LAVRENTIEV REGULARIZATION + RITZ APPROXIMATION = UNIFORM FINITE ELEMENT ERROR ESTIMATES FOR DIFFERENTIAL EQUATIONS WITH ROUGH COEFFICIENTS

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Abstract. We consider a parametric family of boundary value problems for the diffusion equation with the diffusion coefficient equal to a small constant in a subdomain. Such problems are not uniformly well-posed when the constant gets small. However, in a series of papers, Bakhvalov and Knyazev have suggested a natural splitting of the problem into two well-posed problems. Using this idea, we prove a uniform finite element error estimate for our model problem in the standard parameter-independent Sobolev norm. We consider a traditional finite element method with only one additional assumption, namely, that the boundary of the subdomain with the small coefficient does not cut any finite element.

One interpretation of our main theorem is in terms of regularization. Our FEM problem can be viewed as resulting from a Lavrentiev regularization and a Ritz–Galerkin approximation of a symmetric ill-posed problem. Our error estimate can then be used to find an optimal regularization parameter together with the optimal dimension of the approximation subspace.

Key words. Galerkin, Lavrentiev, Ritz, Tikhonov, jump coefficient, error estimate, finite elements, regularization

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1. Introduction. A particularly challenging class of problems arises with models described by partial differential equations (PDE's) with highly discontinuous coefficients. Many important physical problems are of this nature. In particular, they arise in the design and study of composite materials built from essentially different components, e.g. [9, 24, 7, 21, 15].

The *fictitious domain/embedding method* is another source of PDE's with highly discontinuous coefficients, cf., e.g., [25, 1, 12, 20]. In this method, the domain of the original boundary value problem is embedded into a larger one, where a new artificial boundary value problem is constructed. In the new, fictitious part of the domain the coefficients of PDE are chosen to be close to zero, if the original boundary condition is of Neumann type, or very large, in the Dirichlet case.

There are several difficulties associated with the numerical solution of PDE's with a large jump in the coefficients as the problems are not uniformly well-posed with respect to the jump. The most serious difficulty is that an approximation, e.g., by a finite element method (FEM), of the PDE's may be very inaccurate. There are two main reasons for that: the lack of smoothness of the solution and the jump in the coefficients.

It is well known, that solutions of problems with rough coefficients are generally nonsmooth, and then the usual finite element method cannot provide an accurate ap-

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proximation, see, e.g., [14]. One way of approximating a nonsmooth function involves using extra special trial functions, to augment the standard FEM space, in order to approximate the nonsmooth part of the function. Finding such trial functions will usually require some knowledge of the form of the singularity, which is not always available. Another possibility is to employ refined meshes in the areas of low regularity; these areas are often known in advance.

In this paper, however, the lack of smoothness in the solution is not our concern. We shall assume that the solution can somehow be approximated by a trial subspace in a suitable Sobolev space. In our FEM example, we shall assume, for simplicity, that the coefficients are piece-wise smooth and that the jumps in the coefficients appear across a smooth interface between subdomains. Standard assumptions on the right-hand side will allow us to conclude that our solution is piece-wise smooth and, thus, relatively well approximated even by traditional finite elements.

Here we instead address a second difficulty, attributable to the large jump in the coefficients, concentrating on a case of a small coefficient in a subdomain, which results in a problem which is not uniformly elliptic.

The standard approach of deriving a FEM error estimate for selfadjoint PDE problems is based on the fact that the FEM approximation is the best approximation in the energy norm. In our case of highly discontinuous coefficients, the energy norm is not uniformly equivalent (with respect to the jump) to a natural parameter-independent Sobolev norm. This standard approach will fail to give an error estimate in a non-weighted Sobolev space with a constant, independent of the jump. Therefore, energy norm based estimates are naturally considered for problems with highly discontinuous coefficients; there are also similar results based on a least squares technique, see [19], in weighted Sobolev spaces. Energy norm based error estimates are perfectly adequate when the Sobolev norm of the solution goes to infinity with the jump, in other words, when the solution is not uniformly smooth. As a simple example, we can consider a homogeneous Dirichlet problem for a diffusion equation with a small diffusion coefficient in a subdomain and with a fixed, parameter-independent right-hand side. Then, not even the H^1 norm of the solution is uniformly bounded, and there is little hope to obtain uniform error estimates in H^1 norm. However, the assumption that the right-hand side is independent of the jump is not always satisfactory from the physical point of view, e.g., it does not allow us to consider a cavity in a perforated domain as a limit case of a subdomain with a small, but positive, diffusion coefficient. If we instead assume that the right-hand side is also small in the subdomain where the diffusion coefficient is small, then the solution is uniformly bounded in a related parameter-independent Sobolev norm, in this case, the H^1 norm; see [3] for the exact formulation of such an assumption. It is known, see, e.g., [18, 3], that under such an assumption a problem with a cavity can indeed be treated as a limit of a problem with a vanishing coefficient in the corresponding subdomain. Equations determining homogenized properties of composite materials, e.g., [7, 21], provide an example of practical problems where our assumption is satisfied.

In such a case, the energy norm will clearly deteriorate in the subdomain where the coefficient is small, and energy norm-based error estimates do not provide much information on the accuracy in the subdomain. Our goal is to derive new error estimates, which would use the assumption that the amplitude of the right-hand side is consistent with the value of the coefficients in subdomains such that the solution is uniformly bounded in an appropriate parameter-independent Sobolev norm. We also want to take advantage of the assumption that the jump in the coefficients is "regular." While in our example the operator is not uniformly elliptic (with respect to the jump) and the problem is not uniformly well posed, we can nevertheless treat the problem as essentially well-posed because of our special assumptions on the discontinuity in coefficients.

Our approach does not cover truly degenerate elliptic equations, see, e.g., [17], with arbitrary discontinuous coefficients and/or ultimate failure of the ellipticity condition. For such equations, the use of the energy norm, or a properly weighted Sobolev norm seems vital.

As an application of our abstract result, we derive a FEM error estimate for the diffusion equation with a large jump in coefficients between subdomains.

Our main result, being stated for the case involving a small coefficient, can be trivially reformulated for the case of a large coefficient, and provides a uniform error estimate in the absence of locking. Thus, it can be applied, e.g., to estimate the error of a conforming FEM for displacements for linear elasticity equations in the incompressible limit, provided non-locking elements are used. Some problems with a large parameter that appeared in the penalty method can also be treated using our approach.

