Collineations and Characterizations of Generalized Quadrangles of Order $(q+1, q-1)$

S. E. Payne and M. A. Miller

March 1998

Abstract

For each q-arc Ω^- of $PG(2,q)$, $q = 2^e$, there is a well-known construction of a GQ $\mathcal{S}$ with parameters $(q+1, q-1)$. We give a new characterization of $\mathcal{S}$ that makes it easy to see that there is only one case in which the collineation group of $\mathcal{S}$ contains elements not induced by collineations of the ambient $PG(3,q)$. This is the case in which the hyperoval containing Ω^- is regular (i.e., contains a conic) and the nucleus of the conic is not in Ω^-. The situation for the known GQ naturally arising with parameters $(q-1, q+1)$ is also surveyed.

1 Introduction and Review

We assume that the reader is familiar with the concept of generalized quadrangle (GQ) of order (s,t), the term *regular pair* (of points or of lines) and other basic terms of the theory of GQ. These can all be found in the monograph [PT84]. However, we do review certain constructions that will be central to our work here.

Every known GQ with parameters (s,t) where $s = t \pm 2$ is obtained from one of the known GQ of order q by expanding about a regular point or regular line (see the next subsection). We study a conceptually more general class of examples with two main goals in mind. The first goal is to contribute to the eventual discovery of a new class of GQ by characterizing the known GQ among those with certain similar properties. Until now all attempts to construct new examples have failed, so we look for ever stronger characterization theorems. In this context we remark that G. Mason [Ma97] suggests a possible new construction of $GQ(2^e, 2^e)$ with $e = 2k+1 \geq 3$ that would lead to new $GQ(2^e - 1, 2^e + 1)$.

The second goal is to provide a reference that gives a satisfactory description of the collineation group of each of the known GQ with $s = t \pm 2$. The basic knowledge has been available for many years. For example, the article by Bichara, Mazzocca and Somma ([BMS80]), as well as the two articles by De Soete and Thas ([DST84] and [DST86b]), easily yield the information that the collineation group of the GQ derived from a hyperoval and denoted S^+ below is completely determined by the stabilizer in $P\Gamma L(4,q)$, $q = 2^e$, of the associated hyperoval. The case for the known GQ of order $(q+1, q-1)$ derived from a q-arc in $PG(2,q)$, $q = 2^e$, is a little more subtle. The question of when two such GQ are isomorphic is essentially given in [Pa85], but the related (but different) question of what is the full collineation group is not really addressed. We hope to provide a convenient reference for at least one description of the full collineation group. Up to duality, only one class of GQ is known with $s = t \pm 2$ and s even. For $s < t$ these examples have $(s,t) = (q-1, q+1)$ with q an odd prime power. We revisit this example (first constructed by Ahrens and Szekeres [AS69]). Along the way we correct some inaccuracies that appear in the interesting paper [GJS94] by Grundhöffer, Joswig and Stroppel.

1.1 Expansion about a regular point

Let $\mathcal{S} = (P, B, I)$ be a generalized quadrangle of order q with pointset P, lineset B and incidence relation I. Suppose that the point x is regular, i.e., $|\{x, y\}^{\perp\perp}| = 1 + q$ for all $y \in P \setminus x^{\perp}$. It follows that a GQ $\mathcal{S}_x^+$ of order $(q-1, q+1)$ can be constructed as follows (see [PT84] or [Pa71a]): Put $P_x^+ = P \setminus x^{\perp}$, $B_x^1 = \{L \in B : x \not{I} L\}$, $B_x^2 = \{x, y\}^{\perp\perp} : y \in P \setminus x^{\perp}\}$, $B_x^+ = B_x^1 \cup B_x^2$. If I^+ is the natural incidence on $P_x^+ \times B_x^+$ induced by I and containment, then $\mathcal{S}_x^+ = (P_x^+, B_x^+, I_x^+)$ is a GQ with parameters $(q-1, q+1)$. ($\mathcal{S}_x^+$ is sometimes denoted by $P(\mathcal{S}, x)$.)

The reverse process of recovering $\mathcal{S} = (P, B, I)$ from $\mathcal{S}_x^+ = (P_x^+, B_x^+, I_x^+)$ was given in [Pa72b] and reviewed in detail in [Pa90]. In this article, however, we want to consider this reverse process in detail in the point-line dual setting, as described in [Pa85]. Hence we review that situation here.

1.2 Expansion about a regular line

Let L be a regular line of the GQ $\mathcal{S} = (P, B, I)$ of order q (i.e., $|\{L, M\}^{\perp\perp}| = 1+q$ for all $M \in P \setminus L^{\perp}$). Then a GQ $\mathcal{S}_L = (P_L, B_L, I_L)$ of order $(q+1, q-1)$ may be constructed as follows: $B_L = \{M \in B : M \not\sim L\} = B \setminus L^{\perp}$. Also, put $P_L = P_L^1 \cup P_L^2$, where $P_L^1 = \{x \in P : x \text{ not on } L\}$, $P_L^2 = \{L, M\}^{\perp\perp} \setminus \{L\} : M \in B_L\}$. If $M \in B_L$ and $x \in P_L^1$, then $x I_L M$ provided $x I M$. If $M \in B_L$ and $x \in P_L^2$, then $x I_L M$ if and only if $M \in x$. Here $\mathcal{S}_L$ is said to be obtained by "expanding $\mathcal{S}$ about L," and it is sometimes denoted by $P(\mathcal{S}, L)$.

Continuing with the notation just established, let the points incident in $\mathcal{S}$ with the regular line L be labeled $x_0, x_1, \ldots, x_q$, and put $O_i = \{x \in x_i^{\perp} : x \text{ not on } L\}$, $i = 0, 1, \ldots, q$. Then each O_i is an *ovoid* of $\mathcal{S}_L$, i.e., a set of q^2 pair-wise noncollinear points of $\mathcal{S}_L$, a set of points such that each line of $\mathcal{S}_L$ is incident with a unique point of O_i. Moreover, $P_L^2 = O_\infty$ is also an ovoid of $\mathcal{S}_L$, and $\mathcal{O} = \{O_\infty, O_0, \ldots, O_q\}$ is a *fan*, i.e., a set of ovoids of $\mathcal{S}_L$ that partition the points of P_L. The ovoid O_∞ is said to be *pivotal* for $\mathcal{O}$ because of the following special property it satisfies: for $x, y \in O_\infty$ with $x \neq y$, the pair (x, y) is regular in $\mathcal{S}_L$ and its perp in $\mathcal{S}_L$ belongs entirely to some other member O_j of $\mathcal{O}$. It follows that the span (or "perp perp") of $\{x, y\}$ in $\mathcal{S}_L$ is contained in O_∞. Now $\mathcal{S}$ can be reconstructed (up to isomorphism) from $\mathcal{S}_L$ via a process we call *constricting* $\mathcal{S}_L$ *about the ovoid* O_∞.

1.3 Constriction about a regular ovoid

([Pa71b]; [Pa72b]) Let $\mathcal{S} = (P, B, I)$ be a GQ of order $(q+1, q-1)$ whose pointset P is partitioned into ovoids $P = O_\infty \cup O_0 \cup O_1 \cup \cdots \cup O_q$ so that O_∞ is pivotal for $\mathcal{O} = \{O_\infty, O_0, \ldots, O_q\}$. A GQ $\mathcal{S}_\infty = (P_\infty, B_\infty, I_\infty)$ of order (q, q) is constructed as follows: $P_\infty = (P \backslash O_\infty) \cup \{(O_0), (O_1), \ldots, (O_q)\}$; $B_\infty = B \cup \{\{x, y\}^\perp : x, y \in O_\infty,\ x \neq y\} \cup \{L_\infty\}$. Then I_∞ is defined in a natural way: $L_\infty I_\infty (O_i)$, $i = 0, 1, \ldots, q$. If $M = \{x, y\}^\perp$, $x, y \in O_\infty$, $x \neq y$, and if $z \in P \backslash O_\infty$, then $z I_\infty M$ provided $z \in M$. If $z = (O_j)$, then $z I_\infty M$ provided $M \subseteq O_j$. If $M \in B$ and $x \in P_\infty$, then $x I_\infty M$ provided $x I M$. It is easy to check that L_∞ is regular in $\mathcal{S}_\infty$. Moreover, the point (O_i) of $\mathcal{S}_\infty$, $0 \leq i \leq q$, is regular in $\mathcal{S}_\infty$ iff O_i is pivotal for $\mathcal{O}$.

1.4 Slanted symplectic GQ

In all known examples of a GQ of order q, q is a prime power. Moreover, if q is an odd prime power, there is only one example known of a GQ $\mathcal{S}$ with a regular point. In this case $\mathcal{S}$ is the GQ often denoted $W(q)$ arising from a symplectic polarity of $PG(3, q)$, and this example has all points regular. In the finite case, these GQ with q odd were actually constructed using a different method by R. W. Ahrens and G. Szekeres [AS69] in 1969. For $q = 2^h$, examples of GQ with parameters $(q-1, q+1)$ were constructed by both M. Hall, jr. [Ha71] and by Ahrens and Szekeres in [AS69], and these examples were then generalized by S. E. Payne in [Pa71a]. In the next section these constructions with $q = 2^h$ will be reviewed in greater detail. Moreover, the remaining sections of the paper provide a kind of revision of [Pa85] that both provides a few new results and makes it easy to determine what is the collineation group of the known examples of GQ with parameters $(q+1, q-1)$, $q = 2^h$.

For the moment we concentrate on the example $P(W(q), x)$. In fact, this example exists over each field F (finite or infinite), and the above construction works all the time, as noted by Grundhöffer, Joswig and Stroppel in [GJS94]. But of course the collineation group of $W(F)$ is transitive on points, so $P(W(F), x)$ is independent of the point x (up to isomorphism). The article [GJS94] gives an interesting and rather thorough treatment of these GQ and isomorphisms between them. Unfortunately, its explicit description (p. 145 of [GJS94]) of their collineation groups is incorrect. Hence we give a brief discussion here. First we note that the cases for $|F| < 5$ are truly exceptional. The unique $GQ(3,5)$ is studied thoroughly in [Pa90]. For example, the GQ $P(W(4),x)$ is flag-homogeneous but not a Moufang GQ. The case with $|F| = 3$ has been studied many times (as the point-line dual of the twenty seven lines on a cubic surface), and is treated once more in [GJS94].

To make comparison with [GJS94] easy, let F be any field with at least five elements and let the symplectic polarity ρ of $PG(3,F)$ be given by the alternating form

$$f(u,v) = u \begin{pmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{pmatrix} v^T.$$

Let x be the point with coordinates $(1,0,0,0)$. The group Γ of [GJS94] is then the stabilizer of the point x in the projective symplectic group preserving the alternating form, i.e., $\Gamma = (PSp_4F)_x$. Then Γ is induced by the group of linear maps $u \mapsto uB$, for all matrices B of the form

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ y & a_1 & a_2 & 0 \\ z & a_3 & a_4 & 0 \\ b & c & d & |A| \end{pmatrix}.$$

In this description, $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$ is any matrix in $GL(2,F)$, $|A|$ is its determinant, and

$$\begin{pmatrix} y \\ z \end{pmatrix} = |A|^{-1} A \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix}.$$

Here b, c, d are arbitrary elements of F.

