPACK [33].

2.4 Smoothing Functions

Since the function H defined in (5) is not C^2, we cannot apply a homotopy algorithm to it directly. Instead we must form a smooth approximation of H. In recent years, numerous techniques have emerged for solving the nonsmooth equation $H(x) = 0$ which are based on the notion of smoothing (see, for example, [8] and the references therein).

The basic idea of these techniques is to approximate the function H by a family of smooth approximations H_μ with *smoothing parameter* μ. Under suitable assumptions, the solutions to the perturbed systems $H_\mu = 0$ form a smooth trajectory, leading to a solution of the original problem. The smoothing methods generate a sequence of iterates that follow this trajectory. However, these methods decrease μ monotonically, so do not share the strong global convergence properties of the homotopy methods.

Definition 2.12 *Given a nonsmooth function $\varphi : \mathbb{R}^p \to \mathbb{R}$, a* smoother *for φ is a continuous function $\tilde{\varphi} : \mathbb{R}^p \times \mathbb{R}_+ \to \mathbb{R}$ with the following properties:*

1. *$\tilde{\varphi}(x, 0) = \varphi(x)$;*
2. *$\tilde{\varphi}$ is twice continuously differentiable with respect to x on $\{(x, \mu) \in \mathbb{R}^p \times \mathbb{R}_+ \mid \mu > 0\}$.*

We shall find it convenient to make the following weak assumption on the smoother:

Assumption 2.13 *There exists a function $\xi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $\lim_{\mu \downarrow 0} \xi(\mu) = 0$ such that*

$$|\tilde{\varphi}(x, \mu) - \tilde{\varphi}(x, 0)| \leq \xi(\mu)$$

for all $x \in \mathbb{R}^p$ and $\mu \in \mathbb{R}_+$.

Numerous smoothers have been proposed in the literature [9, 10, 11, 12, 20, 21, 25, 29, 34]. Many of the early smoothers were unified by the family of smoothing functions described by Chen and Mangasarian [11]. More recently, Gabriel and Moré [20] introduced a more general family of smoothers for the mixed complementarity problem, which includes the Chen-Mangasarian family.

The Gabriel-Moré smoother for the median function is defined in terms of a density function $\nu : \mathbb{R} \to \mathbb{R}_+$ with bounded absolute mean, that is

$$\int_{-\infty}^{\infty} |s|\nu(s)ds =: \kappa < \infty.$$

A smoother for the function $\text{mid}(l, u, \cdot)$ is then defined as follows:

$$p_\mu(f) := \tilde{p}(f, \mu) := \int_{-\infty}^{\infty} \text{mid}(l, u, f - \mu s)\nu(s)ds.$$

Another smoother of interest is a smoother for the Fischer-Burmeister function proposed by Kanzow [21]:

$$\phi_\mu(a, b) := \tilde{\phi}(a, b, \mu) := a + b - \sqrt{a^2 + b^2 + 2\mu}. \tag{10}$$

Using (2) gives the Kanzow MCP smoother

$$\psi_\mu(l, u, x, f) := \tilde{\psi}(l, u, x, f, \mu) := \phi_\mu(x - l, -\phi_\mu(u - x, -f)). \tag{11}$$

In the case of the Gabriel and More' smoother, Assumption 2.13 is met with $\xi(\mu) = \kappa\mu$ [20][Lemma 2.4]. For the Kanzow MCP smoother, an appropriate ξ is given by the following proposition:

Proposition 2.14 *The Kanzow smoother $\tilde{\psi}$ defined by (11) satisfies*

$$\left|\tilde{\psi}(\mu, l, u, x, f) - \psi(l, u, x, f)\right| \leq 3\sqrt{2\mu}.$$

Proof It is easy to show that for all $a, b, c \in \mathbb{R}$, $|\phi_\mu(a,b) - \phi_\mu(a,c)| \leq 2|b-c|$ for all $\mu \in \mathbb{R}_+$. It is also easy to show that $|\phi_\mu(a,b) - \phi(a,b)| \leq \sqrt{2\mu}$. Thus,

$$\begin{aligned}
\left|\tilde{\psi}(\mu, l, u, x, f) - \psi(l, u, x, f)\right| &= |\phi_\mu(x-l, -\phi_\mu(u-x, -f)) - \phi(x-l, -\phi(u-x, -f))| \\
&= |\phi_\mu(x-l, -\phi_\mu(u-x, -f)) - \phi_\mu(x-l, -\phi(u-x, -f)) \\
&\quad +\phi_\mu(x-l, -\phi_\mu(u-x, -f)) - \phi(x-l, -\phi(u-x, -f))| \\
&\leq 2\left|-\phi_\mu(u-x, -f) + \phi(u-x, -f)\right| + \sqrt{2\mu} \\
&\leq 2\sqrt{2\mu} + \sqrt{2\mu} = 3\sqrt{2\mu}.
\end{aligned}$$

□

3 Algorithmic Framework

The basic idea behind our algorithm is to employ the damped Newton method from Figure 1 until it fails. Such failure may be a result of the iterates converging to a local minimum of the merit function θ. When the Newton method fails, we then apply a homotopy method to solve a smooth approximation to the equations. That is, we solve the equation $H_\mu(x) = 0$, where $(H_\mu)_i(x) := \psi_\mu(l_i, u_i, x_i, F_i(x))$. It is not necessary to solve this smooth equation exactly; we are only interested in generating a point $\tilde{x}$ for which θ is decreased. Under mild assumptions, the homotopy method will find such a point provided that 1) the smoothing parameter is not too large, and 2) the stopping tolerance for the homotopy method is sufficiently small. The general algorithm is given in Figure 2.

