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Abstract: In this paper we begin with a standard form of the linear programming problem. We replace each constant in the problem with a fuzzy number. We then reformat the objective and constraints into an unconstrained fuzzy function by penalizing the objective for possible constraint violations. The range of this fuzzy function lies in the space of fuzzy numbers. The objective is then redefined as optimizing the expected midpoint of the image of this fuzzy function. We show that this objective defines a concave function which, therefore, can be maximized globally. We present an algorithm for finding the optimum.

Keywords: Fuzzy number, Fuzzy function, Possibility Distribution, Linear Programming

1. Introduction

In constrained optimization problems where uncertainty is characterized using possibility distributions, it may be unavoidable and/or advantageous to consider solutions that have a non-zero possibility of violating one or more of the constraints. This can be done by considering the cost of a constraint violation in the problem formulation. In the following discussion we examine such a formulation of the linear programming problem where the constant terms in the problem may not be known precisely. To incorporate this type of uncertainty into the model each constant in the problem is replaced with a fuzzy number (fuzzy sets on R with all $\alpha - cuts$ closed intervals) where we interpret fuzzy numbers as possibility distributions (see [11]). Next we incorporate the possibility of a constraint violation into the model by replacing each constraint with a term in the objective function that reduces the objective by the cost of the violation. From this an unconstrained fuzzy function optimization problem arises. The image of this fuzzy function is a fuzzy number (see [5]). Thus our objective is to, in some sense, find the optimal fuzzy number. To do this we adopt a view of possibility distributions as cumulative subjective probability distributions ([6]) and assume that our utility for a given interval of possible values is its midpoint. Thus we calculate the expected midpoint of a fuzzy number and use this as our basis of comparison. Of course the midpoint is simply one of several ways to obtain an optimal fuzzy number. From this point of view, which is the approach of this paper, the objective is to maximize the expected midpoint of the fuzzy number that represents the possible outcomes for a given action. A gradient ascent algorithm for finding the solution to this problem is developed.

2. Notation

In this paper we will interpret membership degrees as levels of possibility. Throughout this paper a fuzzy subset is denoted by the symbol $\sim$ over a letter. For example if X is a set, then $\tilde{x}$ may be used to denote a fuzzy subset of X and $\tilde{x}_\alpha$ will denote the α-level of possibility ($\alpha-$cut) for $\tilde{x}$, i.e. it is the crisp set $\{x \mid \mu_{\tilde{x}}(x) \geq \alpha \}$ for $\alpha \in (0,1]$ and cls$\{x \mid \mu(x)>0 \}$ for $\alpha=0$ where cls(A) denotes the closure of set A and $\mu_{\tilde{x}}$:X$\rightarrow$[0,1] is the membership function of $\tilde{x}$. If $\tilde{x}$ is a fuzzy number, we will let $\tilde{x}_\alpha=[x_\alpha^-, x_\alpha^+]$ be the closed interval which is the α-cut for $\tilde{x}$ where x_α^- and $x_\alpha^+$are its left and right endpoints respectively. We note that the real line is a subset of the space of fuzzy numbers, i.e. a real number is a fuzzy number where every α-cut is equal to the number. For our purposes here, if x is a vector of real numbers, then $\tilde{x}$ will be a vector of fuzzy numbers.

Let $\widetilde{A}$ denote a matrix of fuzzy numbers, i.e. each coefficient, $\widetilde{a}_{ij}$, is a fuzzy number with $\alpha - cut$ $(\widetilde{a}_{ij})_\alpha = [(\widetilde{a}_{ij})_\alpha^-, (\widetilde{a}_{ij})_\alpha^+]$. We let $\widetilde{A}_\alpha$ be the matrix whose coefficients are the $\alpha - cuts$ of the coefficients of $\widetilde{A}$, i.e. it is a matrix of closed intervals. Let $\widetilde{A}_\alpha^-$ and $\widetilde{A}_\alpha^+$ denote the matrices whose coefficients are the left and right endpoints of the entries of $\widetilde{A}_\alpha$, e.g. the ij'th entry of $\widetilde{A}_\alpha^+$ is $(\widetilde{a}_{ij})_\alpha^+$. We will let $(\widetilde{A}_i)_\alpha^+$ and $(\widetilde{A}^j)_\alpha^+$ denote the i'th row and the j'th column of matrix $\widetilde{A}_\alpha^+$ respectively.

Finally we define $EA(\tilde{x})=\frac{1}{2}\int_0^1 (x_\alpha^- + x_\alpha^+)d\alpha$, the expected midpoint of fuzzy number $\tilde{x}$. The cumulative subjective probability interpretation of possibilities is used for this paper (see [6]).

3. Problem Formulation

The following is the form of the linear programming problem considered herein:

Maximize $c^T x$
Subject to $Ax \leq b$
$x \geqq 0$

Where c and $x \in R^n$ and $b \in R^m$.