It is now easy to see that the stabilizer in Γ of the point p with coordinates $(0,0,0,1)$ has the pleasant description as the group induced by all matrices of the form

$$B = \begin{pmatrix} 1 & & \\ & A & \\ & & |A| \end{pmatrix}.$$

(There are two things wrong with the description of Γ given in [GJS94]. One of them is that their formula for $w = (y, z)^T$ is missing a factor x in their notation. The other one has to do with whether or not nonsquares belong to the field F. If λ is a nonsquare, and if $A = \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}$, and if we put $b = c = d = 0$, so w is a column of two zeros, then

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix}$$

is a matrix of Γ fixing the point p, but it does not arise in their description of Γ_p.)

It is also clear that all the field automorphisms commute with the polarity ρ and fix both points x and p. Hence the group of automorphisms of $P(W(F), x)$ fixing the point p (as would have been shown in [GJS94] if Γ had been determined correctly) is $\Gamma_p \rtimes Aut(F) \cong GL(2, F) \rtimes Aut(F)$. The full group stabilizing $P(W(F), x)$ is transitive on the points of $P(W(q), x)$, so it is the above group times an elementary abelian factor of order q^3, and it has two orbits on the lines of $P(W(F), x)$.

2 Generalized Quadrangles Associated with Arcs and Ovals in $PG(2,q)$

¿From now on we assume that q is a power of 2. Let π be a desarguesian projective plane of order q embedded as a hyperplane in projective space $G = PG(3,q)$, $q = 2^e$, $e \geq 1$. It is well known that a q-arc (respectively, $(q+1) - arc$) of π extends (uniquely if $q \geq 4$) to a hyperoval (i.e., a $(q+2)$-arc) of π (cf. [Th87]). So we may start with an arbitrary hyperoval Ω^+ (and delete two points if we want a q-arc).

2.1 GQ from hyperovals

(R. W. Ahrens and G. Szekeres [AS69]; M. Hall, Jr. [Ha71]) Let P^+ be the set of points of $G \setminus \pi$, and B^+ the set of lines of G not contained in π but meeting π in a (unique!) point of Ω^+. If I^+ is the incidence relation naturally induced by that of G, then $\mathcal{S}^+ = \mathcal{S}(\Omega^+) = (P^+, B^+, I^+)$ is a GQ with parameters $(s,t) = (q-1, q+1)$.

2.2 GQ from ovals

(J. Tits [De68], p. 304; slightly modified for the case q a power of 2) Let $x \in \Omega^+$, so that $\Omega_x = \Omega^+ \setminus \{x\}$ is an oval with nucleus x. Let $\mathcal{O}_x$ be the set of planes of G that contain the point x, and put $P_x = P^+ \cup \mathcal{O}_x$. Let B_x be the set of lines of G that are tangent to Ω_x. Incidence I_x is the natural one: a point of P_x and a line of B_x are incident with respect to I_x provided they are incident as objects of G. Then $\mathcal{S}_x = (P_x, B_x, I_x)$ is a GQ with parameters (q,q). For example, as a point of $\mathcal{S}_x$, π is incident in $\mathcal{S}_x$ with those lines of π incident in G with the point x.

2.3 GQ from q-arcs

(S. E. Payne [Pa71a] and [Pa85]) Let x and y be any two points of Ω^+, so that $\Omega^-_{xy} = \Omega \setminus \{x,y\}$ is a q-arc of π. Let $\mathcal{O}_{xy}$ (respectively, $\mathcal{O}_{yx}$) be the set of planes of G containing x but not y (respectively, containing y but not x). Put $P^-_{xy} = P^+ \cup \mathcal{O}_{xy} \cup \mathcal{O}_{yx}$. Let B^-_{xy} be the set of lines of G not in π but which meet π in a (unique) point of Ω^-_{xy}. Incidence I^-_{xy} is the natural one: a point of P^-_{xy} and a line of B^-_{xy} are incident with respect to I^-_{xy} if and only if they are incident as objects of G. Then $\mathcal{S}^-_{xy} = (P^-_{xy}, B^-_{xy}, I^-_{xy})$ is a GQ with parameters $(s,t) = (q+1, q-1)$.

2.4 Expansion and Constriction in the GQ associated with ovals

Let Ω^+ be an arbitrary hyperoval of of π, and let x be any point of Ω^+. Then π, as a point of $\mathcal{S}_x$, is both regular and coregular (i.e., each line through π is regular). Expanding $\mathcal{S}_x$ about π yields exactly the GQ $\mathcal{S}^+$. Since each line through π in $\mathcal{S}_x$ is regular, each member of the packing of spreads obtained in $\mathcal{S}^+$ is pivotal. In fact, each spread is the set of lines of G not in π but meeting Ω^+ in a given point. Moreover, a pair of nonconcurrent lines of $\mathcal{S}^+$ is regular iff the two lines belong to one of these spreads.

On the other hand, let y be a second point of Ω^+. Expanding $\mathcal{S}_x$ about the line $L = xy$ yields exactly the GQ $\mathcal{S}^-_{xy}$. Let $\pi_1, \ldots, \pi_q$ be the planes of G meeting π at the line $L = xy$, and let $\mathcal{O}_i$ be the set of points of $G \setminus \pi$ contained in π_i, $1 \leq i \leq q$. The set $\mathcal{O}_{yx}$ of "new" points of $\mathcal{S}^-_{xy}$ an ovoid, and the set $\mathcal{O}_{xy}$ is also an ovoid. By construction, $\mathcal{O}_{yx}$ is pivotal for the fan $\mathcal{O} = \{\mathcal{O}_{yx}, \mathcal{O}_{xy}, \mathcal{O}_1, \ldots, \mathcal{O}_q\}$. But also, since π is a regular point in $\mathcal{S}_x$, the set $\mathcal{O}_{xy}$ of points of $\mathcal{S}^-_{xy}$ that were points of $\mathcal{S}_x$ collinear with π is pivotal for $\mathcal{O}$. If even one of the ovoids $\mathcal{O}_i$ is pivotal for the fan $\mathcal{O}$, then the corresponding point of the line L_∞ in $\mathcal{S}_x$ is regular in $\mathcal{S}_x$. But it then follows that each point of L_∞ is regular in $\mathcal{S}_x$ and the oval Ω_x must be a translation oval. Moreover, the self-dual GQ of the form $\mathcal{S}_x$ are precisely those for which Ω_x is a translation oval. We shall review these later in greater detail. Constricting about $\mathcal{O}_{yx}$ gives a GQ isomorphic to $\mathcal{S}_x$. Constricting about $\mathcal{O}_{xy}$ gives a GQ isomorphic to $\mathcal{S}_y$.

3 Regular Ovoids in $GQ(q+1, q-1)$

An ovoid $\mathcal{O}$ of the GQ $\mathcal{S}$ of order $(q+1, q-1)$ will be called **regular** provided that for distinct points $x, y \in \mathcal{O}$ the pair $\{x, y\}$ is regular and $\{x, y\}^{\perp\perp} \subseteq \mathcal{O}$. (Note that this is what was called a "normal" ovoid in [Pa72b].)

Let $\mathcal{O}_\infty$ be a regular ovoid of the GQ $\mathcal{S}$ of order $(q+1, q-1)$, $q = 2^e \geq 4$. This means that $\mathcal{O}_\infty$ is a set of q^2 pairwise noncollinear points of $\mathcal{S}$ such that for distinct points $x, y \in \mathcal{O}_\infty$ the pair $\{x, y\}$ is regular and $\{x, y\}^{\perp\perp} \subseteq \mathcal{O}_\infty$.

3.1 The affine plane $\pi(\mathcal{O}_\infty)$

It follows immediately that there is an affine plane $\pi(\mathcal{O}_\infty)$ of order q whose **points** are the points of $\mathcal{O}_\infty$, whose **lines** are the hyperbolic lines $\{x, y\}^{\perp\perp}$, $x \neq y$, $x, y \in \mathcal{O}_\infty$, and whose incidence is containment. This implies that each hyperbolic line is in a "parallel class" of pairwise disjoint hyperbolic lines that partition $\mathcal{O}_\infty$.

3.2 A fan of ovoids

Let $T_1, \ldots, T_q$ be the pairwise disjoint hyperbolic lines in one parallel class of $\pi(\mathcal{O}_\infty)$. So $\mathcal{O}_\infty = T_1 \cup T_2 \cup \cdots \cup T_q$. Then $T_1^\perp \cup \cdots \cup T_q^\perp$ is an ovoid of $\mathcal{S}$. Moreover, if for each j, $1 \leq j \leq q$, T_j' is arbitrarily chosen to be T_j or T_j', then $T_1' \cup \cdots \cup T_q'$ is an ovoid of $\mathcal{S}$. Let $E_0, E_1, \ldots, E_q$ be the $q+1$ parallel classes of lines of $\pi(\mathcal{O}_\infty)$. Then for $0 \leq i \leq q$, $\mathcal{O}_i = \cup\{T^\perp : T \in E_i\}$ is an ovoid. And $\mathcal{M} = \{\mathcal{O}_\infty, \mathcal{O}_0, \ldots, \mathcal{O}_q\}$ is a **fan** of ovoids, (i.e., a partition of the pointset of $\mathcal{S}$ by ovoids), for which $\mathcal{O}_\infty$ is **pivotal**, i.e., $\mathcal{O}_\infty$ is regular, and for $x, y \in \mathcal{O}_\infty$, $x \neq y$, $\{x, y\}^\perp \subseteq \mathcal{O}_j$ for some $j \in I$. By construction each $\mathcal{O}_i$ is partitioned into hyperbolic lines of size q whose perps partition $\mathcal{O}_\infty$, $0 \leq i \leq q$.

3.3 Intersection of perps with ovoids of the fan

Put $I = \{0, 1, \ldots, q\}$; $\tilde{I} = I \cup \{\infty\}$. The proofs of the following are easily adapted from the proofs given in Section IV of [Pa85].

(i) Let $j \in I$; $b, d \in \mathcal{O}_j$, $b \neq d$. If $\{b,d\}^\perp \cap \mathcal{O}_\infty \neq \emptyset$, then $\{b,d\}^\perp \subseteq \mathcal{O}_\infty$. And the perps of the hyperbolic lines in the parallel class of $\{b,d\}^\perp$ partition $\mathcal{O}_j$.

(ii) If $\mathcal{O}_\infty$, $\mathcal{O}_i$, $\mathcal{O}_j$ are distinct, $a \in \mathcal{O}_\infty$, $b \in \mathcal{O}_i, a \not\sim b$, then $|\{a,b\}^\perp \cap \mathcal{O}_j| = 1$.

(iii) If $\mathcal{O}_\infty$, $\mathcal{O}_i$, $\mathcal{O}_j$ are distinct, $x \in \mathcal{O}_i$, $y \in \mathcal{O}_j, x \not\sim y$, then $|\{x,y\}^\perp \cap \mathcal{O}_\infty| = 1$.

3.4 Constriction about $\mathcal{O}_\infty$

As $\mathcal{O}_\infty$ is pivotal for the fan $\mathcal{M}$, we may constrict $\mathcal{S} = (P, B, I)$ about $\mathcal{O}_\infty$ to obtain a GQ $\mathcal{S}_\infty = (P_\infty, B_\infty, I_\infty)$ using the method described earlier.