Figure 2: Algorithmic Framework

Step 1 [Initialization] Given a starting vector $x^0 \in \mathbb{R}^n$, a parameter $\beta < 1$ and a convergence tolerance $\epsilon > 0$, set $k = 0$.

Step 2 [Attempt Descent Algorithm] Run the nonsmooth damped Newton algorithm from Figure 1 with starting point x^k and with $tol = \epsilon$. This generates a point $\tilde{x}^k$.

Step 3 [Termination check] If $\theta(\tilde{x}^k) < \epsilon$, stop; otherwise continue with step 4.

Step 4 [Generate better starting point] Determine a smoothing parameter $\mu > 0$ such that $\xi(\mu) \leq (\beta/2\sqrt{n}) \left\|H(x^k)\right\|$. Run the homotopy algorithm to solve the smooth equation $H_\mu(x) = 0$ to a tolerance of $\frac{\beta}{2}\left\|H(\tilde{x}^k)\right\|$. If the homotopy algorithm fails, stop. Otherwise, set x^{k+1} equal to the solution.

Step 5 Return to step 2 with k replaced by $k+1$.

The global convergence behavior for this algorithm is established by the following theorem:

Theorem 3.1 *The algorithm in Figure 2 either terminates in Step 3 (at a solution), or fails in Step 4 (during the homotopy method).*

Proof Assume that Step 4 of the algorithm is always successful and that the test in Step 3 of the algorithm always fails. Then since the damped Newton method always terminates in a finite number of iterations, the algorithm will generate an infinite sequence of points $\{x^k\}$. Because of the linesearch criteria in the damped Newton method, $\theta(\tilde{x}^k) \leq \theta(x^k)$. Now,

$$\begin{aligned}\left\|H(x^{k+1})\right\| &\leq \left\|H_\mu(x^{k+1})\right\| + \left\|H(x^{k+1}) - H_\mu(x^{k+1})\right\| \\ &\leq \tfrac{\beta}{2}\left\|H(\tilde{x}^k)\right\| + \sqrt{n}\xi(\mu) \\ &\leq \tfrac{\beta}{2}\left\|H(\tilde{x}^k)\right\| + \tfrac{\beta}{2}\left\|H(\tilde{x}^k)\right\| \\ &\leq \beta\left\|H(\tilde{x}^k)\right\|.\end{aligned}$$

Thus, $\theta(x^{k+1}) \leq \beta^2\theta(\tilde{x}^k) \leq \beta^2\theta(x^k) \leq \beta^{2(k+1)}\theta(x^0)$. Thus, for some finite value of k, $\theta(x^k) < \epsilon$, contradicting the assumption that the test in Step 3 always fails. □

The proposition shows that the success the algorithm relies entirely upon the success of the homotopy method in Step 4. This in turn depends on two questions: 1) Does the homotopy zero curve lead to a solution in finite length, and 2) can the homotopy method successfully track this zero curve. Since we cannot guarantee successful curve tracking, the second question represents a theoretical stumbling block. However, as we previously discussed, sophisticated codes, such as HOMPACK, are available which perform this curve tracking fairly reliably. We therefore focus our attention on the first question.

Theorem 2.11 provides sufficient conditions under which a homotopy zero curve exists which leads to a solution in finite length. We now prove several results which are more specific to the complementarity framework.

Lemma 3.2 *Let ψ be a positively oriented median-bounded MCP function, and let H be defined by (5). If $\mathbb{B}$ is bounded, then*

$$\lim_{\|x\|\to\infty} \frac{x^\top H(x)}{\|x\|} = +\infty. \tag{12}$$

Proof For a given $x \in \mathbb{R}^n$, suppose $x_i < 0$ and $H_i(x) > 0$. Then by positive orientation, $\text{mid}(x_i - l_i, x_i - u_i, F_i(x)) > 0$, which implies that $x_i > l_i$ and $\text{mid}(x_i - l_i, x_i - u_i, F_i(x)) \leq x_i - l_i < |l_i|$. Thus,

$$\begin{aligned} x_i H_i(x) &\geq -|l_i|\,|\psi(l_i, u_i, x_i, F_i(x))| \\ &\geq -|l_i| M |\,\text{mid}(x_i - l_i, x_i - u_i, F_i(x))| \\ &\geq -M l_i^2 \end{aligned}$$

where M is the constant guaranteed by the median-bounded property (see Definition 2.4). Similarly, if $x_i > 0$ and $H_i(x) < 0$, we can show that $x_i H_i(x) \geq -M u_i^2$. Since these are the only two cases in which $x_i H_i(x)$ can be negative, we have that

$$x_i H_i(x) \geq -M b_i^2,$$

where $b_i := \max\{|l_i|, |u_i|, 1\}$. Let $b_{max} := \max_i b_i$, $b_{min} := \min_i b_i$ and $d := \|b\|$.