The technical part of the paper starts with some simple, but relevant, theoretical results for a parametric family of abstract symmetric operator equations with a jump in the coefficients described by a small parameter, see Section 2. Such problems are not uniformly well-posed, in the traditional sense, when the constant gets small. However, they can be split naturally into two well-posed problems as suggested in [3, 4]. Using the projection method for the original problem is equivalent to applying the projection method to these two problems separately.

By exploring this idea and analyzing the fine structure of the error, we prove, in Section 3, a new error estimate of the Ritz–Galerkin method applied to our operator equation. The estimate does not use the energy norm, and all the constants are explicitly displayed and shown to be independent of the parameter. Under the assumption that the amplitude of the corresponding right-hand side is small consistently with the parameter, our bound for the error is parameter independent.

It is interesting to note that our finite dimensional problem can be viewed as a result of using the Lavrentiev regularization – an analog of the Tikhonov regularization for symmetric problems – for the Ritz–Galerkin approximation of an ill-posed problem with a null-space, see, e.g., the recent books [26], pp. 162-173, and [8], Chapter 4. Our error estimate can be used to find the optimal regularization parameter together with the optimal dimension of the approximation subspace. A similar result is known in the regularization theory, cf. [22], pp. 75-78, but is based on different assumptions and techniques. However, the question of the choice of the regularization parameter for a general symmetric ill-posed problem is outside of the scope of the present paper.

In Section 4 we provide a simple example of our general theory. We apply our main result to estimate the error of the FEM approximation of a parametric family of boundary value problems for the diffusion equation with the diffusion coefficient equal to a small constant in a subdomain. For simplicity, we consider a traditional FEM with the only additional assumption that the boundary of the subdomain with the small coefficient does not cut any finite element. In such a setting our abstract estimate becomes a uniform FEM error estimate for the model problem in the standard parameter-independent Sobolev norm. For example, our theory provides an error estimate for the well known case of a self-intersecting interface with a constant independent of the jump, but dependent on the mesh size. Sufficient smoothness of the interface is important.

Finally, we recall that the numerical solution of the resulting algebraic system is also difficult as the matrix of the system is not uniformly well-posed with respect to the jump in coefficients. For a number of iterative methods, using preconditioners, the larger the jumps of the coefficients, the slower the convergence. However, it has been shown in the continuous case, cf. [2, 3, 6], that if a special initial guess is used, then the rate of convergence does not depend on the size of the jumps even if a standard preconditioned iterative method is used. A similar result has been established in [16, 4] for algebraic systems of linear equations with a symmetric coefficient matrix. Such methods are closely related to the *capacitance matrix methods*, see, e.g., [23, 11]. Many *domain decomposition methods* can also be used for the effective solution of problems with rough coefficients and can provide uniform convergence. A relationship of different methods of this sort has been discovered in [5].

2. An abstract equation. In a real Hilbert space, we consider an abstract linear system $(A+\omega I)u = f$ with a bounded symmetric nonnegative definite operator A, and a positive parameter $\omega, 0 < \omega \leq 1$, where I is the identity operator. The condition number of the operator $A+\omega I$ may tend to infinity as ω tends to zero; our system may not be uniformly well-posed. The operator A may have a non-zero null-space $\mathbf{Ker}(A)$.

In the present paper, we call a linear subset, which is *not necessarily closed*, a subspace. If such a linear subset is closed, we call it a *closed subspace*. We note that the kernel of a bounded operator is always a closed subspace.

The subspace $\mathbf{Im}(A)$, the range of A, and its closure, $\overline{\mathbf{Im}(A)}$, which is also the orthogonal complement to the kernel $\mathbf{Ker}(A)$, will play a key role in the present paper.

We first make a couple of simple, but important observations, based on results from [3, 4].

LEMMA 2.1. *The subspace* $\mathbf{Im}(A)$ *and its closure* $\overline{\mathbf{Im}(A)}$ *are invariant with respect to the operator* $A+\omega I$

LEMMA 2.2. *Let*

$$f = f_0 + f_1,\ f_0 \in \mathbf{Ker}(A),\ f_1 \in \overline{\mathbf{Im}(A)}.$$

Then,

$$u = u_1 + \frac{1}{\omega} f_0,$$

where $u_1 \in \overline{\mathbf{Im}(A)}$ *is the solution of*

$$(A + \omega I)u_1 = f_1. \tag{1}$$

Lemma 2.3. *Let us assume that*

$$A \geq cI \text{ on } \mathbf{Im}(A), \tag{2}$$

i.e.

$$(Av, v) \geq c(v, v), \forall v \in \mathbf{Im}(A).$$

Then, $\mathbf{Im}(A) = \overline{\mathbf{Im}(A)}$,

$$A + \omega I \geq (c + \omega)I \text{ on } \overline{\mathbf{Im}(A)},$$

and the problem (1) is well-posed on $\overline{\mathbf{Im}(A)}$ *uniformly in* ω, $0 < \omega \leq 1$. *This means that, for any given* $f_1 \in \overline{\mathbf{Im}(A)}$, *there exists a unique solution* $u_1 \in \overline{\mathbf{Im}(A)}$ *and that*

$$\|u_1\| \leq \frac{1}{c + \omega}\|f_1\|.$$

Informally speaking, under assumption (2) the problem $(A + \omega I)u = f$ is not truly ill-posed as it can be split into two problems which are uniformly well-posed in ω, $0 < \omega \leq 1$.

Let us also mention that we do not use assumption (2) later in the paper. Instead, we assume a discrete analog of it.

3. The Ritz–Galerkin method for the abstract equation. Let P be an orthogonal projector on a closed trial subspace $\mathbf{Im}(P)$. We do not assume that $\mathbf{Im}(P)$ is necessarily finite dimensional, though in FEM applications it is usually the case.

Applying the Ritz–Galerkin method to the original equation $(A + \omega I)u = f$ gives

$$P(A + \omega I)v = Pf, \; v \in \mathbf{Im}(P). \tag{3}$$

Let $B = PA|_{\mathbf{Im}(P)}$; then $0 \leq B = B^* \leq \|A\|I$. We can rewrite (3) as $(B + \omega I)v = Pf$ in $\mathbf{Im}(P)$.