- **Points** of $\mathcal{S}_\infty$: $P_\infty = (P \setminus \mathcal{O}_\infty) \ \cup \ \{(\mathcal{O}_0), (\mathcal{O}_1), \ldots, (\mathcal{O}_q)\}$.
- **Lines** of $\mathcal{S}_\infty$: $B_\infty = B \cup \{T^\perp : T$ is a hyperbolic line of $\mathcal{O}_\infty\} \cup \{L_\infty\}$.
- **Incidence** of $\mathcal{S}_\infty$: A line of B is incident with the same points in P_∞ as it was in $\mathcal{S}$, except for the point of $\mathcal{O}_\infty$ which has been removed. If T is a hyperbolic line of $\mathcal{O}_\infty$, then $T^\perp$ is incident with the q points contained in it together with the unique point $(\mathcal{O}_i)$ for which $T^\perp \subseteq \mathcal{O}_i$. The line L_∞ is incident with the $q+1$ points $(\mathcal{O}_i)$, $0 \leq i \leq q$.

3.5 Regularity in $\mathcal{S}_\infty$ and some isomorphic affine planes

Recall the following results from the point-line dual of Lemma 2.6 of [Pa72b].

Theorem 3.5.1 *The line L_∞ is a regular line in $\mathcal{S}_\infty$. Moreover, $(\mathcal{O}_i)$ is a regular point of L_∞ in $\mathcal{S}_\infty$ if and only if $\mathcal{O}_i$ is a regular ovoid of $\mathcal{S}$ which is pivotal for the fan $\mathcal{M}$.*

We will soon be able to establish slightly more: Given that $\mathcal{O}_\infty$ is pivotal for the fan $\mathcal{M} = \{\mathcal{O}_\infty, \mathcal{O}_0, \ldots, \mathcal{O}_q\}$, if one of the ovoids $\mathcal{O}_i$, $i \in I$, is merely regular, then it is pivotal for the fan $\mathcal{M}$.

For the moment, consider the plane $\pi(\mathcal{O}_\infty)$. Let $\mathcal{E}_0 = \{T_1, T_2, \ldots, T_q\}$ and $\mathcal{E}_1 = \{R_1, R_2, \ldots, R_q\}$ be two distinct parallel classes of lines in $\pi(\mathcal{O}_\infty)$.

Obs. 3.5.2 *Each point of $\pi(\mathcal{O}_\infty)$ is on a uniqe hyperbolic line T_i and a unique hyperbolic line R_j, so we may label these points as $x_{i,j} = T_i \cap R_j$.*

Hence $T_i^\perp = \{x_{i,j}, x_{i,h}\}^\perp$ for any $h \neq j$, and $R_j^\perp = \{x_{i,j}, x_{k,j}\}^\perp$ for any $k \neq i$. We now wish to abuse the notation a bit, using the "perp" notation for whichever GQ we find convenient. The context should make it clear what the notation means. The next observation follows with a little thought.

Obs. 3.5.3 *In the GQ $\mathcal{S}_\infty$, $\{T_i^\perp, R_j^\perp\}^\perp$ is the set of lines of $\mathcal{S}$ which contain $x_{i,j}$, together with the line L_∞.*

Moreover,

Obs. 3.5.4 $\{T_i^\perp, R_j^\perp\}^{\perp\perp} = \{\{x_{i,j}, x_{i,h}\}^\perp, \{x_{i,j}, x_{k,j}\}^\perp\}^{\perp\perp}$ *for any $h \neq j$ and $k \neq i$. In other words, $\{T_i^\perp, R_j^\perp\}^{\perp\perp} =$ $\{l^\perp : l$ is incident with $x_{i,j}$ in $\pi(\mathcal{O}_\infty)\}$.*

¿From the regularity of L_∞ in $\mathcal{S}_\infty$ there is an affine plane $\pi(L_\infty) = (\mathcal{P}(L_\infty), \mathcal{L}(L_\infty))$ whose points are are the spans (in $\mathcal{S}_\infty$) of lines meeting L_∞ at distinct points, and whose lines are the lines of $\mathcal{S}_\infty$ different from L_∞ but meeting it at distinct points. Incidence is containment. Hence $\mathcal{P}(L_\infty) = \{\{l^\perp, m^\perp\}^{\perp\perp} : l, m$ are lines in distinct parallel classes of $\pi(\mathcal{O}_\infty)\} = \{\{T_i^\perp, R_j^\perp\}^{\perp\perp} : 1 \leq i, j \leq q\}$. Also, $\mathcal{L}(L_\infty) = \{l^\perp :$ l is a line of $\pi(\mathcal{O}_\infty)\}$.

Theorem 3.5.5 *The affine planes $\pi(\mathcal{O}_\infty)$ and $\pi(L_\infty)$ are isomorphic.*

Proof: Define ψ from $\mathcal{P}(\mathcal{O}_\infty)$ to $\mathcal{P}(L_\infty)$ by $\psi(x_{i,j}) = \{T_i^\perp, R_j^\perp\}^{\perp\perp}$. To see that ψ preserves collinearity, first suppose that $l = T_i = \{x_{i,1}, \ldots, x_{i,q}\}$. So $\psi(l) = \{\{T_i^\perp, R_1^\perp\}^{\perp\perp}, \{T_i^\perp, R_2^\perp\}^{\perp\perp}, \ldots,$ $\{T_i^\perp, R_q^\perp\}^{\perp\perp}\}$, i.e., exactly the set of points in $\mathcal{P}(L_\infty)$ on the line $T_i^\perp$. Now suppose that $l \notin \mathcal{E}_0$, say $l = \{x_{1,r_1}, x_{2,r_2}, \ldots, x_{q,r_q}\}$. So $\psi(l) = \{\psi(x_{1,r_1}), \psi(x_{2,r_2}), \ldots, \psi(x_{q,r_q})\} = \{\{T_1^\perp, R_{r_1}^\perp\}^{\perp\perp}, \{T_2^\perp, R_{r_2}^\perp\}^{\perp\perp}, \ldots,$

$\{T_q^\perp, R_{r_q}^\perp\}^{\perp\perp}\} = \{\{T_1^\perp, l^\perp\}^{\perp\perp}, \{T_2^\perp, l^\perp\}^{\perp\perp}, \ldots, \{T_q^\perp, l^\perp\}^{\perp\perp}\}$. But this is exactly the set of points in $\mathcal{P}(L_\infty)$ on the line $l^\perp \in \mathcal{L}(L_\infty)$. So ψ maps lines to lines. Thus ψ is an isomorphism between $\pi(\mathcal{O}_\infty)$ and $\pi(L_\infty)$. ∎

Now suppose that the ovoid $\mathcal{O}_0$ is also pivotal for the fan $\mathcal{M}$, so that the point $(\mathcal{O}_0)$ of L_∞ is regular in $\mathcal{S}_\infty$. From the regularity of $(\mathcal{O}_0)$ there is a standard construction of a projective plane $\pi^*((\mathcal{O}_0))$ whose points are the points of $\mathcal{S}_\infty$ collinear with $(\mathcal{O}_0)$, and whose lines are the spans $\{x, y\}^{\perp\perp}$ for which x and y are points of $\mathcal{S}_\infty$ collinear with $(\mathcal{O}_0)$. Now form an affine plane $\pi((\mathcal{O}_0)) = (\mathcal{P}((\mathcal{O}_0)), \mathcal{L}((\mathcal{O}_0)))$ by removing the line L_∞ and all of its points from the plane $\pi^*((\mathcal{O}_0))$. This new affine plane has pointset $\mathcal{P}((\mathcal{O}_0)) = \{x : x \in \mathcal{O}_0\}$ and lineset $\mathcal{L}((\mathcal{O}_0)) = \{\{x, y\}^{\perp\perp} : x, y \in \mathcal{O}_0\}$. But these are exactly the point– and linesets of $\pi(\mathcal{O}_0)$, and the incidences are the same. So here we see that the follownig theorem is proved.

Theorem 3.5.6 *The two affine planes $\pi((\mathcal{O}_0))$ and $\pi(\mathcal{O}_0)$ are identical (not merely isomorphic).*

In the case under consideration here, i.e., that both $\mathcal{O}_\infty$ and $\mathcal{O}_0$ are pivotal for $\mathcal{M}$ in $\mathcal{S}$, so that both L_∞ and $(\mathcal{O}_0)$ are regular in $\mathcal{S}_\infty$, we know that $\mathcal{S}_\infty$ is an **amalgamation of planes** (cf., [Pa72a]). Moreover, by the results of this section, $\mathcal{S}_\infty$ is the amalgamation of two desarguesian planes exactly when the two affine planes $\pi(\mathcal{O}_\infty)$ and $\pi(\mathcal{O}_0)$ are both desarguesian (cf. [Pa72a] and [Pa77]). We shall return to this theme later.

3.6 Regularity in $\mathcal{S}$

Thas and van Maldeghem [TvM9?] have shown that no GQ of order $(q+1, q-1)$ with $q \geq 4$ can have all of its points regular. On the other hand, the known examples with $q = 2^e$ have at least two regular ovoids, so there are many pairs of points that are regular. In this subsection we want to investigate the types of regular pairs of points that may exist. Our results lead to the final proof of Section 3, namely that if $\mathcal{S}$ has at least three regular ovoids with two of them pivotal for the same fan $\mathcal{M}$, then an ovoid of $\mathcal{S}$ is regular if and only if it belongs to $\mathcal{M}$, in which case it is pivotal for $\mathcal{M}$.

Theorem 3.6.1 *Suppose that* $\{x_1, x_2\}$ *is a regular pair of points in* $\mathcal{S}$ *with* $x_1 \not\sim x_2$. *Put* $T = \{x_1, x_2\}^{\perp\perp} = \{x_1, \ldots, x_q\}$, *and* $T^\perp = \{y_1, \ldots, y_q\}$. *Then exactly one of the following must occur:*

(i) There are distinct $i, j \in \tilde{I}$ *such that* $T \subseteq \mathcal{O}_i$ *and* $T^\perp \subseteq \mathcal{O}_j$.

(ii) Exactly one of T, $T^\perp$ *has a unique point in common with* $\mathcal{O}_\infty$. *Without loss of generality we assume* $T^\perp \cap \mathcal{O}_\infty = \{y_1\}$ *and* $T \cap \mathcal{O}_\infty = \emptyset$. *In this case there is a unique* $\mathcal{O}_i$ *for which* $T^\perp \setminus \{y_1\} \subseteq \mathcal{O}_i$. *And* $|T \cap \mathcal{O}_j| = 1$ *for each* $j \neq \infty$, i.

Proof: Since $\mathcal{O}_\infty$ is pivotal, if either T or $T^\perp$ has two points in common with $\mathcal{O}_\infty$, then it is contained in $\mathcal{O}_\infty$ and its perp is contained in some $\mathcal{O}_i$. If $T \cap \mathcal{O}_\infty = \emptyset = T^\perp \cap \mathcal{O}_\infty$, then by part (iii) of Subsection 3.3 there must be distinct $i, j \in I$ for which $T \subseteq \mathcal{O}_i$ and $T^\perp \subseteq \mathcal{O}_j$. In the remaining case, exactly one of T, $T^\perp$ has a unique point in common with $\mathcal{O}_\infty$. Without loss of generality we assume that $T^\perp \cap \mathcal{O}_\infty = \{y_1\}$. Also, there is some $i \in I$ for which $T^\perp \setminus \{y_1\} \subseteq \mathcal{O}_i$, since otherwise by Subsection 3.3 T would have to meet $\mathcal{O}_\infty$, clearly an impossibility. Also $y_2, \ldots, y_q$ must be in perps of distinct hyperbolic lines of $\mathcal{O}_\infty$, which means they are on distinct lines ($\neq L_\infty$) through $(\mathcal{O}_i)$ in $\mathcal{S}_\infty$. By part (ii) of Subsection 3.3 $|T \cap \mathcal{O}_j| = 1$ for each $j \neq \infty, i$.