For a given x, let $k := \|x\|/d$. Then if $k > 1$, there exists an index j such that $|x_j| \geq k b_j$. If x_j is positive, then $\text{mid}(x_j - l_j, x_j - u_j, F_j(x)) > (k-1)b_j$, so

$$x_j H_j(x) > (k b_j)(M(k-1)b_j) > k(k-1)b_{min}^2.$$

In similar fashion, we can show that this inequality holds if x_j is negative. Thus,

$$\begin{aligned} x^\top H(x) &= \sum_{i \neq j} x_k H_i(x) + x_j H_j(x) \\ &> -N_1 + k(k-1) b_{min}^2, \end{aligned}$$

where $N_1 := nMb_{max}^2$. Finally, we get

$$\begin{aligned} \lim_{\|x\| \to \infty} \frac{x^\top H(x)}{\|x\|} &> \lim_{\|x\| \to \infty} -\frac{N_1}{\|x\|} + \frac{k(k-1)b_{min}^2}{\|x\|} \\ &= \lim_{k \to \infty} \frac{k-1}{d} b_{min}^2 \\ &= +\infty. \end{aligned}$$

□

Theorem 3.3 *Let ψ be a positively oriented median-bounded MCP function, and let $\tilde{\psi}$ be a smoother for ψ satisfying Assumption 2.13. Suppose $\mathbb{B}$ is bounded, choose $\mu > 0$, and let $H_\mu : \mathbb{R}^n \to \mathbb{R}^n$ be defined by*

$$(H_\mu)_i(x) := \psi_\mu(l_i, u_i, x_i, F_i(x)).$$

Then, H_μ satisfies condition (9) and therefore the conclusions of Theorem 2.11 hold.

Proof

$$\begin{aligned} x^\top H_\mu(x) &= x^\top H(x) + x^\top (H_\mu(x) - H(x)) \\ &\geq x^\top H(x) - \|x\| \, \|H_\mu(x) - H(x)\| \\ &\geq x^\top H(x) - \|x\| \, \xi(\mu). \end{aligned}$$

Dividing both sides by $\|x\|$ and taking the limit as $\|x\| \to \infty$, we have

$$\begin{aligned} \lim_{\|x\| \to \infty} \frac{x^\top H_\mu(x)}{\|x\|} &\geq \lim_{\|x\| \to \infty} \frac{x^\top H(x)}{\|x\|} - \xi(\mu) \\ &= +\infty \quad \text{by Lemma 3.2.} \end{aligned}$$

Thus, for r sufficiently large, we have $x^\top H(x) > 0$ whenever $\|x\| = r$, so (9) holds. □

It is not at all difficult to find MCP functions and corresponding smoothers which satisfy the assumptions of this theorem. With these in hand, the theorem gives a strong result: *if $\mathbb{B}$ is bounded, then the homotopy zero curve being tracked in Step 4 of Figure 2 leads to a solution $\bar{x}$ of $H_\mu(x) = 0$. Furthermore if $\nabla_x H_\mu(\bar{x})$ is nonsingular, then this zero curve is of finite length. Thus, the algorithm will not fail in step 4 as long as the curve tracking is peformed reliably.*

4 Implementation

We implemented the algorithm using the following choices: For an MCP function, we generalized the Fischer-Burmeister function according to (2). As a smoother for this function, we used (11). finally, to track the homotopy zero curves, we used the FIXPDF algorithm from HOMPACK.

4.1 Generalization of the Fischer-Burmeister Function to the MCP

To generalize the Fischer-Burmeister function to the MCP framework, we used the function $H : \mathbb{R}^n \to \mathbb{R}^n$ given by

$$H_i(x) := \psi(l_i, u_i, x_i, F_i(x)), \tag{13}$$

where obvious limits are used to define the function when either bound is infinite; thus, if $l_i = -\infty$, then $H_i(x) := -\phi(u_i - x_i, -F_i(x))$, if $u_i = \infty$, then $H_i(x) := \phi(x_i - l_i, F_i(x))$, and if $l_i = -\infty$ and $u_i = \infty$, then $H_i(x) := F_i(x)$. Observe that if $l_i = 0$ and $u_i = \infty$, then $H_i(x) = \min(x_i, F_i(x))$.

To use the generalized Newton method from Figure 1 to find a zero of H, we need to establish that H is semismooth. The following theorem was proved in [4].

Theorem 4.1 *If f is continuously differentiable on $\mathbb{R}^n$, then the following hold:*

1. *The function H defined by (13) is semismooth on $\mathbb{R}^n$.*
2. *If f is twice continuously differentiable with Lipschitz continuous Hessian, then H is strongly semismooth everywhere.*
3. *The natural merit function $\theta := \frac{1}{2}H^\top(\cdot)H(\cdot)$ is continuously differentiable, with gradient given by $\nabla\Psi(x) = V^\top H(x)$, where V is any element of $\partial H(x)$.*

Observe that Step 2 of the generalized Newton algorithm requires choosing an element of $\partial_B H(x^k)$ or $\partial_B H_\mu(x^k)$. We now address the question of how to calculate such an element. To do this, we shall need the following lemma, which generalizes [16, Proposition 3.1].

Lemma 4.2

$$\partial H_\mu(x) \subset \{D_a(x) + D_b(x)\nabla f(x)\}.$$

Here $D_a(x)$ and $D_b(x)$ are $n \times n$ diagonal matrices whose ith diagonal elements are given by

$$(D_a)_{ii}(x) := a_i(x) + b_i(x)c_i(x), \quad (D_b)_{ii}(x) := b_i(x)d_i(x),$$

where

$$\begin{aligned} a_i(x) &= 1 - \frac{x_i - l_i}{\sqrt{(x_i - l_i)^2 + \phi_\mu(u_i - x_i, -F_i(x))^2 + 2\mu}}, \\ b_i(x)) &= 1 + \frac{\phi(u_i - x_i, -F_i(x))}{\sqrt{(x_i - l_i)^2 + \phi_\mu(u_i - x_i, -F_i(x))^2 + 2\mu}}, \end{aligned} \tag{14}$$