If there are uncertainties about any of the components of A and/or b the possibility of a constraint violation cannot be avoided unless the problem restricts x to the worst/best possible case (optimistic, pessimistic lp - see [13]). To take into account the possibility of a constraint violation each constraint is replaced with a penalty term in the objective function together with the corresponding uncertainty in the coefficients. The actual penalty term will be problem dependent though its generic representation is developed and analyzed. For this paper we will treat constraints as resources and assume that if a resource is exceeded it can be replenished at a cost that is linear with respect to the amount of the violation. The incorporation of truly hard constraints is easily handled within our approach but will not be considered further in this paper. The exception to this is that $x \geq 0$ is considered a crisp constraint. In other words, we will replace the following constraint:

$A_i x \leq b_i$

by subtracting the following penalty term from the objective function,

$d_i \max(0, A_i x - b_i)$

where each $d_i > 0$ is the cost per unit of violation of the right-hand side value. The objective function now takes the following form, where the maximum is taken component wise and $d \in R^m$ and each component is positive:

$f(x) = c^T x - d^T \max(0, Ax - b)$

Now we are in a position to replace each component of c and b and each coefficient of A with a fuzzy number to get the following:

$\tilde{f}(x) = \tilde{c}^T x - \tilde{d}^T \max(0, \tilde{A}x - \tilde{b})$

where max is handled component wise using the extension principle (see [10]). If we restrict x to a compact subset R^n, then $\tilde{f}$ defines a fuzzy function. It is shown in [5] that the image at any vector x, $\widetilde{f}(x)$, is a fuzzy number. This fuzzy number is completely characterized by it's $\alpha - cuts$:

$$\widetilde{f}(x)_\alpha = \left\{c^T x - d^T \max(0, Ax - b) \mid c, d, A, b \in \widetilde{c}_\alpha, \widetilde{d}_\alpha, \widetilde{A}_\alpha \text{ and } \widetilde{b}_\alpha \text{ respectively}\right\}$$

This fuzzy number provides the possibility distribution for the outcome of taking action x. We wish to find the most favorable possibility distribution over all possible actions. As we discussed in our paper [6], using the expected midpoint of the fuzzy number as the basis of comparing two fuzzy numbers makes sense for a decision maker whose utility for an interval of possible values is the midpoint of the interval. This provides the additional advantage of turning the range space into a Banach space for most applications (see [6]). Our new optimization problem is as follows:

$$\text{maximize } \mathrm{EA}(\tilde{\mathrm{f}}(\mathrm{x})) = \mathrm{EA}(\tilde{\mathrm{c}}^{\mathrm{T}}\mathrm{x} - \tilde{\mathrm{d}}^{\mathrm{T}}\max(0,\tilde{\mathrm{A}}\mathrm{x}-\tilde{\mathrm{b}}))$$

This objective function requires the α-cuts of our fuzzy number $\tilde{\mathrm{f}}(\mathrm{x})$ given the definition of expected average. But this is straightforward given the alpha cuts of our fuzzy coefficients because of the linearity of our original problem and the nonnegativity of x. Therefore,

$$\tilde{\mathrm{f}}_\alpha^+(\mathrm{x}) = (\tilde{\mathrm{c}}_\alpha^+)^{\mathrm{T}}\mathrm{x} - (\tilde{\mathrm{d}}_\alpha^-)^{\mathrm{T}}\max(0,\tilde{\mathrm{A}}_\alpha^-\mathrm{x}-(\tilde{\mathrm{b}}_\alpha^+))$$
$$\tilde{\mathrm{f}}_\alpha^-(\mathrm{x}) = (\tilde{\mathrm{c}}_\alpha^-)^{\mathrm{T}}\mathrm{x} - (\tilde{\mathrm{d}}_\alpha^+)^{\mathrm{T}}\max(0,\tilde{\mathrm{A}}_\alpha^+\mathrm{x}-(\tilde{\mathrm{b}}_\alpha^-))$$

These two numbers define the right and left end-points of the α-cut of the fuzzy function evaluated at x where $\tilde{\mathrm{f}}_\alpha^+(\mathrm{x})$ is called the **optimistic** value of $\widetilde{f}(x)$ at possibility level α and $\tilde{\mathrm{f}}_\alpha^-(\mathrm{x})$ is called the **pessimistic** value. The modified problem now becomes:

$$\text{Maximize } \mathrm{EA}(\tilde{\mathrm{f}}(\mathrm{x})) = \tfrac{1}{2}\int_0^1 (\tilde{\mathrm{f}}_\alpha^-(\mathrm{x})+\tilde{\mathrm{f}}_\alpha^+(\mathrm{x}))d\alpha =$$

$$\frac{1}{2}\int_0^1 ((\tilde{c}_\alpha^-)^{\mathrm{T}}x + (\tilde{c}_\alpha^+)^{\mathrm{T}}x - (\tilde{d}_\alpha^-)^{\mathrm{T}}max(0, \tilde{A}_\alpha^- x - (\tilde{b}_\alpha^+)) - (\tilde{d}_\alpha^+)^{\mathrm{T}}max(0, \tilde{A}_\alpha^+ x - (\tilde{b}_\alpha^-))d\alpha \tag{3.1}$$

4. Properties of the Fuzzy Optimization Problem

Theorem 4.1. *The fuzzy optimization problem as defined by 3.1 is concave.*

Proof:

To see that mapping $EA(\tilde{f}(x)):R^n \rightarrow R$ defined above is concave only the terms involving the maximum operator need be considered. Let $\beta \in (0,1)$, x and $y \in R^n$ and A_i a row of matrix A. Due to the properties of the integral all that needs to be shown is that

$\max(0, A_i^T(\beta x+(1-\beta)y)-b) \leq \beta\max(0,A_i^T x-b)+(1-\beta)\max(0,A_i^T x-b)$.