Hence in $\mathcal{S}_\infty$, $\{y_2, \ldots, y_q\} \subseteq \{(\mathcal{O}_i), x_1, \ldots, x_q\}^\perp$ and $\{y_2, \ldots, y_q\}^\perp = \{(\mathcal{O}_i), x_1, \ldots, x_q\}$. Let T_0 be the line of $\mathcal{S}_\infty$ through $(\mathcal{O}_i)$, $T_0 \neq L_\infty$, and $T_0 \cap \{y_2, \ldots, y_q\} = \emptyset$. The lines of $\mathcal{S}$ through y_1 together with L_∞ form one ruling of a grid Γ. For ease of notation, suppose that $\mathcal{O}_i = \mathcal{O}_0$, and $x_j \in \mathcal{O}_j$, $1 \leq j \leq q$. Then the other ruling of Γ consists of $y_1^\perp \cap \mathcal{O}_k = T_k$, $0 \leq k \leq q$, and $x_j \in \mathcal{O}_j \cap y_1^\perp$. ■

Remark: A consequence of the preceding proof is the following: If $y_1 \in \mathcal{O}_\infty$ and $y_2 \in \mathcal{O}_i$ and if $\{y_1, y_2\}$ is regular in $\mathcal{S}$, then $\{y_1, y_2\}^{\perp\perp} \setminus \{y_1\} \subseteq \mathcal{O}_i$.

Corollary 3.6.2 *With the notation adopted above, if for some* $i \in I$, $\mathcal{O}_i$ *is regular, then it is pivotal for* $\mathcal{M}$.

Proof: If $\mathcal{O}_i$ is regular for some $i \in I$, and x, y are distinct points of $\mathcal{O}_i$, then $\{x, y\}$ is regular and $\{x, y\}^{\perp\perp} \subseteq \mathcal{O}_i$. This does not permit the second case of Theorem 3.6.1 to hold, hence the first case must hold. ■

Now suppose that $\mathcal{O}$ is a regular ovoid of $\mathcal{S}$ possibly different from $\mathcal{O}_\infty$. Recall that $\pi(\mathcal{O}_\infty)$ and $\pi(\mathcal{O})$ are affine planes whose "lines" are the hyperbolic lines contained in them. Clearly $\pi(\mathcal{O}_\infty) \cap \pi(\mathcal{O})$ is a subspace of $\pi(\mathcal{O}_\infty)$ and of $\pi(\mathcal{O})$. Hence we have proved the following.

Theorem 3.6.3 *Any regular ovoid of $\mathcal{S}$ different from $\mathcal{O}_\infty$ meets $\mathcal{O}_\infty$ in exactly* $0,\ 1$ *or q points.*

Theorem 3.6.4 *If $\mathcal{O}$ is a regular ovoid with $|\mathcal{O} \cap \mathcal{O}_\infty| = q$, then $\mathcal{O} = T_1 \cup T_2^\perp \cup \cdots \cup T_q^\perp$, where $\mathcal{O} \cap \mathcal{O}_\infty = T_1$, and $\mathcal{O}_\infty = T_1 \cup T_2 \cup \cdots \cup T_q$ is a parallel class partition of the points of $\mathcal{O}_\infty$, so $\mathcal{O}_k = T_1^\perp \cup T_2^\perp \cup T_q^\perp$ for some $k \in I$.*

Proof: Suppose that $\mathcal{O}_\infty = \{z_1, \ldots, z_q, x_{q+1}, \ldots, x_{2q}, \ldots, x_{q^2}\}$, where $T_1 = \{z_1, \ldots, z_q\}$, $T_2 = \{x_{q+1}, \ldots, x_q\}$, $\ldots$, $T_q = \{x_{(q-1)q+1}, \ldots, x_{q^2}\}$ are the disjoint hyperbolic lines of one parallel class of $\mathcal{O}_\infty$, and $\mathcal{O} = \{z_1, \ldots, z_q, y_{q+1}, \ldots, y_{q^2}\}$, with $\mathcal{O} \cap \mathcal{O}_\infty = T_1$. Let $y \in \mathcal{O} \cap \mathcal{O}_k$ for some $k \neq \infty$. Then $\{z_i, y\}^{\perp\perp} \setminus \{z_i\} \subseteq \mathcal{O}_k$ by Theorem 3.6.1, so $\cup_{i=1}^{q} \left(\{z_i, y\}^{\perp\perp} \setminus \{z_i\}\right)$ gives $1 + q(q-2) = q^2 - 2q + 1$ points of $\mathcal{O} \cap \mathcal{O}_k$. Since $2(q^2 - 2q + 1) > q^2 - q$ for $q \geq 3$, $\mathcal{O} \setminus \mathcal{O}_\infty$ must be contained in just one $\mathcal{O}_k$, i.e., $|\mathcal{O} \cap \mathcal{O}_k| = q^2 - q$, and hence $\mathcal{O} \cap \mathcal{O}_k = \{y_{q+1}, \ldots, y_{q^2}\}$. Clearly no y_j is collinear with any z_i. So $y_j^\perp \cap \mathcal{O}_\infty = \{x_{i1}, \ldots, x_{iq}\}$ is disjoint from T_1. So the parallel class of $\mathcal{O}_\infty$ containing T_1 must contain $q-1$ hyperbolic lines $T_2, \ldots, T_q$ whose perps cover $\mathcal{O} \cap \mathcal{O}_k$. Let T_1' be the points of $\mathcal{O}_k \setminus (\mathcal{O} \cap \mathcal{O}_k)$. T_1' must meet the same lines that T_1 does, forcing $T_1' = T_1^\perp$.

So we have the following:

$$\mathcal{O}_\infty = T_1 \cup \cdots \cup T_q,$$

$$\mathcal{O}_k = T_1^\perp \cup \cdots \cup T_q^\perp,$$

$$\mathcal{O} = T_1 \cup T_2^\perp \cup \cdots \cup T_q^\perp.$$

This completes the proof of the theorem. ■

Remark: Continuing with the notation of the above proof, let $x_1 \in T_1$, $x_2 \in T_2^\perp$. By hypothesis $\{x_1, x_2\}^{\perp\perp} \subseteq \mathcal{O}$. If two points of $\{x_1, x_2\}^{\perp\perp}$ belong to the same $T_j^\perp$, then of course $\{x_1, x_2\}^{\perp\perp}$ would have to equal $T_j^\perp$. So $T = \{x_1, x_2\}^{\perp\perp}$ has one point in each of $T_1, T_2^\perp, \ldots, T_q^\perp$. Now let $\{x_1, x_2\}^\perp = \{w_1, \ldots, w_q\} = T^\perp$. Clearly $T^\perp \cap \mathcal{O}_\infty = T^\perp \cap \mathcal{O}_k = \emptyset$, since both $\mathcal{O}_\infty$ and $\mathcal{O}_k$ are ovoids. If two points of $T^\perp$ belong to the same $\mathcal{O}_j$, then since $T \cap \mathcal{O}_\infty \neq \emptyset$, it would have to be that $T \subseteq \mathcal{O}_\infty$. Hence $|T^\perp \cap \mathcal{O}_j| = 1$ for each $j \neq \infty, k$.

Theorem 3.6.5 *If $\mathcal{O}$ is a regular ovoid with $|\mathcal{O} \cap \mathcal{O}_\infty| = 1$, then $|\mathcal{O} \cap \mathcal{O}_i| = q-1$ for $0 \leq i \leq q$. If $\mathcal{O} \cap \mathcal{O}_\infty = \{z\}$ and $x \in \mathcal{O} \cap \mathcal{O}_i$, then $\{z, x\}^{\perp\perp} \setminus \{z\} = \mathcal{O} \cap \mathcal{O}_i$.*

Proof: Suppose $\mathcal{O} \cap \mathcal{O}_\infty = \{z\}$. From the remark following Theorem 3.6.1 we see that if $x \in \mathcal{O} \setminus \mathcal{O}_\infty$, say $x \in \mathcal{O}_i$, then $\{z,x\}^{\perp\perp} \setminus \{z\} \subseteq \mathcal{O}_i$. First suppose that $\{z,x\}^{\perp\perp} \setminus \{z\} \subseteq \mathcal{O} \cap \mathcal{O}_i$, $\{z,y\}^{\perp\perp} \setminus \{z\} \subseteq \mathcal{O} \cap \mathcal{O}_i$, and $z \notin \{x,y\}^{\perp\perp}$, so $\{x,y\}^{\perp\perp} \cap \mathcal{O}_\infty = \emptyset$. Say:

$$\{z,x\}^{\perp\perp} = \{z, x = x_2, \ldots, x_q\}; \{z,y\}^{\perp\perp} = \{z, y = y_2, \ldots, y_q\}.$$

Since $\{x,y\}^{\perp\perp} \cap \mathcal{O}_\infty = \emptyset$, if $\{x,y\}^\perp \cap \mathcal{O}_\infty = \emptyset$ also, then $\{x,y\}^\perp \subseteq \mathcal{O}_k$ for some $k \neq i$, ∞, and $\{x,y\}^{\perp\perp} \subseteq \mathcal{O}_i$. On the other hand, if $\{x,y\}^\perp \cap \mathcal{O}_\infty \neq \emptyset$, then $\{x,y\} \subseteq \mathcal{O}_i$ implies $\{x,y\}^\perp \subseteq \mathcal{O}_\infty$ and $\{x,y\}^{\perp\perp} \subseteq \mathcal{O}_i$. Either way we have the following: For $x, y \in \mathcal{O} \cap \mathcal{O}_i$, $\{x,y\}^{\perp\perp} \subseteq (\mathcal{O} \cap \mathcal{O}_i) \cup \{z\}$. So $(\mathcal{O} \cap \mathcal{O}_i) \cup \{z\}$ is a subspace of $\mathcal{O}$. Hence if $|\mathcal{O} \cap \mathcal{O}_i| \geq q$, then $\mathcal{O} = (\mathcal{O} \cap \mathcal{O}_i) \cup \{z\}$.

So suppose $\mathcal{O}_i = T_1^\perp \cup \cdots \cup T_q^\perp$, where $\mathcal{O}_\infty = T_1 \cup \cdots \cup T_q$. Say $T_1 = \{z_1, \ldots, z_q\}$ and z_1 is the point of $\mathcal{O}_i$ for which $(\mathcal{O}_i \setminus \{z_1\}) \cup \{z\} = \mathcal{O}$. Then the lines covered by z_1 are the same lines as those covered by z, an obvious impossibility. Hence it must be that $|(\mathcal{O} \cap \mathcal{O}_i)| = q-1$ for each $i = 0, 1, \ldots, q$. And if $x \in \mathcal{O} \cap \mathcal{O}_i$, then $\{z,x\}^{\perp\perp} \setminus \{z\} = \mathcal{O} \cap \mathcal{O}_i$ for each $i = 0, \ldots, q$. ■

Theorem 3.6.6 *Let $\mathcal{O}$ be a regular ovoid with $|\mathcal{O} \cap \mathcal{O}_\infty| = 0$. Then exactly one of the following two possibilities must hold:*

(i) $\mathcal{O}$ is some $\mathcal{O}_i \in \mathcal{M}$, in which case $\mathcal{O}_i$ is pivotal for the fan $\mathcal{M}$;

(ii) There is a unique $\mathcal{O}_i$, $0 \leq i \leq q$, for which $|\mathcal{O} \cap \mathcal{O}_\infty| = |\mathcal{O} \cap \mathcal{O}_i| = 0$, and $|\mathcal{O} \cap \mathcal{O}_j| = q$ for $j \neq i, \infty$. In this case $\mathcal{O} \cup \{(\mathcal{O}_i)\}$ is an ovoid of $\mathcal{S}_\infty$.