if $(x_i - l_i, F_i(x), \mu) \neq (0,0,0)$, or

$$(a_i(x), b_i(x)) \in \left\{(1 - \xi, 1 - \rho) \in \mathbb{R}^2 \mid \|(\xi, \rho)\| \leq 1\right\} \tag{15}$$

if $(x_i - l_i, F_i(x), \mu) = (0,0,0)$; and

$$\begin{aligned} c_i(x) &= \frac{x_i - u_i}{\sqrt{(x_i - u_i)^2 + F_i(x)^2 + 2\mu}} + 1, \\ d_i(x)) &= \frac{F_i(x)}{\sqrt{(x_i - u_i)^2 + F_i(x)^2 + 2\mu}} + 1 \end{aligned} \tag{16}$$

if $(x_i - u_i, F_i(x), \mu) \neq (0,0,0)$, or

$$(c_i(x), d_i(x)) \in \left\{(\xi + 1, \rho + 1) \in \mathbb{R}^2 \mid \|(\xi, \rho)\| \leq 1\right\} \tag{17}$$

if $(x_i - u_i, F_i(x), \mu) = (0,0,0)$.

Note that in (14) and (16), if either l_i or u_i is infinite, then the obvious limits are used to define the fractions. Thus, if $l_i = -\infty$, then $(a_i(x), b_i(x)) = (0, 1)$, and if $u_i = \infty$, then $(c_i(x), d_i(x)) = (0, 1)$.

Proof For simplicity of notation, we drop the subscript μ from H_μ. By [14, Proposition 2.6.2(e)],

$$\partial H(x) \subset (\partial H_1(x) \times \cdots \times \partial H_n(x)).$$

Thus, it suffices to prove that for each i,

$$\partial H_i(x) \subset \{(a_i(x) + b_i(x)c_i(x))e^{i^\top} + b_i(x)d_i(x)\nabla F_i(x)\}, \tag{18}$$

where $a_i(x), b_i(x), c_i(x), d_i(x)$ satisfy (14)–(17).

To prove this result, let $g_i : \mathbb{R}^n \to \mathbb{R}$ be defined by $g_i(x) := -\phi(u_i - x_i, -F_i(x))$, and let $h_i : \mathbb{R}^n \to \mathbb{R}^2$ be defined by $h_i(x) := (x_i - l_i, g_i(x))$. We then have that $H_i(x) = \phi(h_i(x))$. Our first step is to show that $\partial H_i(x) = \partial\phi(h_i(x))\partial h_i(x)$.

We consider two cases. In the first case, suppose that $\mu \neq 0$ or $h_i(x) \neq (0, 0)$. It follows that ϕ is continuously differentiable at $h_i(x)$. Furthermore, since F is continuously differentiable, and ϕ is Lipschitz, h_i is locally-Lipschitz at x. Thus, by [14, Theorem 2.6.6], $\partial H_i(x) = \partial\phi(h_i(x))\partial h_i(x)$.

In the second case, suppose that $\mu = 0$ and $h_i(x) = (0, 0)$. It then follows that $u_i - x_i = u_i - l_i > 0$, so ϕ is continuously differentiable at $(u_i - x_i, -F_i(x))$, and therefore h_i is continuously differentiable at x. By the corollary to [14, Proposition 2.2.1], h_i is strictly differentiable at x. Furthermore, since ϕ is Lipschitz and convex [18], then by [14, Proposition 2.3.6(b)], ϕ is regular everywhere. Thus, by [14, Theorem 2.3.9(iii)], $\partial H_i(x) = \partial\phi(h_i(x))\partial h_i(x)$.

We now look at the terms $\partial\phi(h_i(x))$ and $\partial h_i(x)$. It is easily shown that

$$\partial\phi(a, b) = \begin{cases} \left\{\left(1 - \dfrac{a}{\sqrt{a^2 + b^2 + 2\mu}}, 1 - \dfrac{b}{\sqrt{a^2 + b^2 + 2\mu}}\right)\right\} & (a, b, \mu) \neq 0 \\ \{(1 - \xi, 1 - \rho) \mid \|\xi, \rho\| \leq 1\} & (a, b, \mu) = 0. \end{cases}$$

Also,

$$\partial h_i(x)^\top = \left\{(e^i, \sigma^i) \,\middle|\, \sigma^i \in \partial g_i(x)\right\},$$

where e^i is the ith column of the identity matrix. Thus,

$$\partial H_i(x) = \left\{a_i(x)e^{i^\top} + b_i(x)\sigma^i \,\middle|\, \sigma^i \in \partial g_i(x), a_i(x), b_i(x) \text{ satisfy (14) and (15)}\right\}.$$

By similar arguments, we get

$$\partial g_i(x) = \left\{-c_i(x)e^{i^\top} - d_i(x)\nabla F_i(x) \mid c_i(x), d_i(x) \text{ satisfy (16) and (17)}\right\}.$$

Combining these last two relations, we see that (18) is satisfied as an equality. □

Figure 3 describes a simple procedure for calculating an element of $\partial_B H_\mu(x)$.

Theorem 4.3 *The matrix V calculated by the procedure given in Figure 3 is an element of $\partial_B H(x)$.*

Proof If $\mu \neq 0$, then the result follows from Lemma 4.2. Therefore, assume $\mu = 0$. In similar fashion to the proof of [15, Theorem 7.1], we build a sequence of points $\{y^k\}$ where $H(x)$ is differentiable and such that $\nabla H(y^k)$ tends to V. The theorem then follows by the definition of the B-subdifferential.