Let $A_i^T x=z$ and $A_i^T y=w$. Then the above is equivalent to

$\max(0,\beta z+(1-\beta)w-b) \leq \beta\max(0,z-b)+(1-\beta)\max(0,w-b)$

If z,w>b then all terms are greater than zero and equality holds. If z,w<b then all terms are less than zero so the maximum is zero and again equality holds. Assume without loss of generality that z>b and w<b. Then

L.H.S. $= \max(0,\beta z+(1-\beta)w-b) \leq \max(0,\beta z+(1-\beta)b-b)=\beta(z-b)$ = R.H.S. □.

Since the function is concave, a solution should be obtainable using a gradient ascent technique if the problem is bounded. A test for boundedness of 3.1 is as follows:

Theorem 4.2. *If the union of the feasible sets for all possible crisp formulations of the original linear programming problem is bounded then the fuzzy optimization problem is bounded if and only if for all j= 1 to n*

$$EA(\tilde{c}_j) \leq \frac{1}{2}\int_0^1 \left[(\tilde{d}_\alpha^-)^T(\widetilde{A}^j)_\alpha^- + (\tilde{d}_\alpha^+)^T(\widetilde{A}^j)_\alpha^+\right] d\alpha = EA(\tilde{d}^T\widetilde{A}^j)$$

Proof:

If the union of all possible formulations are bounded then we can focus on the objective function for those x that are outside of this union. For this region of R^n each maximum in equation. (3.1) is $\widetilde{A}x - \widetilde{b}$ and equation (3.1) becomes

$$\frac{1}{2}\int_0^1 ((\tilde{c}_\alpha^-)^T x + (\tilde{c}_\alpha^+)^T x - (\tilde{d}_\alpha^-)^T(\tilde{A}_\alpha^- x - (\tilde{b}_\alpha^+)) - (\tilde{d}_\alpha^+)^T(\tilde{A}_\alpha^+ x - (\tilde{b}_\alpha^-))d\alpha$$

Thus

$$\partial(EA(\tilde{f}(x)))/\partial x_j = \frac{1}{2}\int_0^1 (\tilde{c}_j)_\alpha^- + (\tilde{c}_j)_\alpha^+ - (\tilde{d}_\alpha^-)^T(\tilde{A}^j)_\alpha^- - (\tilde{d}_\alpha^+)^T(\tilde{A}^j)_\alpha^+ d\alpha \quad (4.1)$$

and the condition of the theorem follows by requiring that the gradient be nonpositive.□

This theorem implies that the partial with respect to x_j is negative when all constraints are fully violated. This means that the expected increase in the original objective function for an increase in variable x_j $(EA(\tilde{c}_j))$ must be less than the expected increase in cost of replenishing the resources needed to obtain the increase $(EA(\tilde{d}^T\tilde{A}^j))$.

We can use the following theorem to find a reasonable initial point. The absolute (crisp) upper bound on x is given in the next theorem.

Theorem 4.3. *If the fuzzy optimization problem is bounded, then* $\forall j = 1, n$

$$x_j \leq \max\left\{(b_i)_0^+/(a_{ij})_0^- \mid (a_{ij})_0^- \neq 0, i = 1, m\right\}$$

Proof:

This is an immediate consequence of the prior theorem since if x_j does not satisfy this condition then the maximum operator in equation (3.1) disappears and the partial at x_j as given by equation (4.1) will be negative.□

5. A Gradient Ascent Algorithm

This section provides the details of a line search algorithm for finding the maximum of the fuzzy optimization problem. The first step in the algorithm determines if any of the components of the terms $\max(0, \tilde{A}_\alpha^- x - (\tilde{b}_\alpha^+))$ and $\max(0, \tilde{A}_\alpha^+ x - (\tilde{b}_\alpha^-))$ become active for some $\alpha \in (0, 1)$. The second step of the algorithm calculates the values of these α's. The third step utilizes the calculated α's to determine the gradient of the objective function. This gradient is then used as the direction in which to search for the optimal value.

Step One

The objective of this step is to determine which constraints are violated at some $\alpha - cut$ for a given x. Let the set Ω consisting of the indices for the constraints that will be violated for some $\alpha \in (0, 1)$ using the pessimistic values of $\tilde{A}$ and $\tilde{b}$. Let Ψ consist of the indices for the constraints that are violated for all α using the pessimistic values. Let sets Γ and Λ be the same tests but using the optimistic values of $\tilde{A}$ and $\tilde{b}$. Formally, these sets are defined as follows:

$$\Omega = \left\{i \mid (\tilde{A}_i)_0^+ x \geq (\tilde{b}_i)_0^-) \text{ and } (\tilde{A}_i)_1^+ x < (\tilde{b}_i)_1^-)\right\},$$

$$\Psi = \left\{i \mid (\tilde{A}_i)_1^+ x \geq (\tilde{b}_i)_1^-)\right\},$$

$\Gamma=\left\{i \mid (\widetilde{A}_i)_1^- x \geq (\tilde{b}_i)_1^+) \text{ and } (\widetilde{A}_i)_0^- x<(\tilde{b}_i)_0^+)\right\}$ and
$\Lambda=\left\{i \mid (\widetilde{A}_i)_0^- x \geq (\tilde{b}_i)_0^+)\right\}$.