Proof: Suppose $\mathcal{O} \cap \mathcal{O}_\infty = \emptyset$. Let x, y be distinct points of $\mathcal{O}$, so $T = \{x,y\}^{\perp\perp} \subseteq \mathcal{O}$, and $T \cap \mathcal{O}_\infty = \emptyset$. If $T^\perp \cap \mathcal{O}_\infty = \emptyset$, then $T \subseteq \mathcal{O}_i$ for some $i \neq \infty$, and $T^\perp \subseteq \mathcal{O}_k$ for some $k \neq i, \infty$. If $T^\perp \cap \mathcal{O}_\infty \neq \emptyset$, then either $T^\perp \subseteq \mathcal{O}_\infty$ and $T \subseteq \mathcal{O}_i$ for some i, or $|T^\perp \cap \mathcal{O}_\infty| = 1$ and $T^\perp \setminus \mathcal{O}_\infty \subseteq \mathcal{O}_i$ for some $i \in I$ while $|T \cap \mathcal{O}_j| = 1$ for each $j \neq i, \infty$. In all cases, if x, y are distinct points of $\mathcal{O} \cap \mathcal{O}_i$ for some i, then $\{x,y\}^{\perp\perp} \subseteq \mathcal{O} \cap \mathcal{O}_i$. Hence $\mathcal{O} \cap \mathcal{O}_i$ is a subspace of $\mathcal{O}$, and must have size $0, 1, q$ or q^2. One possibility is that $\mathcal{O} = \mathcal{O}_i \in \mathcal{M}$, for some i with $0 \leq i \leq q$.

Suppose that $\mathcal{O} \notin \mathcal{M}$, so $|\mathcal{O} \cap \mathcal{O}_i| = 0, 1$ or q, for each $i \neq \infty$. Let a_i be the number of $\mathcal{O}_j$ with $|\mathcal{O} \cap \mathcal{O}_j| = i, \;\; i = 0, 1, q$, for $0 \leq j \leq q$. Clearly each $a_i \geq 0$, and we have

$$(i) \quad a_0 + a_1 + a_q = q + 1,$$

$$(ii) \quad 0 \cdot a_0 + 1 \cdot a_1 + q \cdot a_q = q^2,$$

from which $a_1 = q(q - a_q)$. Put $k = q - a_q$, so $a_1 = kq$. It follows that $q - k = a_q = q + 1 - a_0 - a_1$, implying $a_1 + a_0 = k + 1 = a_0 + kq$, or $k(q-1) = 1 - a_0$, which must be nonnegative. This allows only two possibilities: $a_0 = 1$ or 0. If $a_0 = 1$, then $k = 0 = a_1$ and $a_q = q$. In this case $\mathcal{O}$ is disjoint from $\mathcal{O}_\infty$ and from one other member of $\mathcal{M}$, say $\mathcal{O}_0$, and meets each of the others $\mathcal{O}_1, \ldots, \mathcal{O}_q$ in q points. For the other case, suppose $a_0 = 0$. Here $k = 1$ and $q = 2$. So for $q \geq 4$ we have only the first case, i.e., $\mathcal{O}$ meets each of $\mathcal{O}_1, \ldots, \mathcal{O}_q$ in a "line" of $\mathcal{O}$. It is now straightforward to check that $\mathcal{O} \cup \{(\mathcal{O}_i)\}$ is an ovoid of $\mathcal{S}_\infty$. ■

3.7 Two pivotal members in the fan M

In this subsection we suppose that in addition to having $\mathcal{O}_\infty$ pivotal for the fan $\mathcal{M}$, the ovoid $\mathcal{O}_0$ is regular, so that in fact it is also pivotal for $\mathcal{M}$.

Theorem 3.7.1 *Suppose $\{x,y\}$ is a regular pair of points with $x \not\sim y$. Put $T = \{x,y\}^{\perp\perp}, \;\; T^\perp = \{x,y\}^\perp$. Then there must exist distinct $i, j \in \tilde{I}$ for which $T \subseteq \mathcal{O}_i, \;\; T^\perp \in \mathcal{O}_j$.*

Proof: Suppose not. Then one of $T^\perp$, T has a unique point in common with $\mathcal{O}_\infty$, say $|T^\perp \cap \mathcal{O}_\infty| = 1$. Then there is some $i \neq \infty$ for which $|T^\perp \cap \mathcal{O}_i| = q - 1$. Clearly $\mathcal{O}_i$ is not regular, so $i \neq 0$. Then $|T \cap \mathcal{O}_j| = 1$ for each $j \neq i, \infty$, including $j = 0$. But $|T \cap \mathcal{O}_0| = 1$ says there must be some $k, \;\; |T \cap \mathcal{O}_k| = q - 1$, an impossibility for $q \geq 4$. ■

Theorem 3.7.2 *Suppose $\mathcal{O}$ is a regular ovoid of $\mathcal{S}$. Then $\mathcal{O} \in \mathcal{M}$.*

Proof: Suppose $\mathcal{O}$ is a regular ovoid with $\mathcal{O} \notin \mathcal{M}$. Then $|\mathcal{O} \cap \mathcal{O}_\infty| = 0, 1$ or q, and $|\mathcal{O} \cap \mathcal{O}_0| = 0, 1$, or q.

Case (i) $|\mathcal{O} \cap \mathcal{O}_\infty| = q$. In this case there is a k with $|\mathcal{O} \cap \mathcal{O}_k| = q^2 - q$, so $k \neq 0$ and $|\mathcal{O} \cap \mathcal{O}_0| = 0$. But Theorem 3.6.6 applied to $\mathcal{O}_0$ in place of $\mathcal{O}_\infty$ says there is some j with $|\mathcal{O} \cap \mathcal{O}_j| = 0$ and $|\mathcal{O} \cap \mathcal{O}_m| = q$ for $m \neq 0, j$. Hence this case cannot arise.

Case (ii) $|\mathcal{O} \cap \mathcal{O}_\infty| = 1$. Here $|\mathcal{O} \cap \mathcal{O}_i| = q - 1$ for $0 \leq i \leq q$. But $|\mathcal{O} \cap \mathcal{O}_0| \neq q - 1$ if $q > 2$, so this case cannot arise.

Case (iii) $|\mathcal{O} \cap \mathcal{O}_\infty| = 0$. By symmetry we may suppose that we also have $|\mathcal{O} \cap \mathcal{O}_0| = 0$. But if $\mathcal{O} \notin \mathcal{M}$ then also $|\mathcal{O} \cap \mathcal{O}_j| = q$ for $j \neq 0, \infty$. But suppose $x \in \mathcal{O} \cap \mathcal{O}_j$, $y \in \mathcal{O} \cap \mathcal{O}_k$, $j \neq k$, and $j, k \neq i, \infty$. Then $\{x, y\}^{\perp\perp}$ does not belong to a single member of $\mathcal{M}$, contradicting Theorem 3.7.1. Hence this case does not arise, completing the proof. ■

4 Characterizing the $GQ(q+1, q-1)$ Arising from q-Arcs

In [Pa85] the $GQ(q+1, q-1)$ constructed from q-arcs of a hyperoval (as in Subsection 2.3) were characterized by seven properties $A_1, A_2, \ldots, A_7$. The proof utilized a characterization of the $GQ(q-1, q+1)$ derived from a hyperoval as in Subsection 2.1 that was given by M. De Soete and J. A. Thas in [DST86b]. When [DST87] appeared, the need for A_7 in the proof given in [Pa85] was removed. Moreover, the properties A_1, A_2 and A_4 hold automatically in our present situation (i.e., $\mathcal{O}_\infty$ is pivotal for the fan $\mathcal{M}$ and q is even). Hence we need only to consider properties A_3, A_5, and A_6.

4.1 Property A_3

We continue to suppose that S is a $GQ(q+1, q-1)$ with $\mathcal{O}_\infty$ pivotal for the fan $\mathcal{M}$ and q is even.

For our present purposes we do not need to review all the terminology and subsidiary results surrounding property A_3 that were given in [Pa85]. A simple statement of this property will suffice. Basically, it says that the ovoid $\mathcal{O}_0$ plays a special role.

A_3. Let L_1, M_1 be nonconcurrent lines of $\mathcal{S}$ meeting lines L_2, M_2 at four distinct points of $P \setminus (\mathcal{O}_\infty \cup \mathcal{O}_0)$. Let a_j be the point of L_j in $\mathcal{O}_\infty$, $j = 1, 2$. And let b_j be the point of M_j in $a_j^\perp$, $j = 1, 2$. Then $b_1 \in \mathcal{O}_0$ if only if $b_2 \in \mathcal{O}_0$.

Note: A_3 has $\mathcal{O}_\infty$ and $\mathcal{O}_0$ playing symmetric roles. The main result we need from [Pa85] is the following.

Theorem 4.1.1 *Suppose that $\mathcal{S}$ satisfies property A_3. In the GQ $\mathcal{S}_\infty$ of order q obtained by constricting $\mathcal{S}$ about the pivotal ovoid $\mathcal{O}_\infty$, the point $(\mathcal{O}_0)$ is a coregular point (i.e., each line through $(\mathcal{O}_0)$ is regular). By 1.5.2 of [PT84], since q is even and $(\mathcal{O}_0)$ is coregular, it must also be true that $(\mathcal{O}_0)$ is regular. Hence we now also have that $\mathcal{O}_0$ is pivotal for the fan $\mathcal{M}$. This means that in all the results we have obtained so far, the roles of $\mathcal{O}_\infty$ and $\mathcal{O}_0$ are completely interchangeable.*

¿From now on we suppose that $\mathcal{S}$ satisfies A_3. Since $(\mathcal{O}_0)$ is a regular point of $\mathcal{S}_\infty$, we may expand $\mathcal{S}_\infty$ about the point $(\mathcal{O}_0)$ to obtain a GQ $\mathcal{S}_\infty^0$ of order $(q-1, q+1)$ as follows:

$\mathcal{S}_\infty^0 = (P_\infty^0, B_\infty^0, I_\infty^0)$, where $P_\infty^0 = \mathcal{O}_1 \cup \mathcal{O}_2 \cup \cdots \cup \mathcal{O}_q$, and $B_\infty^0 = B \cup \{T^\perp : T$ is a hyperbolic line of $\mathcal{O}_\infty$ or of $\mathcal{O}_0$, but $T^\perp \cap (\mathcal{O}_\infty \cup \mathcal{O}_0) = \emptyset\}$. For $x \in P_\infty^0$ and $L \in B$, $x\ I_\infty^0\ L$ if and only if xIL. For $x \in P_\infty^0$, $L = T^\perp \in B_\infty^0 \setminus B$, $x\ I_\infty^0\ T^\perp$ if and only if $x \in T^\perp$.