Figure 3: Procedure to evaluate an element of $\partial_B H(x)$

Step 1 Set $\beta_l := \{i \mid x_i - l_i = 0 = F_i(x)\}$ and $\beta_u := \{i \mid u_i - x_i = 0 = F_i(x)\}$

Step 2 Choose $z \in \mathbb{R}^n$ such that $z_i \neq 0$ for all $i \in \beta_l \bigcup \beta_u$.

Step 3 For each i, if $i \notin \beta_u$, or $\mu \neq 0$, set

$$\begin{aligned} c_i(x) &:= \frac{x_i - u_i}{\sqrt{(x_i - u_i)^2 + F_i(x)^2 + 2\mu}} + 1 \\ d_i(x) &:= \frac{F_i(x)}{\sqrt{(x_i - u_i)^2 + F_i(x)^2 + 2\mu}} + 1; \end{aligned}$$

else if $\mu = 0$ and $i \in \beta_u$, set

$$\begin{aligned} c_i(x) &:= \frac{z_i}{\|(z_i, \nabla F_i(x) z)\|} + 1 \\ d_i(x) &:= \frac{\nabla F_i(x) z}{\|(z_i, \nabla F_i(x) z)\|} + 1. \end{aligned}$$

Step 4 For each i, if $i \notin \beta_l$ or $\mu \neq 0$, set

$$\begin{aligned} a_i(x) &:= 1 - \frac{x_i - l_i}{\sqrt{(x_i - l_i)^2 + \phi(u_i - x_i, -F_i(x))^2 + 2\mu}} \\ b_i(x) &:= 1 - \frac{\phi(u_i - x_i, -F_i(x))}{\sqrt{(x_i - l_i)^2 + \phi(u_i - x_i, -F_i(x))^2 + 2\mu}}; \end{aligned}$$

else if $\mu = 0$ and $i \in \beta_l$, set

$$\begin{aligned} a_i(x) &:= 1 - \frac{z_i}{\|(z_i, c_i(x) z_i + d_i(x) \nabla F_i(x) z)\|} \\ b_i(x) &:= 1 - \frac{c_i(x) z_i + d_i(x) \nabla F_i(x) z)}{\|(z_i, c_i(x) z_i + d_i(x) \nabla F_i(x) z)\|}. \end{aligned}$$

Step 5 For each i, set

$$V_i := (a_i(x) + b_i(x) c_i(x)) e^{i^\top} + b_i(x) d_i(x) \nabla F_i(x).$$

Let $y^k := x + \epsilon_k z$, where z is the vector of Step 2 of Figure 3 and $\{\epsilon_k\}$ is a sequence of positive numbers converging to 0. For $i \notin \beta_l \bigcup \beta_u$, either $x_i \neq l_i$ and $x_i \neq u_i$, or $F_i(x) \neq 0$; and for $i \in \beta_l \bigcup \beta_u$, $z_i \neq 0$. Thus, if ϵ_k is small enough, either $y_i^k \neq l_i$ and $y_i^k \neq u_i$, or $F_i(y^k) \neq 0$. In either case, H is differentiable at y^k.

We now show that for each i, $\lim_{k\to\infty} \nabla H_i(y^k) = V_i(x)$. If either l_i or u_i is infinite, the result is given by [15, Theorem 7.1] by a simple change of variables. Thus, without loss of generality, we assume that l_i and u_i are both finite.

By Lemma 4.2, $\nabla H_i(y^k)$ is given by

$$(a_i(y^k) + b_i(y^k)c_i(y^k))e^i + b_i(y^k)d_i(y^k)\nabla F_i(y^k)$$

where a_i, b_i, c_i, d_i are defined by (14) and (16).

We now consider three cases.

Case 1: $i \notin \beta_l \bigcup \beta_u$**:** In this case, by continuity, $\lim_{k\to\infty} \nabla H_i(y^k) = V_i$.

Case 2: $i \in \beta_u$**:** In this case, $x_i = u_i$, so $y_i^k - u_i = \epsilon_k z_i$, so

$$\begin{aligned} c_i(y^k) &= \frac{\epsilon_k z}{\|(\epsilon_k z, F_i(y^k))\|} + 1 \\ d_i(y^k) &= \frac{F_i(y^k)}{\|(\epsilon_k z, F_i(y^k))\|} + 1 \end{aligned} \tag{19}$$

Since f is continuously differentiable, we can use a Taylor series expansion to get

$$F_i(y^k) = F_i(x) + \epsilon_k \nabla F_i(\zeta^k) z \quad \text{with } \zeta^k \in [x, y^k].$$

Substituting this expression into (19), we see that

$$\begin{aligned} \lim_{k\to\infty} c_i(y^k) &= \frac{z_i}{\|(z_i, \nabla F_i(x) z)\|} + 1 \\ \lim_{k\to\infty} d_i(y^k) &= \frac{\nabla F_i(x) z}{\|(z_i, \nabla F_i(x) z)\|} + 1 \end{aligned}$$

Thus, $\lim_{k\to\infty} \nabla H_i(y^k) = V_i$.