Step Two

For each of the constraints that will be violated for some $\alpha \in (0,1)$ we identify the α at which the constraint is first violated:

For each i$\in \Omega$ let α_i^+ solve $((\widetilde{A}_i)^+_{\alpha_i^+})^T x=(\tilde{b}_i)^-_{\alpha_i^+})$, and

for each i$\in \Gamma$, let α_i^- solve $((\widetilde{A}_i)^-_{\alpha_i^-})^T x=(\tilde{b}_i)^+_{\alpha_i^-})$.

Step Three

Calculate the gradient of ea($\tilde{f}$(x)) by the following formula,

$$\partial(\mathrm{ea}(\tilde{f}(x)))/\partial x_j=\mathrm{ea}(\tilde{c}_j)$$
$$-\textstyle\sum_{i\in\Omega} \frac{1}{2} \int_0^{\alpha_j^+} (\tilde{d}_i)_\alpha^+(\tilde{a}_{ij})_\alpha^+ d\alpha$$
$$-\textstyle\sum_{i\in\Psi} \frac{1}{2} \int_0^1 (\tilde{d}_i)_\alpha^+(\tilde{a}_{ij})_\alpha^+ d\alpha$$
$$-\textstyle\sum_{i\in\Gamma} \frac{1}{2} \int_{\alpha_j^-}^1 (\tilde{d}_i)_\alpha^-(\tilde{a}_{ij})_\alpha^- d\alpha$$
$$-\textstyle\sum_{i\in\Lambda} \frac{1}{2} \int_0^1 (\tilde{d}_i)_\alpha^-(\tilde{a}_{ij})_\alpha^- d\alpha$$

Step Four

Test gradient for sufficiently close to zero.
If sufficiently close, stop otherwise continue.

Step Five

Line search in the direction of the gradient.
Return to step one.

The above algorithm requires special handling of the constraint violation due to the use of the maximum operator in the integral. Alternatively, max(0,x) could be replaced by

$$\frac{\sqrt{x^2+\epsilon^2}+x}{2}$$

where ϵ is a very small constant.

Appendix B provides formulas for implementing the gradient ascent algorithm when the coefficients of the original linear programming problem are replaced by trapezoidal fuzzy numbers (see [10]).

6. Example

The following crisp problem is from Leunberger ([16]).

Maximize	$2x_1+x_2$
Subject to	$x_1+\frac{8}{3}x_2 \leq 4$
	$x_1+x_2 \leq 2$
	$2x_1 \leq 3$
	$x_1,x_2 \geq 0$

The solution to the crisp problem is 3.5 at (1.5,.5).

Assume that the penalty for violating the three constraints are 3,2,3 per unit of violation respectively. We replace each term in the problem with the triangular fuzzy number with $\alpha - cut$ as follows:

$\tilde{w}_\alpha = [w - .5 + .5\alpha, w + .5 - .5\alpha]$.

Thus we replace the number 2 with the fuzzy number $\tilde{2}$ with $\alpha - cut$ $\tilde{2}_\alpha = [1.5 + .5\alpha, 2.5 - .5\alpha]$. We interpret this to mean that the probability is α that $\tilde{2}_\alpha$ contains the range of possible values for $\tilde{2}$. For example, the probability is .5 that the range of possible values for $\tilde{2}$ is a subset of $[1.75, 2.25]$. With these replacements our penalized fuzzy function is:

$$\begin{aligned}
&\tilde{f}(x) = \tilde{2}x_1 + \tilde{1}x_2 \\
&-\tilde{3}\max[0, \tilde{1}x_1 + \tilde{\tfrac{8}{3}}x_2 - \tilde{4}] \\
&-\tilde{2}\max[0, \tilde{1}x_1 + \tilde{1}x_2 - \tilde{2}] \\
&-\tilde{3}\max[0, \tilde{2}x_1 - \tilde{3}].
\end{aligned}$$

Our reformulated problem becomes:

Maximize $\frac{1}{2}\int_0^1 \left[\tilde{f}_\alpha^+(x) + \tilde{f}_\alpha^-(x)\right] d\alpha$ where

$$\begin{aligned}
&\tilde{f}_\alpha^+(x) = (2.5 - .5\alpha)x_1 + (1.5 - .5\alpha)x_2 \\
&-(2.5 + .5\alpha)\max[0, (.5 + .5\alpha)x_1 + (\tfrac{8}{3} - .5 + .5\alpha)x_2 - (4.5 - .5\alpha)] \\
&-(1.5 + .5\alpha)\max[0, (.5 + .5\alpha)x_1 + (.5 + .5\alpha)x_2 - (2.5 - .5\alpha)] \\
&-(2.5 + .5\alpha)\max[0, (1.5 + .5\alpha)x_1 - (3.5 - .5\alpha)]
\end{aligned}$$

and

$$\begin{aligned}
&\tilde{f}_\alpha^-(x) = (1.5 + .5\alpha)x_1 + (.5 + .5\alpha)x_2 \\
&-(3.5 - .5\alpha)\max[0, (1.5 - .5\alpha)x_1 + (8/3 + .5 - .5\alpha)x_2 - (3.5 + .5\alpha)] \\
&-(2.5 - .5\alpha)\max[0, (1.5 - .5\alpha)x_1 + (1.5 - .5\alpha)x_2 - (1.5 + .5\alpha)] \\
&-(3.5 - .5\alpha)\max[0, (2.5 - .5\alpha)x_1 - (2.5 + .5\alpha)].
\end{aligned}$$