We use the same approach as in the previous section when treating this equation. Let

$$Pf = (Pf)^0 + (Pf)^1, \; (Pf)^0 \in \mathbf{Ker}(B), \; (Pf)^1 \in \overline{\mathbf{Im}(B)}.$$

Then,

$$v = v^1 + \frac{1}{\omega}(Pf)^0,$$

where v^1 satisfies

$$P(A+\omega I)v^1 = (Pf)^1,\ v^1 \in \overline{\mathbf{Im}(B)}. \tag{4}$$

The following lemma is an analog of Lemma 2.1.

LEMMA 3.1. *The subspace* $\mathbf{Im}(B)$ *and its closure* $\overline{\mathbf{Im}(B)}$ *are invariant with respect to the operator* B *in* $\mathbf{Im}(P)$ *and to the operator* $P(A+\omega I)$.

LEMMA 3.2.

$$\mathbf{Ker}(B) = \mathbf{Ker}(A) \cap \mathbf{Im}(P).$$

Proof. Clearly,

$$\mathbf{Ker}(B) \supseteq \mathbf{Ker}(A) \cap \mathbf{Im}(P).$$

Let $u \in \mathbf{Im}(P)$, such that $PAu = 0$. Then

$$0 = (PAu, u) = (Au, u) = \|A^{1/2}u\|^2.$$

Thus, $A^{1/2}u = 0$ and $u \in \mathbf{Ker}(A)$. □

LEMMA 3.3. *Vector* $(Pf)^0 \in \mathbf{Ker}(A) \cap \mathbf{Im}(P)$ *is the best approximation of the vector* f_0 *in the subspace* $\mathbf{Ker}(A) \cap \mathbf{Im}(P)$, *i.e.*

$$(f_0 - (Pf)^0, v) = 0,\ \forall\ v \in \mathbf{Ker}(A) \cap \mathbf{Im}(P). \tag{5}$$

Proof. Indeed,

$$((Pf)^0, v) = (Pf, v) = (f, v) = (f_0, v) + (f_1, v) = (f_0, v)$$

as $(Pf)^0 \in \mathbf{Ker}(A) \cap \mathbf{Im}(P)$ and $v \in \mathbf{Ker}(A) \cap \mathbf{Im}(P)$. □

LEMMA 3.4.

$$\overline{\mathbf{Im}(B)} = \overline{P\mathbf{Im}(A)}.$$

Proof. We have

$$\mathbf{Im}(B) = \mathbf{Im}(PAP) \subseteq \mathbf{Im}(PA) = P\mathbf{Im}(A),$$

but this does not complete the argument.

Instead of the statement of the Lemma, we shall prove the following equivalent formula

$$\mathbf{Ker}(PAP) = \mathbf{Ker}(AP).$$

Trivially,

$$\mathbf{Ker}(PAP) \supseteq \mathbf{Ker}(AP).$$

Now, let $PAPu = 0$ for some vector u. Then

$$0 = (PAPu, u) = (APu, Pu) = \|A^{1/2}Pu\|^2.$$

Therefore, $A^{1/2}Pu = 0$ and $APu = 0$. □

We have to make an important *Assumption*, analogous to (2),

$$B \geq \tilde{c}I \text{ on } \mathbf{Im}(B), \tag{6}$$

i.e.

$$(Bv, v) \geq \tilde{c}(v, v), \forall v \in \mathbf{Im}(B).$$

It is important to understand that *assumption (6) always holds for finite dimensional approximations*, i.e. when $\dim \mathbf{Im}(P) < \infty$. However, for essentially ill-posed problems the constant goes to zero when we make the subspace $\mathbf{Im}(P)$ larger in order to improve the approximation. In our later example, assumption (6) will be equivalent to a discrete extension theorem valid uniformly in the mesh size parameter. Consequently, our example is not truly ill-conditioned.

LEMMA 3.5. *Under assumption (6), the subspace* $\mathbf{Im}(B)$ *is closed,* $\mathbf{Im}(B) = \overline{\mathbf{Im}(B)}$, *and*

$$\mathbf{Im}(B) = P\overline{\mathbf{Im}(A)} \tag{7}$$

Proof. The subspace $\mathbf{Im}(B)$ is closed by the Closed Graph Theorem of functional analysis. Then (7) follows from the previous lemma and the inclusion

$$\mathbf{Im}(B) \subseteq P\mathbf{Im}(A)$$

already given in Lemma 3.4 □

The following lemma can be established by using standard arguments.

LEMMA 3.6. *Under assumption (6), the following inequality holds*

$$P(A + \omega I) = B + \omega I \geq (\tilde{c} + \omega)I \text{ on } \mathbf{Im}(B),$$

and problem (4) is well-posed on $\mathbf{Im}(B)$ *uniformly in* ω. *The latter means that there exists an inverse operator* $(B + \omega I)^{-1} : \mathbf{Im}(B) \to \mathbf{Im}(B)$ *which is bounded uniformly in* ω, $0 < \omega \leq 1$, *such that*

$$\|(B + \omega I)^{-1}\|_{\mathbf{Im}(B) \to \mathbf{Im}(B)} \leq \frac{1}{\tilde{c} + \omega} < \frac{1}{\tilde{c}},$$

and for any given $f_1 \in \overline{\mathbf{Im}(A)}$ *there exists a unique solution* $v^1 \in \mathbf{Im}(B)$ *such that*

$$\|v^1\| \leq \frac{1}{\tilde{c} + \omega}\|Pf_1\|.$$

Our goal is to estimate the norm of the difference of the exact solution u of our original equation $(A+\omega I)u = f$ and its Ritz approximation v given by (3):

$$\|u - v\| = \|u_1 - v^1 + \frac{1}{\omega}\{f_0 - (Pf)^0\}\|.$$

We first need some simpler estimates.