Case 3: $i \in \beta_l$**:** In this case, $x_i = l_i$ and $F_i(x) = 0$. Clearly, $x_i \neq u_i$, so ϕ is continuously differentiable in a neighborhood of $(u_i - x_i, -F_i(x))$. Thus, using an argument similar to the above we get

$$\lim_{k\to\infty} a_i(y^k) = \frac{z_i}{\|(z_i, \nabla \phi(u_i - x_i, -F_i(x)) z)\|} - 1 \tag{20}$$

$$\lim_{k\to\infty} b_i(y^k) = \frac{\nabla \phi(u_i - x_i, -F_i(x)) z)}{\|(z_i, \nabla \phi(u_i - x_i, -F_i(x)) z)\|} - 1 \tag{21}$$

Finally, $\nabla \phi(u_i - x_i, -F_i(x)) = c_i(x)e^i + d_i(x)\nabla F_i(x)$, where $c_i(x)$ and $d_i(x)$ are given by (16). Substituting this expression into (20) and (21), we see that $\lim_{k\to\infty} \nabla H_i(y^k) = V_i$.

□

4.2 Tracking the Homotopy Zero Curve

The above discussion describes how to use the Fischer-Burmeister MCP function and associated Kanzow MCP smoother within our algorithmic framework. It remains to discuss how to track the homotopy zero curve of H_μ. To do this, we used the FIXPDF routine from HOMPACK. FIXPDF tracks the zero curve using an ordinary differential equation based algorithm. There are two user-defined parameters which govern how accurately the zero curve is tracked: *arctol* specifies the local error allowed the ODE solver when following the zero curve, and *eps* specifies the local error allowed the ODE solver when very near the solution. We used choices of $arctol$=10^{-4} and eps=10^{-6}. However, if the algorithm failed, we restarted with $arctol$=10^{-5}.

To reduce problems near the solution, we modified FIXPDF so that it would terminate whenever a point was discovered with a sufficiently improved merit function value. Thus, rather than following the zero curve all the way to the solution, we stopped as soon as a point $\hat{x}^k$ was generated with $\theta(\hat{x}^k) \leq \zeta\theta(\tilde{x}^k)$, where $\zeta \in (0,1)$. For our testing we chose $\zeta = 0.1$.

4.3 Scaling

One potential difficulty with the homotopy algorithm is that if the Jacobian matrix is poorly conditioned at the solution, it can be very difficult to track the zero curve. Indeed, if the Jacobian is singular at the solution, the zero curve may have infinite arc length. To address this difficulty, we incorporated the scaling method used by Chen and Mangasarian [11] into our algorithm. Prior to each call to the generalized Newton method, the algorithm examines the diagonal elements of $\nabla f(x^k)$. If $|\nabla f(x^k))_{ii}| > 100$, then f_i is scaled by a factor of $10/|\nabla f(x^k))_{ii}|$.

4.4 Computational Results

The above algorithmic framework was implemented in MATLAB using a MEX interface to call HOMPACK. We used parameter values $\sigma = .1$, $\alpha = 0.5$, $m_{max} = 30$, $\beta = .5$, and $\epsilon = 10^{-12}$. The algorithm was run on all of the problems with fewer than 110 variables in the MCPLIB and GAMSLIB problem libraries, using all starting points provided by the MATLAB interface [17]. Additionally, the algorithm was also run on the 125 variable vonthmcp problem, which is known to be particularly challenging. The results are given in Tables 1 and 2. For each problem, we list the size of the problem (that is, the number of variables), the starting point used from the MATLAB interface, the number of calls to the homotopy algorithm, the number of Jacobian evaluations required, and the final value of θ for the unscaled problem. Notice that because the stopping criteria was applied to the scaled problem, some of the final unscaled θ values are larger than 10^{-12}.

All of the test problems were successfully solved; however, in some cases very large numbers of Jacobian evaluations were required. In developing the method, we also tested the algorithm on all of the problems from MCPLIB without using the scaling method. The unscaled method solved all of the problems with the exception of pgvon106, ehl_k40, ehl_k60, ehl_k80, and ehl_kost.

5 Conclusions

The algorithm described in this paper represents a qualitatively different approach for solving complementarity problems. Because of its basis in probability-one homotopy algorithms, it has strong global convergence theory which suggests it may be successful on problems which are too difficult for other methods. The fact that the method was able to solve all of the test problems

Table 1: MCPLIB Test Problems

Problem Name	size	st. pt.	homotopy calls	Jacobian evaluations	θ final
bertsekas	15	1	2	787	7.94e-21
bertsekas	15	2	3	1174	3.86e-20
bertsekas	15	3	0	36	7.70e-19
billups	1	1	1	139	9.43e-23
choi	13	1	0	5	1.31e-13
colvdual	20	1	0	16	5.94e-14
colvdual	20	2	1	449	3.54e-20
colvnlp	15	1	0	16	4.03e-13
colvnlp	15	2	0	15	2.24e-23
cycle	1	1	0	3	1.80e-15
ehl_k40	41	1	1	487	1.08e-17
ehl_k60	61	1	0	17	1.13e-15
ehl_k80	81	1	0	15	1.23e-18
ehl_kost	101	1	0	14	3.43e-19
ehl_kost	101	2	0	14	3.43e-19
ehl_kost	101	3	0	14	3.43e-19
explcp	16	1	0	24	4.55e-20
freebert	15	1	1	719	1.04e-16
freebert	15	2	7	8868	1.60e-13
freebert	15	3	1	729	2.01e-17
freebert	15	4	1	734	2.62e-21
freebert	15	5	7	8801	1.09e-18
freebert	15	6	1	681	2.04e-17
gafni	5	1	0	12	8.47e-13
gafni	5	2	0	8	1.33e-13
gafni	5	3	0	7	5.81e-15
hanskoop	14	1	1	107	9.92e-16
hanskoop	14	2	1	105	1.13e-21
hanskoop	14	3	1	99	1.39e-17
hanskoop	14	4	1	96	5.22e-14
hanskoop	14	5	1	184	1.57e-14
hydroc06	29	1	0	5	8.42e-24
hydroc20	99	1	0	10	1.79e-21
jel	6	1	0	8	2.06e-13
josephy	4	1	0	7	1.31e-23
josephy	4	2	0	7	5.12e-23
josephy	4	3	0	14	1.10e-16
josephy	4	4	0	4	5.87e-16
josephy	4	5	0	4	1.00e-13
josephy	4	6	0	7	4.39e-15
kojshin	4	1	0	10	1.32e-15
kojshin	4	2	0	7	1.80e-15