The image of this fuzzy function at (1.5,.5), a fuzzy number, is shown in Figure 6.1. The expected midpoint of this fuzzy number, EA($\tilde{f}(1.5, .5)$), is 1.5192. This is not the optimal value of the fuzzy optimization problem. The optimal value of the

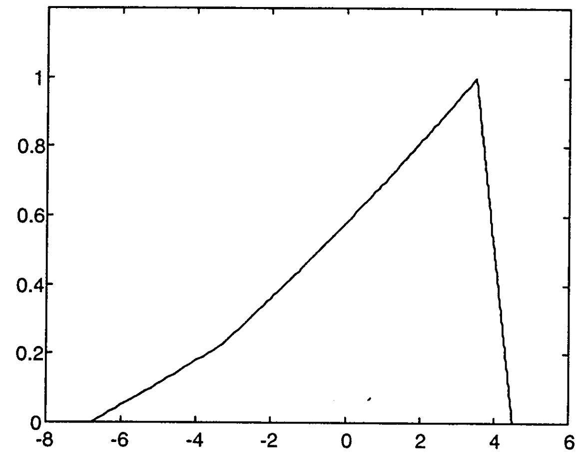

Figure 6.1: Image of the fuzzy function at (1.5,.5).

fuzzy optimization problem is found at (1.1,.4372) where $\mathrm{EA}(\widetilde{f}(1.1,.4372)) = 2.2794$ (see appendix A for the details on the implementation of the ascent algorithm for this problem).

Figure 6.2 shows the fuzzy number $\widetilde{f}(1.5,.5)$ and the fuzzy number $\widetilde{f}(1.1,.4372)$ which is optimal with respect to the EA functional. From Figure 6.2 you can see the trade-off that is evident in this formulation of fuzzy programing. Recall that the optimistic values of our fuzzy function are the right hand sides of each α-level of the fuzzy number image and the pessimistic values are the left hand sides. The optimistic values of $\widetilde{f}(1.1,.4372)$ are less than the optimistic values of $\widetilde{f}(1.5,.5)$ but this is more than offset by increases in the pessimistic values. Using the view of possibility levels as cumulative subjective probabilities one can also state that there is only a 10% subjective probability that a negative result is possible at (1.1,.4372) compared to a 60% subjective probability for (1.5,.5).

7. Conclusion

A possibilistic formulation of the fuzzy linear programming problem using penalty methods is possible and solvable. This formulation allows the user to incorporate model uncertainty into the solution obtained from the model. It also provides a

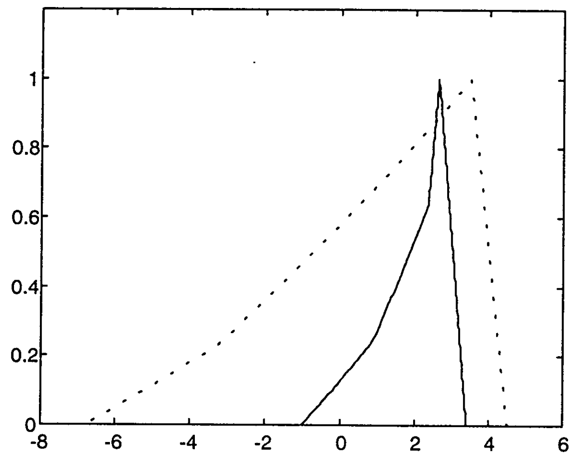

Figure 6.2: Image of the fuzzy function at (1.5,.5) and at the optimal point (1.1, .4372).

possibility distribution showing the fuzzy set of possible outcomes for the action taken.

8. Appendix A

The test for boundedness and the details of the application of the gradient ascent algorithm to our example is presented here. We will need the following for our algorithm:

$$A_0^- = \begin{bmatrix} .5 & 2.1667 \\ .5 & .5 \\ 1.5 & 0 \end{bmatrix} \quad A_1^{+\text{ or -}} = \begin{bmatrix} 1 & 2.6667 \\ 1 & 1 \\ 2 & 0 \end{bmatrix} \quad A_0^+ = \begin{bmatrix} 1.5 & 3.1167 \\ 1.5 & 1.5 \\ 2.5 & 0 \end{bmatrix}$$

$$b_0^- = \begin{bmatrix} 3.5 \\ 1.5 \\ 2.5 \end{bmatrix} \quad b_1^{+\text{ or }-} = \begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix} \quad b_0^+ = \begin{bmatrix} 4.5 \\ 2.5 \\ 3.5 \end{bmatrix}$$

$$d_0^- = \begin{bmatrix} 2.5 \\ 1.5 \\ 2.5 \end{bmatrix} \quad d_1^{+\text{ or }-} = \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix} \quad d_0^+ = \begin{bmatrix} 3.5 \\ 2.5 \\ 3.5 \end{bmatrix}$$