LEMMA 3.7. *The following estimate holds:*

$$\|Pf_1 - (Pf)^1\| \leq \|f_0 - (Pf)^0\|. \tag{8}$$

Proof. We have

$$Pf_1 - (Pf)^1 = Pf_1 - \{Pf - (Pf)^0\} = (Pf)^0 - Pf_0 = P\{(Pf)^0 - f_0\}.$$

Thus,

$$\|Pf_1 - (Pf)^1\| = \|P\{(Pf)^0 - f_0\}\| \leq \|f_0 - (Pf)^0\|.$$

□

LEMMA 3.8. *Under assumption (6), let $w^1 \in \mathbf{Im}(P)$ be the Ritz approximation of $u_1 \in \overline{\mathbf{Im}(A)}$, the solution of (1), i.e.*

$$P(A+\omega I)w^1 = Pf_1. \tag{9}$$

Then, the following estimate holds:

$$\|u_1 - w^1\| \leq \left(1 + \frac{\|(I-P)AP\|}{\tilde{c}+\omega}\right)\|u_1 - Pu_1\|. \tag{10}$$

Proof. By the triangle inequality,

$$\|u_1 - w^1\| \leq \|u_1 - Pu_1\| + \|Pu_1 - w^1\|.$$

We now estimate the second term.

We first notice that $Pu_1 \in \mathbf{Im}(B)$ and $Pf_1 \in \mathbf{Im}(B)$ by Lemma 3.5 as $u_1 \in \mathbf{Im}(A)$ and $f_1 \in \mathbf{Im}(A)$. Then, $w^1 \in \mathbf{Im}(B)$ by Lemmas 3.1 and 3.6. Thus, $Pu_1 - w^1 \in \mathbf{Im}(B)$ and, using Lemma 3.6,

$$\|Pu_1 - w^1\| = \|(B-\omega I)^{-1}(A-\omega I)(Pu_1 - w^1)\| \leq \frac{1}{\tilde{c}+\omega}\|(A-\omega I)(Pu_1 - w^1)\| =$$

$$\frac{1}{\tilde{c}+\omega}\sup_{v\in\mathbf{Im}(B),v\neq 0}\frac{|((A+\omega I)(Pu_1 - w^1), v)|}{\|v\|}.$$

By using the equality

$$((A+\omega I)(u_1 - w^1), v) = 0,\ v \in \mathbf{Im}(P),$$

which follows from the definition of w^1 as the Ritz approximation of u_1 in $\mathbf{Im}(P)$, we can estimate the numerator:

$$|((A+\omega I)(Pu_1 - w^1), v)| = |((A+\omega I)(Pu_1 - u_1), v)|$$

$$= ((PA(I-P)(Pu_1 - u_1), v)| \leq \|(I-P)AP\| \|Pu_1 - u_1\| \|v\|.$$

The statement of the lemma follows immediately. □

LEMMA 3.9. *For w^1, defined in the previous lemma, we also have the estimate*

$$\|v^1 - w^1\| \leq \frac{1}{\tilde{c}+\omega} \|Pf_1 - (Pf)^1\| \tag{11}$$

assuming that (6) holds.

Proof. Comparing equations (4) and (9), which define v^1 and w^1, shows that the operator is the same, but that the right-hand sides may differ,

$$(B+\omega I)(v^1 - w^1) = Pf_1 - (Pf)^1.$$

Now, by Lemma 3.5,

$$Pf_1 \in P\overline{\mathbf{Im}(A)} = \mathbf{Im}(B).$$

Therefore, by Lemma 3.1, $w^1 \in \mathbf{Im}(B)$ and so is v^1. Finally, Lemma 3.6 gives the estimate of the lemma. □

Summarizing Lemmas 3.7-3.9, we have our main result.

THEOREM 3.1. *Under assumption (6), the following error estimate holds:*

$$\|u - v\| \leq C_1 \mathrm{dist}\{u_1; \mathbf{Im}(P)\} + C_2 \mathrm{dist}\left\{\frac{1}{\omega} f_0; \mathbf{Im}(P) \cap \mathbf{Ker}(A)\right\}, \tag{12}$$

$$C_1 = 1 + \frac{\|(I-P)AP\|}{\tilde{c}+\omega} \leq 1 + \frac{\|(I-P)AP\|}{\tilde{c}},\ C_2 = \frac{\omega}{\tilde{c}+\omega} + 1 \leq \frac{1}{\tilde{c}+1} + 1 < 2,$$

where u is the exact solution of $(A+\omega I)u = f$ and v is its Ritz approximation given by (3).

REMARK 3.1. *Let us assume that we can approximate u_1 and f_0 accurately by properly choosing P, and that the constant $\tilde{c}$ does not tend to 0. Then, the first term can be made small uniformly in ω, $0 < \omega \leq 1$. If we make the natural assumption that the amplitude of f_0 is consistent with $\omega \to 0$, i.e. $f_0 = O(\omega)$, which is equivalent to the assumption that the exact solution*

$$u = u_1 + \frac{1}{\omega} f_0$$

is uniformly bounded, see Lemmas 2.2-2.3, then the second term in our theorem can also be made uniformly small. Thus, the theorem provides a ω-uniform error estimate of the Ritz solution for a problem which is not formally ω-uniformly well-posed.

Remark 3.2. *The theorem can also be applied to a problem with a large parameter written in the following form*

$$\left(I + \frac{1}{\omega}A\right) u = g.$$

We simply take f to satisfy $g = \omega f$, and use a scaling to return to the previous problem. If g is parameter-independent, which is usually the case in practice, then the amplitude of f_0 is consistent with $\omega \to 0$ automatically, and we get a uniform estimate. Moreover, the amplitudes of f_1 and u_1 are then also consistent with $\omega \to 0$. This allows us to rewrite the estimate (12) in the following form:

$$\|u - v\| \leq C_3 \mathrm{dist}\{\frac{1}{\omega}u_1; \mathbf{Im}(P)\} + C_2 \mathrm{dist}\left\{\frac{1}{\omega}f_0; \mathbf{Im}(P) \cap \mathbf{Ker}(A)\right\}, \tag{13}$$

where,

$$C_3 = \omega C_1 = \omega + \|(I - P)AP\|\frac{\omega}{\tilde{c} + \omega} \leq 1 + \|(I - P)AP\|,\ C_2 < 2.$$

This shows that assumption (6) is no longer needed. Let us note, that in some practical applications the subspace $\mathbf{Im}(P) \cap \mathbf{Ker}(A)$ is too small, maybe even trivial, to provide a good approximation for $f_0 \in \mathbf{Ker}(A)$. Such a situation is known as locking. Thus, our inequality (13) gives a uniform error estimate if there is no locking.