Table 1: MCPLIB Test Problems

Problem Name	size	st. pt.	homotopy calls	Jacobian evaluations	θ final
kojshin	4	3	0	32	1.47e-22
kojshin	4	4	0	4	5.98e-15
kojshin	4	5	0	5	1.70e-14
kojshin	4	6	0	5	5.26e-13
mathinum	3	1	0	4	4.33e-17
mathinum	3	2	0	5	8.96e-13
mathinum	3	3	0	11	7.65e-14
mathinum	3	4	0	7	1.09e-16
mathisum	4	1	0	5	6.92e-19
mathisum	4	2	0	7	7.56e-24
mathisum	4	3	0	8	8.30e-23
mathisum	4	4	0	6	2.70e-21
methan08	31	1	0	4	6.94e-23
nash	10	1	0	8	3.34e-21
nash	10	2	0	10	4.06e-15
pgvon105	105	1	1	961	2.85e-20
pgvon105	105	2	4	3028	8.18e-19
pgvon105	105	3	1	1163	4.50e-14
pgvon106	106	1	2	4522	4.33e-11
pies	42	1	2	833	6.62e-14
powell	16	1	0	8	4.31e-15
powell	16	2	0	11	3.41e-14
powell	16	3	1	169	1.31e-20
powell	16	4	0	11	2.58e-17
powell_mcp	8	1	0	6	2.09e-23
powell_mcp	8	2	0	6	8.91e-13
powell_mcp	8	3	0	8	1.10e-15
powell_mcp	8	4	0	7	8.57e-15
scarfanum	13	1	0	13	1.70e-16
scarfanum	13	2	0	13	1.68e-16
scarfanum	13	3	0	11	7.26e-17
scarfasum	14	1	0	8	1.78e-16
scarfasum	14	2	0	10	1.78e-16
scarfasum	14	3	0	11	1.74e-16
scarfbnum	39	1	1	487	3.32e-22
scarfbnum	39	2	1	115	8.94e-20
scarfbsum	40	1	0	17	1.43e-14
scarfbsum	40	2	3	652	6.94e-21
sppe	27	1	0	8	9.35e-15
sppe	27	2	0	6	5.66e-21
tobin	42	1	0	10	1.63e-18
tobin	42	2	0	8	4.15e-14

Table 2: GAMSLIB Test Problems

Problem Name	size	st. pt.	homotopy calls	Jacobian evaluations	θ final
cafemge	101	1	0	10	1.34e-21
cirimge	9	1	0	5	5.46e-11
hansmcp	43	1	0	14	2.23e-17
hansmge	43	1	0	16	6.86e-24
harkmcp	32	1	1	340	3.02e-13
harmge	11	1	2	418	1.13e-17
kehomge	9	1	0	10	3.05e-16
kormcp	78	1	0	3	4.86e-25
oligomcp	6	1	0	7	3.17e-17
sammge	23	1	0	0	0.00e+00
scarfmcp	18	1	0	9	2.98e-20
scarfmge	18	1	0	12	2.28e-19
shovmge	51	1	0	1	1.01e-14
threemge	9	1	0	0	0.00e+00
transmcp	11	1	3	1352	6.26e-14
two3mcp	6	1	0	8	2.06e-13
unstmge	5	1	0	9	3.23e-21
vonthmge	80	1	10	12838	2.69e-11
vonthmcp	125	1	1	1135	1.14e-25
wallmcp	6	1	0	2	1.73e-20

supports this claim. However, the method, at present, is very slow. On a number of test problems, the algorithm required the evaluation of an extremely large number of Jacobian evaluations. When compared to the performance of other recent algorithms [6, 22] on this test library, the homotopy method is not competitive in terms of computer time. Nevertheless, because of its potential to solve more difficult problems, the homotopy method may, in many situations, be more efficient in real time, since it may require less human intervention to produce a solution.

References

[1] J. C. Alexander. The topological theory of an embedding method. In H. Wacker, editor, *Continuation Methods*, pages 37–68. Academic, New York, 1978.

[2] J. C. Alexander, R. B. Kellogg, T.-Y. Li, and J. A. Yorke. Piecewise smooth continuation. manuscript, 1979.

[3] J. C. Alexander, T.-Y. Li, and J.A. Yorke. Piecewise smooth homotopies. In B. C. Eaves, F. J. Gould, H.-O. Peitgen, and M. J. Todd, editors, *Homotopy Methods and Global Convergence*, pages 1–14. Plenum, New York, 1983.

[4] S. C. Billups. *Algorithms for Complementarity Problems and Generalized Equations.* PhD thesis, University of Wisconsin–Madison, Madison, Wisconsin, August 1995.

[5] S. C. Billups. Improving the robustness of complementarity solvers using proximal perturbations. UCD/CCM Report No. 96, Department of Mathematics, University of Colorado at Denver, Denver, Colorado, March 1996.