Note: If the roles of $\mathcal{O}_\infty$ and $\mathcal{O}_0$ were interchanged so as to first construct a GQ $\mathcal{S}_0$ of order q by constricting about $\mathcal{O}_0$ and then constructing a GQ $\mathcal{S}_0^\infty$ of order $(q-1, q+1)$ by expanding about the regular point $(\mathcal{O}_\infty)$, the two GQ $\mathcal{S}_\infty^0$ and $\mathcal{S}_0^\infty$ would be **exactly the same**.

The remainder of the proof of [Pa85] charactizing the $\mathcal{S}$ derived from q-arcs is to use the assumption that $\mathcal{S}$ satisfies A_5 and A_6 to show that $\mathcal{S}_\infty^0 = \mathcal{S}_0^\infty$ satisfies the properties used in [DST87] to show that $\mathcal{S}_\infty^0$ is constructed from a hyperoval as in 2.1. Then $\mathcal{S}$ can be recovered from $\mathcal{S}_\infty^0$ by constricting about an appropriate spread and then expanding about a particular regular line.

4.2 Properties A_5 and A_6

For our present purposes we may also ignore all the notation and subsidiary results surrounding the properties A_5 and A_6 that were given in [Pa85].

A_5 Let (L_1, L_2, L_3) be a centric triad of lines (three pairwise noncurrent lines with a common transversal) with the property that the points of $\mathcal{O}_\infty$ on L_1, L_2, L_3 are all collinear with with all three points of $\mathcal{O}_0$ on L_1, L_2, L_3. Then, if for some $(a, b) \in \mathcal{O}_\infty \times \mathcal{O}_0$ with a and b noncollinear, both L_1 and L_2 are incident with points of $\{a, b\}^\perp$, then L_3 is also incident with a point of $\{a, b\}^\perp$.

A_6 Let x_1, x_2, x_3 be distinct points of some $\mathcal{O}_j$, $1 \le j \le q$. Let (L_1, L_2, L_3) and (M_1, M_2, M_3) be two triads of lines such that L_i meets M_i at x_i, $1 \le i \le 3$. Suppose that the three points of $\mathcal{O}_\infty$ on L_1, L_2, L_3 (resp., M_1, M_2, M_3) are each collinear with the three points of $\mathcal{O}_0$ on L_1, L_2, L_3 (resp., M_1, M_2, M_3). Then (L_1, L_2, L_3) is centric if and only if (M_1, M_2, M_3) is centric.

4.3 Amalgamations of Desarguesian Planes

In [Pa77] it was shown that if a GQ $\mathcal{S}' = (P', B', I')$ of order q has a regular point (∞) incident with a regular line $[\infty]$, for which both the associated projective planes are Desarguesian, then there are two permutations α, β of the elements of $F = GF(q)$, $q = 2^e$, that form an admissible pair and and may be used to coordinatize the GQ $\mathcal{S}'$ as follows.

Without loss of generality we may assume that $0^\alpha = 0^\beta = 0$, and $1^\alpha = 1^\beta = 1$. The pair (α, β) is **admissible** provided that whenever u_1, u_2, u_3 are distinct elements of F and z_1, z_2, z_3 are distinct elements of F (with subscripts taken modulo 3), then

$$0 = \sum_{i=1}^{3} u_i(z_{i+1} - z_{i-1})$$

implies that

$$0 \neq \sum_{i=1}^{3} u_i^\alpha(z_{i+1}^\beta - z_{i-1}^\beta).$$

The points of $\mathcal{S}'$ have labels (∞), (m), (a, b), (m, v, w) for arbitrary $a, b, m, v, w \in F$. And the lines have labels $[\infty]$, $[a]$, $[m, v]$, $[a, b, c]$ for arbitrary $m, v, a, b, c \in F$. Incidence I' is defined as follows:

$$[a, b, c]I'(a, b)I'[a]I'(\infty)I'[\infty]I'(m)I'[m, v]I'(m, v, w),$$

and

$$(m, v, w)I'[a, b, c] \text{ if and only if } w = ma + b \text{ and } v = a^\alpha m^\beta + c.$$

Now expand $\mathcal{S}'$ about the regular line $[\infty]$ to obtain a GQ $\mathcal{S} = (P, B, I)$ of order $(q + 1, q - 1)$. The lines of $\mathcal{S}$ are given by

$$B = \{[a, b, c] : a, b, c \in F\};$$

The points of $\mathcal{S}$ are given by $P = P_1 \cup P_2 \cup P_3$, where

(i) $\quad P_1 = \{m, v, w) : m, v, w \in F\};$

(ii) $\quad P_2 = \{(a, b) : a, b \in F\};$

(iii) $\quad P_3 = \{((a, c)) : a, c \in F\}.$

Then incidence I is defined as follows:

$(m, v, w)\ I\ [a, am + w, a^\alpha m^\beta + v]\ \forall\ m, v, w, a \in F;$

$(a, b)\ I\ [a, b, c]\ \forall\ a, b, c \in F;$

$((a, c))\ I\ [a, b, c]\ \forall\ a, b, c \in F.$

Let the elements of F be labeled in any way as $F = \{m_1, m_2, \ldots, m_q\}$. For each i, $1 \le i \le q$, let $\mathcal{O}_i = \{(m_i, v, w) : v, w \in F\}$. Put $\mathcal{O}_0 = \{(a, b) : a, b \in F\}$, and $\mathcal{O}_\infty = \{((a, c)) : a, c \in F\}$. It follows that $\mathcal{M} = \{\mathcal{O}_i : i \in \tilde{I}\}$ is a fan of ovoids for which $\mathcal{O}_\infty$ and $\mathcal{O}_0$ are pivotal. And of course the two affine planes $\pi(\mathcal{O}_\infty)$ and $\pi(\mathcal{O}_0)$ are Desarguesian.

As a kind of converse, now suppose that $\mathcal{S} = (P, B, I)$ is a GQ of order $(q+1, q-1)$ with a fan $\mathcal{M} = \{\mathcal{O}_i : i \in \tilde{I}\}$ for which both $\mathcal{O}_\infty$ and $\mathcal{O}_0$ are pivotal. Suppose also that the affine planes $\pi(\mathcal{O}_\infty)$ and $\pi(\mathcal{O}_0)$ are Desarguesian. Constrict about $\mathcal{O}_\infty$ to obtain a GQ $\mathcal{S}_\infty$ that is an amalgamation of Desarguesian planes. Hence $\mathcal{S}_\infty$ is coordinatized by an admissible pair (α, β) as above. Since expanding $\mathcal{S}_\infty$ about the regular line L_∞ is the inverse process of constricting $\mathcal{S}$ about $\mathcal{O}_\infty$, we may assume that $\mathcal{S}$ is coordinatized as above.

The next few results come from [Pa77] and provide information about collineations of $\mathcal{S}_\infty$. Recall that a symmetry about a line L is a collineation of $\mathcal{S}_\infty$ that fixes each line meeting L, and the existence of the maximal number q of symmetries about L (i.e., L is an **axis of symmetry**) forces L to be regular.

Theorem 4.3.1 *β is additive if and only if the pair $\{[a], M\}$ is regular for some line M not meeting $[\infty]$, if and only if $[a]$ is an axis of symmetry for all $a \in F$, in which case the symmetries about $[a_0]$ are given as follows (where it suffices to give the action on points not collinear with (∞) and lines not concurrent with $[\infty]$.*

For $\sigma \in F$,
$(m, v, w) \mapsto (m + \sigma, v + a_0^\alpha \sigma, w + a_0\sigma);$

$[a, b, c] \mapsto [a, b + \sigma(a + a_0), c + \sigma(a^\alpha + a_0^\alpha)].$

Theorem 4.3.2 *β is multiplicative if and only if there is a full group of $q - 1$ homologies fixing $(0, 0, 0)$ and ∞ and all points in their perp. In this case the homologies are determined as follows:*

For $0 \ne t \in F$,

$(m, v, w) \mapsto (tm, t^\beta v, tw);$

$[a, b, c] \mapsto [a, tb, t^\beta c].$

Theorem 4.3.3 *α is additive if and only if the pair $\{(m), p\}$ is regular for some point p not collinear with (∞), if and only if (m) is a center of symmetry for all $m \in F$, in which case the symmetries about (m_0) are:*

For $\sigma \in F$:
$(m, v, w) \mapsto (m, v + \sigma^\alpha(m^\beta + m_0^\beta), w + \sigma(m + m_0));$

$[a, b, c] \mapsto [a + \sigma, b + \sigma m_0, c + \sigma^\alpha m_0^\beta].$

Theorem 4.3.4 *α is multiplicative if and only if there is a full group of $q - 1$ homologies fixing $[0, 0, 0]$ and $[\infty]$ and all lines in their perp. In this case the homologies are determined as follows:*

For $0 \neq t \in F$,
$(m, v, w) \mapsto (m, t^\alpha v, tw);$

$[a, b, c] \mapsto [ta, tb, t^\alpha c].$

Theorem 4.3.5 *Let $\mathcal{S}$ and $\mathcal{S}_\infty$ be as above, coordinatized by the admissible pair (α, β). Then the following are equivalent:*

(i) *$(\mathcal{O}_0)$ is a coregular point of $\mathcal{S}_\infty$.*
Note that $(\mathcal{O}_0)$ is coordinatized as (∞) above.

(ii) *β is additive.*

(iii) *$\mathcal{S}$ satisfies property A_3.*

Proof: That (i) and (iii) are equivalent is result V.2 of [Pa85]. That (i) and (ii) are equivalent is essentially contained on page 225 of [Pa77]. ■

Before considering property A_5 we work out the collinearities and concurrencies between the points and the lines of $\mathcal{S}$.

Lemma 4.3.6 *The following collinearities and concurrencies hold in $\mathcal{S}$.*

(*i*) $(a,b) \sim ((x,y))$ *iff* $a = x$, *in which case they are on* $[a,b,y]$.

(*ii*) $(m,v,w) \sim (a,b)$ *iff* $w = am + b$, *in which case* $(m,v,w) = (m,v,am+b)I[a,b,a^\alpha m^\beta + v]I(a,b)$.

(*iii*) $(m,v,w) \sim ((a,c))$ *iff* $v = a^\alpha m^\beta + c$, *in which case* $(m,v,w) = (m, a^\alpha m^\beta + c, w)I[a, am+w, c]I((a,c))$.

(*iv*) $(m,v,w) \sim (m',v',w')$ *if* $m \neq m'$ *and* $\left(\frac{w+w'}{m+m'}\right)^\alpha = \frac{v+v'}{m^\beta + m'^\beta}$, *in which case they lie on the line* $\left[\frac{w+w'}{m+m'}, am+w, a^\alpha m^\beta + v\right]$.

(*v*) $[a,b,c] \sim [a,b,c']$ *at* (a,b).

(*vi*) $[a,b,c] \sim [a,b',c]$ *at* $((a,c))$.

(*vii*) $[a,b,c] \sim [a',b',c']$ *with* $a \neq a'$ *iff* $\left(\frac{b+b'}{a+a'}\right)^\beta = \left(\frac{c+c'}{a^\alpha + a'^\alpha}\right)$, *in which case they meet at* $\left(\frac{b+b'}{a+a'}, \frac{a^\alpha c' + a'^\alpha c}{a^\alpha + a'^\alpha}, \frac{ab' + a'b}{a+a'}\right)$.