Remark 3.3. *The term $\|(I - P)AP\|$ can be replaced by the simple upper bound $\|A\|$.*

The following section gives an example of our general theory.

4. Example: FEM for the diffusion equation. To illustrate our results, we now consider a standard finite element method applied to the diffusion equation in two dimensions with a highly discontinuous diffusion coefficient:

$$\mathrm{div}\,(k \mathrm{grad}\, u - \phi) = 0,\ u \in \overset{0}{W}{}_2^1(\square),\ \phi \in (L_2(\square))^2. \tag{14}$$

where, for simplicity, $\square$ is a polygonal simply connected domain with a Lipschitz boundary $\partial\square$.

Let $\mathcal{D} \subset \square$ be a polygonal simply connected domain with a Lipschitz boundary $\partial\mathcal{D}$, and let the open set $\mathcal{D}^\perp$ be defined by the conditions:

$$\mathcal{D} \cap \mathcal{D}^\perp = \emptyset, \quad \bar{\mathcal{D}} \cup \bar{\mathcal{D}}^\perp = \square.$$

We assume that $\mathcal{D}^\perp$ is also a polygonal simply connected domain with a Lipschitz boundary $\partial\mathcal{D}^\perp$. For simplicity, we also assume that $\partial\mathcal{D} \cap \partial\square$ has a positive Lebesgue measure on $\partial\square$,

$$\mathrm{mes}\{\partial\mathcal{D} \cap \partial\square\} > 0,$$

to ensure that any function in $\overset{0}{W}{}_2^1(\square)$ that is constant in $\mathcal{D}$ vanishes on $\mathcal{D}$.

We assume that the diffusion coefficient k is a bounded measurable scalar function on $\Box$, and highly discontinuous :

$$k = \omega \text{ on } \mathcal{D}^{\perp}, k_0 \leq k - \omega \leq k_1 \text{ on } \mathcal{D}, \tag{15}$$

with some constants $0 < k_0 \leq 1 \leq k_1 < \infty$ and with $0 < \omega \leq 1$.

We now represent equation (14) as in (3). We introduce $\overset{0}{W}{}_2^1(\Box)$ as our Hilbert space with the scalar product

$$(u, v) = (u, v)_{\overset{0}{W}{}_2^1(\Box)} = \int_{\Box} \text{grad}\, u \cdot \text{grad}\ v d\Box,$$

and define our operator A by

$$\int_{\mathcal{D}} (k - \omega) \text{grad}\, u \cdot \text{grad}\, v d\mathcal{D} = (Au, v), \forall u, v \in \overset{0}{W}{}_2^1(\Box).$$

Then,

$$\int_{\Box} k \text{grad}\, u \cdot \text{grad}\, v d\Box =$$

$$\int_{\mathcal{D}} (k - \omega) \text{grad}\, u \cdot \text{grad}\, v d\mathcal{D} + \omega \int_{\Box} \text{grad}\, u \cdot \text{grad}\, v d\Box = ((A + \omega I)u, v),$$

or, in other words,

$$A + \omega I = (\Delta)^{-1} \text{div} k \text{grad}$$

where $\Delta \equiv \text{div}\, \text{grad}$ refers to the Laplacian, defined on functions in $\overset{0}{W}{}_2^1(\Box)$.

The right-hand side f is defined by

$$\int_{\Box} \phi \cdot \text{grad}\, v d\Box = (f, v), \forall v \in \overset{0}{W}{}_2^1(\Box),$$

or, in other words,

$$f = \Delta^{-1} \text{div} \phi.$$

We now conclude that the weak form of equation (14),

$$\int_{\Box} k \text{grad}\, u \cdot \text{grad}\, v d\Box = \int_{\Box} \phi \cdot \text{grad}\, v d\Box, \forall v \in \overset{0}{W}{}_2^1(\Box)$$

is equivalent to our original operator equation $(A + \omega I)u = f$, written in the following form

$$((A + \omega I)u, v) = (f, v), \forall v \in \overset{0}{W}{}_2^1(\Box).$$

It follows from the definition of A that $\mathbf{Ker}(A)$ is $\overset{0}{W}{}_2^1(\mathcal{D}^\perp)$ extended by zero to $\mathcal{D}$, and that $\mathbf{Im}(A)$ is a set of all functions in $\overset{0}{W}{}_2^1(\Box)$ which are harmonic in $\mathcal{D}^\perp$. Thus, to construct an orthogonal sum, say $f = f_0 + f_1$, used in Section 2, we take the restriction of f to the subdomain $\mathcal{D}$ and extend it harmonically into $\mathcal{D}^\perp$. This gives us f_1. Then, we take $f_0 = f - f_1 \in \overset{0}{W}{}_2^1(\mathcal{D}^\perp)$ and extend it by zero in $\mathcal{D}$.

In Section 2, we have established that our abstract problem, though not formally well posed itself, can be split into two well posed problems, if assumption (2) holds. We did not use this assumption in Section 3, but we shall need it later in this section to conclude that the lemmas of Section 2 can be applied in our example. We follow [3] to prove that assumption (2) holds for equation (14) under our assumptions on the subdomains.

We first prove that assumption (2) is equivalent to the following extension theorem:

PROPOSITION 4.1. *There exists a constant $\kappa > 0$ such that for any function $v \in \overset{0}{W}{}_2^1(\Box)$ there is a function $w \in \overset{0}{W}{}_2^1(\Box)$ satisfying $w - v \in \mathbf{Ker}(A)$ and*

$$\int_{\mathcal{D}} \operatorname{grad} w \cdot \operatorname{grad} w d\mathcal{D} \geq \kappa \int_{\Box} \operatorname{grad} w \cdot \operatorname{grad} w d\Box. \tag{16}$$

Proof. Assumption (2) means that

$$(Av, v) \geq c(v, v),\ v \in \mathbf{Im}(A).$$

By (15),

$$k_0 \int_{\mathcal{D}} \operatorname{grad} v \cdot \operatorname{grad} v d\mathcal{D} \leq \int_{\mathcal{D}} (k-\omega) \operatorname{grad} v \cdot \operatorname{grad} v d\mathcal{D} = (Av, v) \leq k_1 \int_{\mathcal{D}} \operatorname{grad} v \cdot \operatorname{grad} v d\mathcal{D}.$$