[6] S. C. Billups, S. P. Dirkse, and M. C. Ferris. A comparison of large scale mixed complementarity problem solvers. *Computational Optimization and Applications*, 7:3–25, 1997.

[7] S. C. Billups and M. C. Ferris. QPCOMP: A quadratic program based solver for mixed complementarity problems. *Mathematical Programming*, 76:533–562, 1997.

[8] B. Chen and X. Chen. A global linear and local quadratic continuation smoothing method for variational inequalities with box constraints. AMR 96/39, School of Mathematics, The University of New South Wales, Sydney, Australia, 1997.

[9] B. Chen and P. T. Harker. A non-interior-point continuation method for linear complementarity problems. *SIAM Journal on Matrix Analysis and Applications*, 14:1168–1190, 1993.

[10] Chunhui Chen and O. L. Mangasarian. Smoothing methods for convex inequalities and linear complementarity problems. *Mathematical Programming*, 78:51–70, 1995.

[11] Chunhui Chen and O. L. Mangasarian. A class of smoothing functions for nonlinear and mixed complementarity problems. *Computational Optimization and Applications*, 5:97–138, 1996.

[12] X. Chen and L. Qi. A parameterized Newton method and a Broyden-like method for solving nonsmooth equations. *Computational Optimization and Applications*, 5:97–138, 1994.

[13] S.-N. Chow, J. Mallet-Paret, and J. A. Yorke. Finding zeros of maps: homotopy methods that are constructive with probability one. *Mathematics of Computation*, 32:887–899, 1978.

[14] F. H. Clarke. *Optimization and Nonsmooth Analysis.* John Wiley & Sons, New York, 1983.

[15] T. De Luca, F. Facchinei, and C. Kanzow. A semismooth equation approach to the solution of nonlinear complementarity problems. *Mathematical Programming*, 75:407–439, 1996.

[16] F. Facchinei and J. Soares. A new merit function for nonlinear complementarity problems and a related algorithm. *SIAM Journal on Optimization*, 7:225–247, 1997.

[17] M. C. Ferris and T. F. Rutherford. Accessing realistic complementarity problems within Matlab. In G. Di Pillo and F. Giannessi, editors, *Nonlinear Optimization and Applications*, pages 141–153. Plenum Press, New York, 1996.

[18] A. Fischer. A special Newton–type optimization method. *Optimization*, 24:269–284, 1992.

[19] A. Fischer. An NCP–function and its use for the solution of complementarity problems. In D. Z. Du, L. Qi, and R. S. Womersley, editors, *Recent Advances in Nonsmooth Optimization*, pages 88–105. World Scientific Publishers, 1995.

[20] S. A. Gabriel and J. J. Moré. Smoothing of mixed complementarity problems. In M. C. Ferris and J. S. Pang, editors, *Complementarity and Variational Problems: State of the Art*, Philadelphia, Pennsylvania, 1997. SIAM Publications.

[21] C. Kanzow. Some noninterior continuation methods for linear complementarity problems. *SIAM Journal on Matrix Analysis and Applications*, 17:851–868, 1996.

[22] Christian Kanzow and Heiko Pieper. Jacobian smoothing methods for general nonlinear complementarity problems. Mathematical Programming Technical Report 97–8, Computer Sciences Department, University of Wisconsin, Madison, Wisconsin, 1997.

[23] L. Qi. Convergence analysis of some algorithms for solving nonsmooth equations. *Mathematics of Operations Research*, 18:227–244, 1993.

[24] L. Qi. Regular pseudo-smooth NCP and BVIP functions and globally and quadratically convergent generalized Newton methods for complementarity and variational inequality problems. AMR 97/14, School of Mathematics, The University of New South Wales, Sydney, Australia, July 1997.

[25] L. Qi and X. Chen. A globally convergent successive approximation method for nonsmooth equations. Technical report, School of Mathematics, University of New South Wales, Sydney, 1993.

[26] H. Sellami. *A Continuation Method for Normal Maps.* PhD thesis, University of Wisconsin – Madison, Madison, Wisconsin, 1994.

[27] H. Sellami and S. M. Robinson. Homotopies based on nonsmooth equations for solving nonlinear variational inequalities. In G. Di Pillo and F. Giannessi, editors, *Nonlinear Optimization and Applications.* Plenum Press, New York, 1996.

[28] H. Sellami and S. M. Robinson. Implementation of a continutation method for normal maps. *Mathematical Programming*, pages 563–578, 1997.

[29] S. Smale. Algorithms for solving equations. In *Proceedings of the International Congress of Mathematicians*, pages 172–195, 1986.

[30] L. T. Watson. An algorithm that is globally convergent with probability one for a class of nonlinear two-point boundary value problems. *SIAM Journal on Numerical Analysis*, 16:394–401, 1979.

[31] L. T. Watson. A globally convergent algorithm for computing fixed points of C^2 maps. *Applied Mathematics and Computation*, 5:297–311, 1979.

[32] L. T. Watson. Solving the nonlinear complementarity problem by a homotopy method. *SIAM Journal on Control and Optimization*, 17:36–46, 1979.

[33] L. T. Watson, S. C. Billups, and A. P. Morgan. Algorithm 652: HOMPACK: A suite of codes for globally convergent homotopy algorithms. *ACM Transactions on Mathematical Software*, 13:281–310, 1987.

[34] I. Zang. A smoothing-out technique for min-max optimization. *Mathematical Programming*, 19:61–71, 1980.