In order to discuss property A_5 we need to consider just when the perp of two noncollinear points of $\mathcal{O}_\infty$ and $\mathcal{O}_0$ has a point on a given line. Using the preceding lemma, the next follows in a straightforward manner.

Lemma 4.3.7 *Let* a, a_1, a_2 *be distinct elements of* F. *Then* $\{(a_1,b_1),((a_2,c_2))\}^\perp$ *has a point on* $[a,b,c]$ *if and only if* $\left(\frac{b_1+b}{a_1+a}\right)^\beta = \frac{c_2+c}{a_2^\alpha + a^\alpha}$, *in which case that point is* $\left(\frac{b_1+b}{a_1+a}, \frac{a_2^\alpha c + a^\alpha c_2}{a_2^\alpha + a^\alpha}, \frac{ab_1 + a_1 b}{a_1+a}\right)$.

Lemma 4.3.8 *For* $0 \neq a \in F$, $\left(\frac{1}{a}\right)^\beta \neq \frac{1}{a^\alpha}$.

Proof: In the definition of admissibility, put $u_1 = 0$, $u_2 = 1$, $u_3 = a \neq 0, 1$; $z_1 = 0$, $z_2 = a^{-1}$, $z_3 = 1$. It follows that $0 = \sum u_i(z_{i+1} - z_{i-1})$, and the condition that $0 \neq \sum u_i^\alpha (z_{i+1}^\beta - z_{i-1}^\beta)$ is exactly the inequality of the lemma. ■

Theorem 4.3.9 *Let* $\mathcal{S}$ *and* $\mathcal{S}_\infty$ *satisfy the hypotheses of Theorem 4.3.5. Also, suppose that* $\mathcal{S}$ *satisfies property* A_3. *Then* $\mathcal{S}$ *satisfies property*

A_5 if and only if β is multiplicative, in which case β is an automorphism of F. In this case, by III.3 of [Pa77] we may assume that β is the identity permutation. Moreover, $\mathcal{S}$ is derived from a q-arc in $PG(2,q)$ as in Subsection 2.3.

Proof: First suppose that A_5 holds. By Lemma 4.3.8 we may choose $a_1 \neq 0, 1$, and then pick a_2 so that $\left(\frac{1}{a_1}\right)^\beta = \frac{1}{a_2^\alpha}$, in which case we know that $0, 1, a_1, a_2$ are all distinct. Then we consider a very special setup for A_5. Consider the centric triad $\{L_1, L_2, L_3\}$, where

$L_1 = [0,0,0]$ with points $((0,0)), (0,0), (y,0,0)$,

$L_2 = [0,1,1]$ with points $((0,1)), (0,1), (y+1,1,1)$,

$L_3 = [0,b,b^\beta]$ with points $((0,b^\beta)), (0,b), (y+b, b^b, b)$,

where $b \neq 0, 1$, and $[1, y, y^\beta]$ is a transversal of $\{L_1, L_2, L_3\}$.

This latter holds for all $y \in F$. Now using the a_1 and a_2 chosen above, for any $b_1 \in F$ choose $c_2 \in F$ so that $\left(\frac{b_1}{a_1}\right)^\beta = \frac{c_2}{a_2^\alpha}$, from which it follows that $\{(a_1, b_1), ((a_2, c_2))\}^\perp$ has a point on $[0,0,0]$. Similarly, $\{(a_1, b_1), ((a_2, c_2))\}^\perp$ has a point on $[0,1,1]$ since $\left(\frac{b_1+1}{a_1}\right)^\beta = \frac{c_2+1}{a_2^\alpha}$ by the additivity of β. By the hypothesis that A_5 holds, we know that $\{(a_1, b_1), ((a_2, c_2))\}^\perp$ must have a point on $[0, b, b^\beta]$, implying that $\left(\frac{b_1+b}{a_1}\right)^\beta = \frac{c_2+b^\beta}{a_2^\alpha}$. Again using the additivity of β, we see that $\left(\frac{b}{a_1}\right)^\beta = \frac{b^\beta}{a_2^\alpha} = b^\beta \cdot \left(\frac{1}{a_1}\right)^\beta$. This must hold for all $b \neq 0, 1$. So $\left(b \cdot \frac{1}{a_1}\right)^\beta = b^\beta \cdot \left(\frac{1}{a_1}\right)^\beta$ for all $b, \frac{1}{a_1} \in F \setminus \{0\}$. It follows that β is multiplicative.

Conversely, suppose that β is multiplicative, so that β is actually an autormorphism of F. In this case by III.3 of [Pa77] we may assume that $\beta = id$ and α is replaced by $\alpha\beta^{-1}$. It is then straightforward to show that the condition that $(\alpha\beta^{-1}, id)$ be admissible is the same as the condition that $\alpha\beta^{-1}$ be a permutation that gives an oval. (Put $z_i = u_i$, $i = 0, 1, 2$, modulo 3.) Hence it suffices to show that if $\beta = id$ and if we write α in place of $\alpha\beta^{-1}$, then the GQ $\mathcal{S}'$ coordinatized by the admissible pair (α, id) is isomorphic to $T_2(O)$ for some oval O. We sketch this step. Let the oval O consist of the points $(1, a, a^\alpha, 0)$, $a \in F$, together with the point $(0,1,0,0)$, and having nucleus equal to $(0,0,1,0)$. Coordinatize the GQ $T_2(O)$ as follows. The point $(m,w,v,1) \in PG(3,q)$ will be coordinatized as (m,v,w) as a point of $\mathcal{S}'$. The plane $[a,1,0,b]^T$ will be coordinatized as (a,b) as a point of $\mathcal{S}'$. The plane $[1,0,0,m]^T$ will be coordinatized as (m) as a point of $\mathcal{S}'$. The plane $[0,0,0,1]$ will be coordinatized as the point (∞) of $\mathcal{S}'$. The line $< (m,w,v,1),(1,a,a^\alpha,0) >=< (0,am+w,a^\alpha m+v,1),(1,a,a^\alpha,0) >$ will be coordinatized as $[a,am+w,a^\alpha m+v]$ as a line of $\mathcal{S}'$. The line $< (m,w,v,1),(0,1,0,0) >=< (m,0,v,1),(0,1,0,0) >$ will be coordinatized as $[m,v]$ as a line of $\mathcal{S}'$. Finally, the line $< (1,a,a^\alpha,0),(0,0,1,0) >$ is coordinatized as $[a]$ and $< (0,1,0,0),(0,0,1,0) >$ is coordinatized as $[\infty]$. It now is straightforward to check that (m,v,w) is incident with $[a,b,c]$ if and only if $b+w=am$ and $c+v=a^\alpha m$, and all the other incidences are also identical to those of $\mathcal{S}'$. ■

At this stage we see that if $\mathcal{S}$ is coordinatized by an admissible pair (α,β) as above and also satisfies A_3 and A_5, then it automatically satisfies A_6 (and A_7, which we no longer need). Basically, A_6 says that the lines of $PG(3,q)$ that are not in π and are not lines of $\mathcal{S}$ are recovered as intersections of grids of lines of $\mathcal{S}$ (see [Pa85]). This helps make it possible to describe the collineations if $\mathcal{S}$.

5 Collineations of the Known GQ(q+1,q-1), $q = 2^e$

5.1 More than two pivotal ovoids

Let $\mathcal{S}$ be a $GQ(q+1, q-1)$ derived from a q-arc Ω^- in $\pi = PG(2, q)$. Let A and B be the points of π for which $\Omega^+ = O^- \cup \{A, B\}$ is a hyperoval. Let $\mathcal{O}_{AB}$ (resp., $\mathcal{O}_{BA}$) be the ovoid of $\mathcal{S}$ consisting of the planes of $PG(3, q)$ through A but not B (resp., through B but not A). If $\pi_1, \ldots, \pi_q$, are the planes of $PG(3, q)$ different from π but containing the line AB, let $\mathcal{O}_i$ be the set of points of π_i not in π. Then $\mathcal{M} = \{\mathcal{O}_{AB}, \mathcal{O}_{BA}\} \cup \{\mathcal{O}_i : 1 \leq i \leq q\}$ is a fan of ovoids of $\mathcal{S}$ for which both $\mathcal{O}_{AB}$ and $\mathcal{O}_{BA}$ are pivotal. Constrict $\mathcal{S}$ about the ovoid $\mathcal{O}_{AB}$ to obtain a GQ of order q with regular point $(\mathcal{O}_{BA})$.

It is clear that any collineation of $\mathcal{S}$ must preserve the fan $\mathcal{M}$. If $\mathcal{O}_{AB}$ and $\mathcal{O}_{BA}$ are the only pivotal members of $\mathcal{M}$, they must be left invariant or interchanged. In this case, each collineation of $\mathcal{S}$ must be induced by a collineation of $PG(3, q)$ that leaves the q-arc O^- invariant. If either $\mathcal{O}_{AB}$ or $\mathcal{O}_{BA}$ is moved to some ovoid other than $\mathcal{O}_{AB}$ or $\mathcal{O}_{BA}$, then we know its image is a regular ovoid and hence must be a member of $\mathcal{M}$ pivotal for $\mathcal{M}$. In this case a second point of L_∞ (in addition to $(\mathcal{O}_{BA})$) is regular, so by Theorem 4.3.3 all points of L_∞ are regular, implying that each member of $\mathcal{M}$ is pivotal. Hence also in this case if (α, id) is the admissible pair used to coordinatize the GQ of order $(q+1, q-1)$, α is additive, and the oval is a translation oval. Moreover, the original GQ $\mathcal{S}$ is the point-line dual of the $GQ(q-1, q+1)$ constructed from the hyperoval Ω^+. The details are worked out in [Pa90] for a slightly different coordinatization, but we review them below for the coordinatization being used here. The point is that there are **concievably more** collineations of $\mathcal{S}$ than just those induced by the collineations of $PG(3, q)$ leaving the q-arc invariant. However, even in this case the collineations of the point-line dual $GQ(q-1, q+1)$ are exactly those induced by the collineations of $PG(3, q)$ leaving invariant the hyperoval. We use this below to conclude that in fact the only case in which S actually admits collineations other than those induced by the collineations of $PG(3, q)$ leaving the q-arc invariant is the case in which the original hyperoval is regular and one of the two points deleted from the conic to give the associated q-arc is the nucleus of the conic, which is always uniquely determined.

Theorem 5.1.1 *Let $\beta \in Aut(F)$ be an automorphism of maximal order and put $\alpha = \beta^{-1}$. Then let Ω^+ be the hyperoval defined by*

$$\Omega^+ = \{(1, a, a^\alpha, 0) : a \in F\} \cup \{(0, 1, 0, 0), (0, 0, 1, 0)\},$$

i.e.,

$$\Omega^+ = \{(x, y, z, 0) \in PG(3, q) : z^\beta = x^{\beta-1}y\} \cup \{(0, 0, 1, 0)\}.$$

Define the q-arc Ω^- by

$$\Omega^- = \{(1, a, a^\alpha, 0) : a \in F\}.$$

Now define a map ρ from the GQ $\mathcal{S}(\Omega^+)$ of order $(q-1, q+1)$ to the GQ $\mathcal{S}(\Omega^-)$ of order $(q+1, q-1)$ that maps points (resp., lines) of $\mathcal{S}(\Omega^+)$ (given as objects of $PG(3, q)$) to lines (resp., points) of $\mathcal{S}(\Omega^-)$ (using the coordinatization given in the previous section) as follows:

$$\rho : (m, w, v, 1) \ \mapsto \ [m, v, w^\alpha],$$

$$\rho :< (0, b, c, 1), (1, a, a^\alpha, 0) > \ \mapsto \ (a^\alpha, b^\alpha, c),$$

$$\rho :< (m, 0, v, 1), (0, 1, 0, 0) > \ \mapsto \ (m, v),$$

$$\rho :< (m, w, 0, 1), (0, 0, 1, 0) > \ \mapsto \ ((m, w^\alpha)).$$

Then ρ is a duality.