Thus, we just need to prove that the inequality of the proposition is equivalent to

$$\int_{\mathcal{D}} \operatorname{grad} v \cdot \operatorname{grad} v d\mathcal{D} \geq \kappa \int_{\Box} \operatorname{grad} v \cdot \operatorname{grad} v d\Box,\ v \in \mathbf{Im}(A). \tag{17}$$

Let us suppose that the proposition holds. We apply inequality (16) to a function $v \in \mathbf{Im}(A)$, using the fact that $v = w$ in $\mathcal{D}$ since $w - v \in \mathbf{Ker}(A)$:

$$\int_{\mathcal{D}} \operatorname{grad} v \cdot \operatorname{grad} v d\mathcal{D} \geq \kappa \int_{\Box} \operatorname{grad} w \cdot \operatorname{grad} w d\Box.$$

Finally,

$$\int_{\Box} \operatorname{grad} w \cdot \operatorname{grad} w d\Box = (w, w) = (v, v) + (w - v, w - v) \geq (v, v) = \int_{\Box} \operatorname{grad} v \cdot \operatorname{grad} v d\Box,$$

as $w = v + (w - v)$ is an orthogonal decomposition.

To prove the reverse, let us consider an arbitrary function $v \in \overset{0}{W}{}_2^1(\Box)$ and represent it as an orthogonal decomposition $v = w + (v - w)$, $w \in \mathbf{Im}(A)$, $v - w \in \mathbf{Ker}(A)$. Then (17) holds for $w \in \mathbf{Im}(A)$ and (16) immediately follows. $\Box$

The following lemma is well known, e.g., [10],

LEMMA 4.1. *Proposition 4.1 holds under our assumptions on the subdomains.*

We note that the proposition is not true, in general, if the boundary of $\mathcal{D}$ is not Lipschitz, cf., e.g., [3]. For periodic boundary value problems, extension theorems can be found in [21, 3].

We also have to prove that the functions u_1 and f_0, of Theorem 3.1, are smooth enough (uniformly in ω) in $\mathcal{D}$ and $\mathcal{D}^\perp$ under natural assumptions, so that we can use standard FEM approximation results.

We first consider $f_0 \in \mathbf{Ker}(A)$, which can be defined by

$$(f - f_0, v) = 0, \forall v \in \mathbf{Ker}(A),$$

i.e. f_0 vanishes in $\mathcal{D}$ and its restriction to $\mathcal{D}^\perp$ satisfies $f_0 \in \overset{0}{W}{}_2^1 (\mathcal{D}^\perp)$ and

$$\int_{\mathcal{D}^\perp} (\mathrm{grad}\, f_0 - \phi) \cdot \mathrm{grad}\, v d\mathcal{D}^\perp = 0, \forall v \in \overset{0}{W}{}_2^1 (\mathcal{D}^\perp).$$

Under our assumptions on $\mathcal{D}^\perp$, this problem is well-posed:

$$\|f_0\|_{W_2^1(\mathcal{D}^\perp)} \leq C\|\phi\|_{(L_2(\mathcal{D}^\perp))^2}.$$

Here and below C denotes a generic constant dependent only on the domains $\square$ and $\mathcal{D}$.

Moreover, if the function ϕ is smooth in $\mathcal{D}^\perp$, then f_0 also has some extra smoothness:

$$\|f_0\|_{W_2^{1+\alpha}(\mathcal{D}^\perp)} \leq C\|\phi\|_{(W_2^\alpha(\mathcal{D}^\perp))^2},$$

for some positive α. Thus, f_0 can be approximated in $\overset{0}{W}{}_2^1 (\mathcal{D}^\perp)$ by a finite element-based subspace.

We also need u_1 to be (uniformly in ω) smooth enough in $\mathcal{D}$ and $\mathcal{D}^\perp$. By Lemmas 2.2-2.3,

$$\|u_1\|_{W_2^1(\square)} \leq C\|\phi\|_{(L_2(\square))^2}.$$

It is outside the scope of the present paper to check if any additional assumptions are required to prove that

$$\|u_1\|_{W_2^{1+\alpha}(\mathcal{D})} \leq C\|\phi\|_{(W_2^\alpha(\square))^2},$$

and

$$\|u_1\|_{W_2^{1+\alpha}(\mathcal{D}^\perp)} \leq C\|\phi\|_{(W_2^\alpha(\square))^2},$$

with some positive α. Instead, we simply assume that these estimates are true in our case.

We now define a finite element method (FEM). Because of the abstract nature of our main error estimate, it can be applied to any conforming FEM that fits inside the general Ritz–Galerkin framework. As an example, we consider the simplest FEM.

Let a standard finite element triangulation be constructed, with shape regular and quasi-uniform triangles and denote the mesh size parameter by h. We consider the subspace of $\overset{0}{W}{}_2^1(\square)$ that consists of continuous piecewise linear functions. Let P be the $\overset{0}{W}{}_2^1(\square)$-orthogonal projector onto $\mathbf{Im}(P)$ which is the FEM subspace just described. Then our abstract Ritz-Galerkin method becomes a standard FEM.

The subspace $\mathbf{Ker}(B)$ is, by Lemma 3.2, a subspace of continuous piecewise linear functions in $\mathbf{Im}(P)$ which vanish on $\mathcal{D}$. The subspace $\mathbf{Im}(B)$ consists of functions in $\mathbf{Im}(P)$ that are discrete harmonic on $\mathcal{D}^\perp$.