Test for Boundedness

We first test to see if our problem is bounded:

For i=1,

ea($\widetilde{c}_1$) = 2 and

$\frac{1}{2}\int_0^1 \left[(\tilde{d}_\alpha^-)^T(\widetilde{A}^1)_\alpha^- + (\tilde{d}_\alpha^+)^T(\widetilde{A}^1)_\alpha^+\right] d\alpha =$

$$\frac{1}{2}\int_0^1 \left(\begin{array}{c} \left(2.5+.5\alpha \quad 1.5+.5\alpha \quad 2.5+.5\alpha \right) \begin{pmatrix} .5+.5\alpha \\ .5+.5\alpha \\ 1.5+.5\alpha \end{pmatrix} + \\ \left(3.5-.5\alpha \quad 2.5-.5\alpha \quad 3.5-.5\alpha \right) \begin{pmatrix} 1.5-.5\alpha \\ 1.5-.5\alpha \\ 2.5-.5\alpha \end{pmatrix} \end{array} \right) d\alpha =$$

$\frac{1}{2}\int_0^1 (23.5 - 3.0\alpha + 1.5\alpha^2) d\alpha = 11.25$

Thus EA($\widetilde{c}_1$) $< \frac{1}{2}\int_0^1 \left[(\tilde{d}_\alpha^-)^T(\widetilde{A}^1)_\alpha^- + (\tilde{d}_\alpha^+)^T(\widetilde{A}^1)_\alpha^+\right] d\alpha.$

For i=2,

ea($\widetilde{c}_2$) = 1 and

$\frac{1}{2}\int_0^1 \left[(\tilde{d}_\alpha^-)^T(\widetilde{A}^2)_\alpha^- + (\tilde{d}_\alpha^+)^T(\widetilde{A}^2)_\alpha^+\right] d\alpha =$

$$\frac{1}{2}\int_0^1 \left(\begin{array}{c} \left(2.5+.5\alpha \quad 1.5+.5\alpha \quad 2.5+.5\alpha \right) \begin{pmatrix} 2.1667+.5\alpha \\ .5+.5\alpha \\ 0 \end{pmatrix} + \\ \left(3.5-.5\alpha \quad 2.5-.5\alpha \quad 3.5-.5\alpha \right) \begin{pmatrix} 2.6667-.5\alpha \\ 1.5-.5\alpha \\ 0 \end{pmatrix} \end{array} \right) d\alpha =$$

$\frac{1}{2}\int_0^1 (19.25 - 1.75\alpha + \alpha^2) d\alpha = 9.3542$

Thus our problem is bounded.

An Upper Bound on x_j

We compute an upper bound on x as follows:

$x_1 \leq \max\{4.5/.5, 2.5/.5, 3.5/1.5\} = 9.0$

and

$x_2 \leq \max\{4.5/2.1667, 2.5/.5\} = 5.0$

Application of the Ascent Algorithm

Let us start at x=(1.5,.5), where EA($\widetilde{f}$(x))=

$\int_0^1 \frac{1}{2}((2.5-.5\alpha)1.5 + (1.5-.5\alpha).5)d\alpha$

$-\int_0^1 \frac{1}{2}(2.5+.5\alpha)\max[0, (.5+.5\alpha)1.5 + (\frac{8}{3} - .5 + .5\alpha).5 - (4.5-.5\alpha)]d\alpha$

$-\int_0^1 \frac{1}{2}(1.5+.5\alpha)\max[0, (.5+.5\alpha)1.5 + (.5+.5\alpha).5 - (2.5-.5\alpha)]d\alpha$

$- \int_0^1 \frac{1}{2}(2.5 + .5\alpha) \max[0, (1.5 + .5\alpha)1.5 - (3.5 - .5\alpha)]d\alpha$
$+ \int_0^1 \frac{1}{2}((1.5 + .5\alpha)1.5 + (.5 + .5\alpha).5)d\alpha$
$- \int_0^1 \frac{1}{2}(3.5 - .5\alpha) \max[0, (1.5 - .5\alpha)1.5 + (\frac{8}{3} + .5 - .5\alpha).5 - (3.5 + .5\alpha)]d\alpha$
$- \int_0^1 \frac{1}{2}(2.5 - .5\alpha) \max[0, (1.5 - .5\alpha)1.5 + (1.5 - .5\alpha).5 - (1.5 + .5\alpha)]d\alpha$
$- \int_0^1 \frac{1}{2}(3.5 - .5\alpha) \max[0, (2.5 - .5\alpha)1.5 - (2.5 + .5\alpha)]d\alpha = 1.5192$

We will look for an improvement in this value by using the gradient ascent algorithm as follows:

Step One:

Determine Ω

$$A_0^+ x = \begin{bmatrix} 3.8334 \\ 3 \\ 3.75 \end{bmatrix} \geq^? b_0^- = \begin{bmatrix} 3.5 \\ 1.5 \\ 2.5 \end{bmatrix} \text{ and}$$

$$A_1^+ x = \begin{bmatrix} 2.8334 \\ 2 \\ 3 \end{bmatrix} <^? b_1^- = \begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix} \begin{matrix} \textit{yes} \\ \textit{no} \\ \textit{no} \end{matrix}$$

$\Omega = \{1\}$

Determine Ψ

$$A_1^+ x = \begin{bmatrix} 2.8334 \\ 2 \\ 3 \end{bmatrix} \geq^? b_1^- = \begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix} \begin{matrix} \textit{no} \\ \textit{yes} \\ \textit{yes} \end{matrix}$$

$\Psi = \{2, 3\}$

Determine Γ

$$A_1^- x = \begin{bmatrix} 2.8334 \\ 2 \\ 3 \end{bmatrix} \geq^? b_1^+ = \begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix} \text{ and}$$

$$A_0^- x = \begin{bmatrix} 1.8334 \\ 1 \\ 2.25 \end{bmatrix} <^? b_0^+ = \begin{bmatrix} 4.5 \\ 2.5 \\ 3.5 \end{bmatrix} \begin{matrix} \textit{no} \\ \textit{yes} \\ \textit{yes} \end{matrix}$$

$\Gamma = \{2, 3\}$

Determine Λ

$$A_0^- x = \begin{bmatrix} 1.8334 \\ 1 \\ 2.25 \end{bmatrix} \geq^? b_0^+ = \begin{bmatrix} 4.5 \\ 2.5 \\ 3.5 \end{bmatrix} \begin{matrix} \textit{no} \\ \textit{no} \\ \textit{no} \end{matrix}$$

$\Lambda = \emptyset$.

Step Two

For each i$\in\Omega$, determine α_j^+ by solving $((\widetilde{A}_i)^+_{\alpha_i^+})^T$x=$(\tilde{b}_i)^-_{\alpha_i^+}$):

i= 1

$(1.5-.5\alpha)1.5+(3.1667-.5\alpha).5=3.5+.5\alpha$, Solution is : $\alpha_1^+=.2222$

For each i$\in\Gamma$, determine α_i^- by solving $((\widetilde{A}_i)^-_{\alpha_i^+})^T$x=$(\tilde{b}_i)^+_{\alpha_i^+}$)

i= 2

$(.5+.5\alpha)1.5+(.5+.5\alpha).5=2.5-.5\alpha$, Solution is : $\alpha_2^-=1.0$

i= 3

$(1.5+.5\alpha)1.5=3.5-.5\alpha$, Solution is : $\alpha_3^-=1.0$

Step Three

Determine the gradient of EA($\widetilde{f}(x)$) :

$\partial(\mathrm{EA}(\tilde{f}(x)))/\partial x_1=2-\int_0^{.2222}\frac{1}{2}(3.5-.5\alpha)(1.5-.5\alpha)d\alpha$
$-\int_0^1\frac{1}{2}(2.5-.5\alpha)(1.5-.5\alpha)d\alpha-\int_0^1\frac{1}{2}(3.5-.5\alpha)(2.5-.5\alpha)d\alpha=-3.6363$

$\partial(\mathrm{EA}(\tilde{f}(x)))/\partial x_2=1-\int_0^{.2222}\frac{1}{2}(3.5-.5\alpha)(\frac{8}{3}+.5-.5\alpha)d\alpha$
$-\int_0^1\frac{1}{2}(2.5-.5\alpha)(1.5-.5\alpha)d\alpha=-1.6075$

We search for an improved solution at $(1.5,.5)+.05(-3.6363,-1.6075)=(1.3182,.4196)$ where EA($\widetilde{f}$(x))=

$\int_0^1\frac{1}{2}((2.5-.5\alpha)1.3182+(1.5-.5\alpha).4196)d\alpha$
$-\int_0^1\frac{1}{2}(2.5+.5\alpha)\max[0,(.5+.5\alpha)1.3182+(\frac{8}{3}-.5+.5\alpha).4196-(4.5-.5\alpha)]d\alpha$
$-\int_0^1\frac{1}{2}(1.5+.5\alpha)\max[0,(.5+.5\alpha)1.3182+(.5+.5\alpha).4196-(2.5-.5\alpha)]d\alpha$
$-\int_0^1\frac{1}{2}(2.5+.5\alpha)\max[0,(1.5+.5\alpha)1.3182-(3.5-.5\alpha)]d\alpha$
$+\int_0^1\frac{1}{2}((1.5+.5\alpha)1.3182+(.5+.5\alpha).4196)d\alpha$
$-\int_0^1\frac{1}{2}(3.5-.5\alpha)\max[0,(1.5-.5\alpha)1.3182+(8/3+.5-.5\alpha).4196-(3.5+.5\alpha)]d\alpha$
$-\int_0^1\frac{1}{2}(2.5-.5\alpha)\max[0,(1.5-.5\alpha)1.3182+(1.5-.5\alpha).4196-(1.5+.5\alpha)]d\alpha$
$-\int_0^1\frac{1}{2}(3.5-.5\alpha)\max[0,(2.5-.5\alpha)1.3182-(2.5+.5\alpha)]d\alpha=2.0648$

This is an improvement over x=(1.5,.5)

9. Appendix B

Following are formulas that can be used to implement the gradient ascent algorithm when all coefficients in the original linear programming problem are replaced by trapezoidal fuzzy numbers of the form (a_1,a_2,a_3,a_4) (see [10]).