The proof is a straightforward matter of checking that incidences are preserved.

First we consider the case that Ω^+ is a regular hyperoval. In this case the natural nucleus is fixed by all stabilizing collineations and is triply transitive on the points of the conic. Here we may suppose that $\beta = 2$. In this case there is a collineation θ of $PG(3, q)$ leaving Ω^+ invariant and defined on points by

$$\theta : (x, y, z, w) \ \mapsto \ (x + y, y, z + y, w).$$

Clearly θ induces a collineation (also called θ) of $\mathcal{S}(\Omega^+)$. Using ρ to transform θ into a collineation $\bar{\theta}$ of $\mathcal{S}(\Omega^-)$, we obtain the following theorem.

Theorem 5.1.2 *Let Ω^- be the q-arc containing the points $(1, a, a^{1/2}, 0)$, and let $\mathcal{S}(\Omega^-)$ be the corresponding $GQ(q+1, q-1)$. Then there is a collineation of $\mathcal{S}(\Omega^-)$ defined as follows:*

$$\bar{\theta} : \ [m, v, w] \ \mapsto \ [m + w^2, v + w^2, w];$$

$$\bar{\theta} : \ (a, b, c) \ \mapsto \ (\tfrac{q}{a+1}, \tfrac{b}{a+1}, c + \tfrac{b^2}{a+1}), \quad a \neq 1;$$

$$\bar{\theta} : \ (1, b, c) \ \mapsto \ (b^2, c + b^2);$$

$$\bar{\theta} : \ (a, b) \ \mapsto \ (1, a^{1/2}, b + a);$$

$$\bar{\theta} : \ ((a, c)) \ \mapsto \ ((a + c^2, c)).$$

Again we leave the matter of checking that incidences are perserved as a straightforward exercise.

The important thing to notice here is that the ovoid of points $((a, c))$ (corresponding to $\mathcal{O}_\infty$ in the general treatment) is left invariant, but the ovoid of points (a, b) (corresponding to $\mathcal{O}_0$ in the general treatment) is interchanged with the ovoid of points of the form $(1, b, c)$. The following result includes an explicit collineation in the regular hyperoval case that makes it clear that all ovoids of the fan other than the one consisting of points of the form $((a, c))$ are in the same orbit of the collineation group of $\mathcal{S}(\Omega^-)$. Moreover, since the nucleus of the regular hyperoval is fixed by all collineations of $PG(3, q)$ that stabilize the hyperoval, in this case the ovoid of points $((a, c))$ is stabilized by all collineations of $\mathcal{S}(\Omega^-)$.

Using the same technique as above to find a collineation of $\mathcal{S}(\Omega^-)$ when the hyperoval contains a translation hyperoval, we find the following collineation.

Theorem 5.1.3 *Let $\alpha \in Aut(F)$ be an automorphism of maximal order so that (α, id) is an admissible pair. For $b, d \in F$ with $d \neq 0$, the map $\theta(b, d)$ given below is a collineation of $\mathcal{S}(\Omega^-)$.*

$$\theta(b,d) : [m,v,w] \mapsto [m, mb+vd, m^\alpha b + wd];$$

$$\theta(b,d) : (a,b,c) \mapsto (ad+b, bd, cd);$$

$$\theta(b,d) : (m,v) \mapsto (m, mb+vd);$$

$$\theta(b,d) : ((m,w)) \mapsto ((m, m^\alpha b + wd)).$$

When $\alpha \notin \{2, \frac{1}{2}\}$, so the hyperoval does not contain a conic, the full collineation group of $\mathcal{S}(\Omega^-)$ leaves invariant each of the two special ovoids with points of the forms (a,b) and $((a,c))$, respectively, even though all the ovoids in the fan are pivotal for the fan, and the other q members of the fan are all in the same orbit, on which the group of collineations acts doubly transitively.

5.2 Recapitulation

Start with the same hypotheses as in the first paragraph of the previous subsection.

It is clear that any collineation of $PG(3,q)$ that leaves invariant the q-arc Ω^- (and hence also leaves invariant the unique hyperoval Ω^+ containing Ω^-) must induce a collineation of $\mathcal{S}$ that leaves invariant each of the two special pivotal ovoids $\mathcal{O}_{AB}$ and $\mathcal{O}_{BA}$, or interchanges them. When these two ovoids are the only regular ovoids (and hence the only pivotal ovoids for the fan $\mathcal{M}$) of $\mathcal{S}$, it is clear that the collineations of $\mathcal{S}$ are precisely those induced by collineations of the ambient $PG(3,q)$ leaving Ω^- invariant. And there are many examples for which both pivotal ovoids are in the same orbit, i.e., for which A and B are in the same orbit of the stabilizer of Ω^-. On the other hand, in the previous subsection we saw that even when there are more than two (and hence $q+2$) pivotal ovoids for the fan $\mathcal{M}$, in all but one case both $\mathcal{O}_{AB}$ and $\mathcal{O}_{BA}$ are left invariant and all collineations of $\mathcal{S}$ are induced by collineations of $PG(3,q)$. The one exception is when the hyperoval Ω^+ contains a conic and one of A and B is the nucleus of Ω^+. In this case one of A, B is fixed by the full collineation group of $\mathcal{S}$ and the remaining $q+1$ ovoids of the fan $\mathcal{M}$ are all in the same orbit. In only this case are there collineations of $\mathcal{S}$ not induced by collineations of the ambient projective space.

References

[AS69] R. W. Ahrens and G. Szekeres, On a combinatorial generalization of 27 lines associated with a cubic surface, *J. Austral. Math. Soc.*, 10 (1969), 485 – 492.

[Bi78a] A. Bichara, Caratterizzazione dei sistemi rigati immersi in $A_{3,q}$, *Riv. Mat. Univ. Parma*, (4) 4 (1978), 277 – 290.

[Bi78b] A. Bichara, Sistemi rigati immersi in uno spazio combinatorio, *Sem. Geom. Comb. Univ. Roma*, 6 (1978).

[BMS78] A. Bichara, F. Mazzocca and C. Somma, Sulla classificazione dei sistemi rigati immersi in $AG(3,q)$, *Sem. Geom. comb. Univ. Roma* 9 (1978).

[BMS80] A. Bichara, F. Mazzocca and C. Somma, On the classification of generalized quadrangles in finite affine spaces $AG(3,2^h)$, *Boll. Un. Mat. Ital.*, (5) 17-B (1980), 298 – 307.

[De68] P. Dembowski, *Finite Geometries*, Springer Verlag, 1968.

[DS87] M. De Soete, Characterizations of $P(Q(4,q),L)$, *J. Geometry*, 29 (1987), 50 – 60.

[DST84] M. De Soete and J. A. Thas, A characterization theorem for the generalized quadrangle $T_2^*(O)$ of order $(s,s+2)$, *Ars Comb.*, 17 (1984), 225 – 242.

[DST86a] M. De Soete and J. A. Thas, R-regularity and characterizations of the generalized quadrangle $P(W(s),(\infty))$, *Ann. Discrete Math.*, 30 (1986), 171 – 184.

[DST86b] M. De Soete and J. A. Thas, Characterizations of the generalized quadrangles $T_2^*(O)$ and $T_2(O)$, *Ars Comb.*, 22 (1986), 171 – 186.

[DST87] M. De Soete and J. A. Thas, A characterization of the generalized quadrangle $Q(4,q)$, q odd, *J. Geometry*, 28 (1987), 57 – 79.

[EP72] M. M. Eich and S. E. Payne, Nonisomorphic block designs derived from generalized quadrangles, *Rendiconti della Accademia Naz. dei Lincei*, Serie VIII, vol. LII(1972), 893 – 902.

[GJS94] T. Grundhöfer, M. Joswig and M. Stroppel, Slanted symplectic quadrangles, *Geometriae Dedicata* 49 (1994), 143 – 154.

[Ha71] M. Hall, Jr., Affine generalized quadrilaterals, *Studies in Pure Math.*, (ed. L. Mirsky), Academic Press (1971), 113 – 116.

[Ma97] G. Mason, Some generalized quadrangles in characteristic 2, in *Mostly Finite Geometries*, (ed. N. L. Johnson), Marcel Dekker (1997), 343 – 352.

[Pa71a] S. E. Payne, Nonisomorphic generalized quadrangles, *J. Algebra*, 18 (1971), 201 – 212.

[Pa71b] S. E. Payne, The equivalence of certain generalized quadrangles, *J. Combin. Theory*, 10 (1971), 284 – 289.

[Pa72a] S. E. Payne, Generalized quadrangles as amalgamations of projective planes, *J. Algebra*, 22 (1972), 120 – 136.

[Pa72b] S. E. Payne, Quadrangles of order $(s-1, s+1)$, *J. Algebra*, 22 (1972), 97 – 119.

[Pa75] S. E. Payne, Skew-translation generalized quadrangles, *Congressus Numerantium*, XIV (1975), 485 – 504.

[Pa76] S. E. Payne, Translation generalized quadrangles of order eight, *Congressus Numerantium*, XVII (1976), 469 – 474.

[Pa77] S. E. Payne, Generalized quadrangles with symmetry II, *Simon Stevin*, 50 (1977), 209 – 245.

[Pa85] S. E. Payne, Hyperovals and generalized quadrangles, in *Finite Geometries* (eds. C. A. Baker and L. M. Batten), Marcel Dekker, 1985, 251 – 270.

[Pa86] S. E. Payne, Hyperovals yield many GQ, *Simon Stevin*, 60 (1986), 211 – 225.

[Pa90] S. E. Payne, The generalized quadrangle with $(s, t) = (3, 5)$, *Congressus Numerantium*, 77 (1990), 5 – 29.

[Pa92] S. E. Payne, Generalized quadrangles derived from groups and having $s = t + 2$, *The Journal of Combinatorial Mathematics and Combinatorial Computing*, 12 (1992), 65 – 76.

[PT84] S. E. Payne and J. A. Thas, *Finite generalized quadrangles*, Pitman, 1984.

[So82] C. Somma, Generalized quadrangles with parallelism, *Ann. Discrete Math.*, 14 (1982), 265 – 282.

[Th87] J. A. Thas, Complete arcs and algebraic curves in $PG(2, q)$, *J. Algebra*, 106 (1987), 451 – 464.

[TvM9?] J. A. Thas and H. van Maldeghem, Lax embedding of generalized quadrangles in finite projective spaces, preprint, 1997.

[VMB93] H. van Maldeghem and I. Bloemen, Generalized hexagons as amalgamations of generalized quadrangles, *Europ. J. Combinatorics*, 14 (1993), 593 – 604.