Now we need to prove that assumption (6) holds in our example. As in the continuous case, we first prove that assumption (6) is equivalent to a finite element extension theorem:

PROPOSITION 4.2. *There exists a constant* $\tilde{\kappa} > 0$ *such that for any function* $v \in \overset{0}{W}{}_2^1(\square) \cap \mathbf{Im}(P)$ *there exists a function* $w \in \overset{0}{W}{}_2^1(\square) \cap \mathbf{Im}(P)$ *satisfying* $w - v \in \mathbf{Ker}(A) \cap \mathbf{Im}(P)$ *and*

$$\int_{\mathcal{D}} \mathrm{grad}\, w \cdot \mathrm{grad}\, w d\mathcal{D} \geq \tilde{\kappa} \int_{\square} \mathrm{grad}\, w \cdot \mathrm{grad}\, w d\square. \tag{18}$$

Proof. Assumption (6) means that

$$(Bv, v) \geq \tilde{c}(v, v),\ v \in \mathbf{Im}(B).$$

By the definition of B, we have $(Av, v) = (Bv, v)$. Then, by (15),

$$k_0 \int_{\mathcal{D}} \mathrm{grad}\, v \cdot \mathrm{grad}\, v d\mathcal{D} \leq \int_{\mathcal{D}} (k - \omega)\mathrm{grad}\, v \cdot \mathrm{grad}\, v d\mathcal{D} = (Bv, v) \leq k_1 \int_{\mathcal{D}} \mathrm{grad}\, v \cdot \mathrm{grad}\, v d\mathcal{D}.$$

Thus, we just need to prove that the inequality of the proposition is equivalent to

$$\int_{\mathcal{D}} \mathrm{grad}\, v \cdot \mathrm{grad}\, v d\mathcal{D} \geq \tilde{\kappa} \int_{\square} \mathrm{grad}\, v \cdot \mathrm{grad}\, v d\square, v \in \mathbf{Im}(B). \tag{19}$$

Let us suppose that the proposition holds. We can then apply inequality (18) to a function $v \in \mathbf{Im}(B)$, using the fact that $v = w$ in $\mathcal{D}$ since $w - v \in \mathbf{Ker}(B) \subset \mathbf{Ker}(A)$:

$$\int_{\mathcal{D}} \mathrm{grad}\, v \cdot \mathrm{grad}\, v d\mathcal{D} \geq \tilde{\kappa} \int_{\square} \mathrm{grad}\, w \cdot \mathrm{grad}\, w d\square.$$

Finally,

$$\int_{\square} \mathrm{grad}\, w \cdot \mathrm{grad}\, w d\square = (w, w) = (v, v) + (w - v, w - v) \geq (v, v) = \int_{\square} \mathrm{grad}\, v \cdot \mathrm{grad}\, v d\square,$$

as $w = v + (w - v)$ is an orthogonal decomposition.

To prove the reverse, let us consider an arbitrary function $v \in \mathbf{Im}(P)$ and represent it as an orthogonal decomposition $v = w + (v - w)$, $w \in \mathbf{Im}(B)$, $v - w \in \mathbf{Ker}(B)$. Then we have (19) for $w \in \mathbf{Im}(B)$ and (16) immediately follows. $\square$

Let us note that Proposition 4.2 always holds for any FEM, since a FEM produces a finite dimensional trial subspace. However, in some cases the constant, $\tilde{\kappa}$, goes to zero with h, the mesh size parameter, even if the continuous analog, Proposition 4.1, holds. Then $\tilde{c}$ decreases too, which affects our error estimate for small ω. Therefore, if possible, the FEM should be constructed such that $\tilde{\kappa}$ in Proposition 4.2 can be chosen to be independent of h.

The following well-known lemma, see, e.g., [1, 27], provides the required result for our example.

LEMMA 4.2. *Assume, in addition, that none of finite elements cut the boundary $\partial\mathcal{D} \cup \partial\mathcal{D}^{\perp} \cup \partial\Box$. Then Proposition 4.2 holds with a constant $\tilde{\kappa}$ independent of h.*

We can therefore use all the results of the previous section. Together with our assumptions on uniform smoothness and standard approximation estimates, it leads to the following error estimate (uniform in ω)

$$\|u - u_h\|_{W_2^1(\Box)} \le Ch^{\alpha}\left(\|\phi\|_{(W_2^{\alpha}(\Box))^2} + \frac{1}{\omega}\|\phi\|_{(W_2^{\alpha}(\mathcal{D}^{\perp}))^2}\right),$$

where u is the exact solution and u_h, earlier denoted by v, is the FEM solution.

5. Numerical experiments for the diffusion equation. We consider three model problems: a square divided into two rectangles (Case I), an L-shaped domain divided into two rectangles (Case II), and a square divided into four squares (Case III).

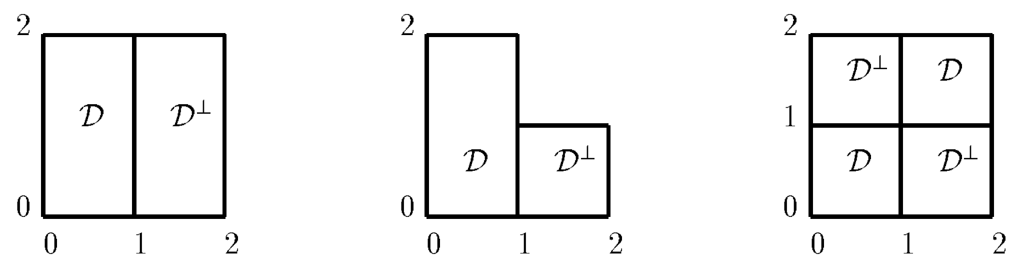

FIG. 1. *Model Domains: Cases I-III*

We assume that

$$k = \omega \text{ on } \mathcal{D}^{\perp}, k = 1 \text{ in } \mathcal{D}, \tag{20}$$

and that $0 < \omega \le 1$.

The exact solution is chosen to be the following function, see also Figure 2,

$$\text{Case I: } u = \begin{cases} \omega x(x-1)y(y-2) & \text{in } \mathcal{D} \\ -(x-1)(x-2)y(y-2) & \text{in } \mathcal{D}^{\perp}. \end{cases}$$

$$\text{Case II: } u = \begin{cases} \omega x(x-1)y(y-1) & \text{in } \mathcal{D},\ 0 \le y \le 1 \\ -\omega x(x-1)(y-1)(y-2) & \text{in } \mathcal{D},\ 1 \le y \le 2 \\ -(x-1)(x-2)y(y-1) & \text{in } \mathcal{D}^{\perp}. \end{cases}$$

$$
\text{Case III: } u = \begin{cases} \omega x(x-1)y(y-1) & \text{in } \mathcal{D},\ 0 \le x \le 1,\ 0 \le y \le 1 \\ \omega(x-1)(x-2)(y-1)(y-2) & \text{in } \mathcal{D},\ 1 \le x \le 2,\ 1 \le y \le 2 \\ -(x-1)(x-2)y(y-1) & \text{in } \mathcal{D}^{\perp},\ 1 \le x \le 2,\ 0 \le y \le 1 \\ -x(x-1)(y-1)(y-2)) & \text{in } \mathcal{D}^{\perp},\ 0 \le x \le 1,\ 1 \le y \le 2. \end{cases}
$$