Let $A1, A2, A3, A4$ be crisp matrices where the entries of A_i are the i'th elements of the trapezoidal numbers that make of the fuzzy numbers in $\widetilde{A}$. For example if each coefficient in $\widetilde{A}$ is the trapezoidal number (5,6,7,8) then A_2 will be a matrix with all entries equal to the number 6. We define B_i, C_i, and D_i for i=1 to 4 in the same way.

With this representation we have the following formulas for the α-levels of $\widetilde{A}$(the formulas for $\widetilde{B}, \widetilde{C}$, and $\widetilde{D}$ are identical):

$$\widetilde{A}_\alpha^- = A1 + (A2 - A1) * \alpha$$

and

$$\widetilde{A}_\alpha^+ = A4 + (A3 - A4) * \alpha$$

Let the A be an mxn matirx.

Calculating α_i^+ and α_i^-

```
For i=1 to m
  if ( A4(i,:)*x>B1(i) )&( A3(i,:)*x<B2(i) )
    α_i^+=(B1(i)-A4(i,:)*x)/( (A3(i,:)-A4(i,:))*x - (B2(i)-B1(i)) )
  elseif A3(i,:)*x>=B2(i)
    α_i^+=1
  else
    α_i^+=0
  end
  if ( A2(i,:)*x>B3(i) )&( A1(i,:)*x<B4(i) )
    α_i^-=( B4(i)-A1(i,:)*x )/( (A2(i,:)-A1(i,:) )*x + ( B4(i)-B3(i) ) )
  elseif A1(i,:)*x>=B4(i)
    α_i^-=0
  else
    α_i^-=1
  end
end
```

Calculating EA($\widetilde{f}$(x))

EA($\widetilde{f}(x)$) =1/4*(C1+C2+C3+C4)'*x -
$\sum_{i=1}^{m}$[1/2 * D1(i) * (A1(i,:)*x-B4(i))
+ 1/2* (D1(i) * ((A2(i,:)-A1(i,:)) *x + B4(i) - B3(i))
+ (D2(i)-D1(i)) * (A1(i,:) *x - B4(i)) *(1/2)

+ 1/2* ((D2(i)-D1(i)) * ((A2(i,:) -A1(i,:)) *x
+ B4(i) - B3(i))) * (1/3)
- (1/2 * D1(i) * (A1(i,:)*x-B4(i)) * α_i^-
+ 1/2* (D1(i) * ((A2(i,:)-A1(i,:)) *x + B4(i) - B3(i))
+ (D2(i)-D1(i)) * (A1(i,:) *x - B4(i))) *1/2 * $(\alpha_i^-)^2$
+ 1/2* ((D2(i)-D1(i)) * ((A2(i,:) -A1(i,:)) *x
+ B4(i) - B3(i))) * 1/3 * $(\alpha_i^-)^3$)
+ 1/2 * D4(i) * (A4(i,:)*x-B1(i)) * α_i^+
+ 1/2* (D4(i) * ((A3(i,:)-A4(i,:)) *x + B1(i) - B2(i))
+ (D3(i)-D4(i)) * (A4(i,:) *x - B1(i))) *1/2 * $(\alpha_i^+)^2$
+ 1/2* ((D3(i)-D4(i)) * ((A3(i,:) -A4(i,:)) *x
+ B1(i) - B2(i))) * 1/3 * $(\alpha_i^+)^3$]

Calculating $\partial(\mathbf{EA}(\tilde{\mathbf{f}}(\mathbf{x})))/\partial \mathbf{x}_j$

$\partial(\mathrm{EA}(\tilde{\mathrm{f}}(\mathrm{x})))/\partial \mathrm{x}_j$ = 1/4*(C1(j)+C2(j)+C3(j)+C4(j)) -
$\sum_{i=1}^{m}$[1/2 * D1(i) * A1(i,j)
+ 1/2* (D1(i) * (A2(i,j)-A1(i,j))
+ (D2(i)-D1(i)) * A1(i,j)) *(1/2)
+ 1/2* ((D2(i)-D1(i)) * (A2(i,j) -A1(i,j))) * (1/3)
- (1/2 * D1(i) * A1(i,j) * α_i^-
+ 1/2* (D1(i) * (A2(i,j)-A1(i,j))
+ (D2(i)-D1(i)) * A1(i,j)) * 1/2 * $(\alpha_i^-)^2$
+ 1/2* ((D2(i)-D1(i)) * (A2(i,j) -A1(i,j))) * 1/3 * $(\alpha_i^-)^3$)
+ 1/2 * D4(i) * A4(i,j) * α_i^+
+ 1/2* (D4(i) * (A3(i,j)-A4(i,j))
+ (D3(i)-D4(i)) * A4(i,j)) *1/2 * $(\alpha_i^+)^2$
+ 1/2* ((D3(i)-D4(i)) * (A3(i,j) -A4(i,j))) * 1/3 * $(\alpha_i^+)^3$]

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