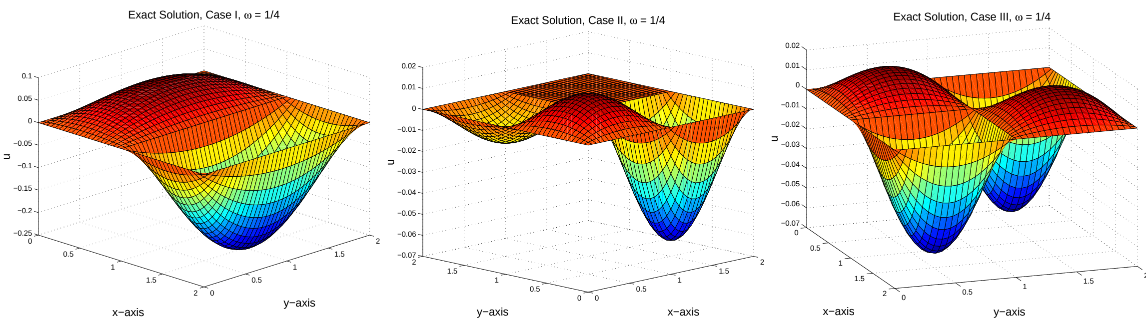

FIG. 2. *Exact Solution, $\omega = 1/4$, Cases I-III*

The Galerkin method used is defined by piecewise linear functions on the standard uniform triangulation, and we use a four-point quadrature scheme, see [13], which is exact for third-order polynomials.

The error in the approximation was computed as

$$
\epsilon = \sqrt{\int_{\square} |\nabla u - \nabla u_h|^2 d\square}
$$

where u is the exact solution and u_h the approximation.

The results are shown in the figures for $1 \le h \le 2^{-5}$ and $1 \le \omega \le 2^{-5}$. Numerical results with smaller ω are practically the same as those with $\omega = 2^{-5}$, within the picture resolution.

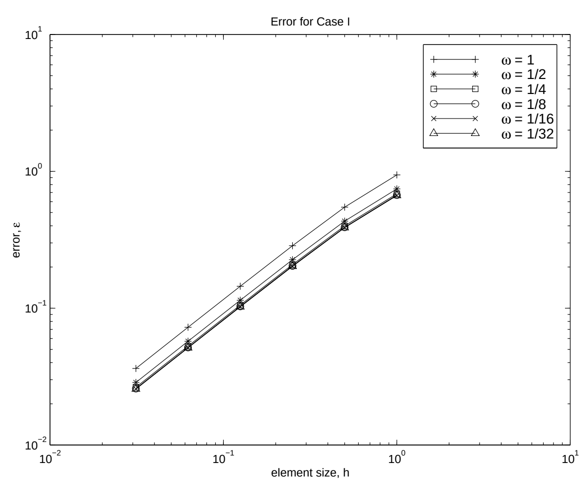

FIG. 3. *Error in Case I*

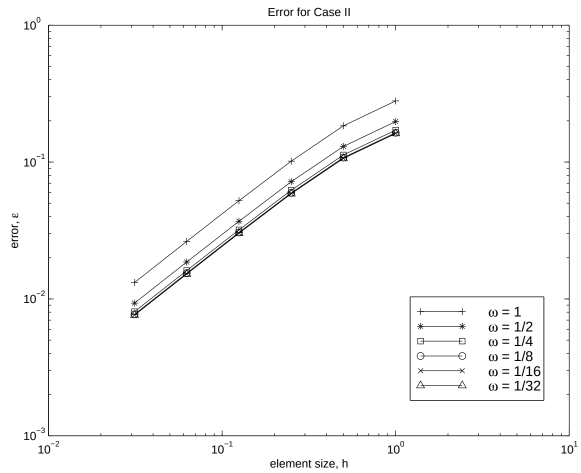

FIG. 4. *Error in Case II*

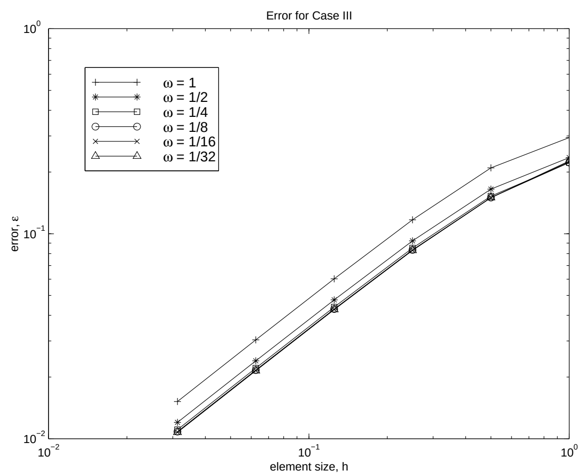

FIG. 5. *Error in Case III*

The figures show that ϵ decreases linearly with h if ω is fixed, as we expect. When h is fixed, decreasing ω appears to cause ϵ to approach a limiting value. The exact solution, as well as the approximate solution, approach limits when $\omega \to 0$. Our numerical tests with $\omega = \frac{1}{32}$ and ω as small 10^{-6} demonstrate that the error, ϵ, approaches a limit as ω decreases.

In our other set of numerical tests, the solution u is chosen to be independent of ω. We noticed that for some functions u, the FEM solutions u_h are also independent of ω, and, trivially, the approximation error does not depend on ω either. We finally found a simple function

$$u = (x + y)x(2 - x)y(2 - y)$$

in Case I that has led to the FEM solution u_h changing with ω. The approximation error for this case is given in Table 1.

	$\omega=1$	$\omega=10^{-1}$	$\omega=10^{-2}$	$\omega=10^{-3}$	$\omega=10^{-4}$	$\omega=10^{-5}$
$h=1$	3.9470	3.9858	4.0025	4.0046	4.0048	4.0048
$h=1/2$	2.3012	2.3086	2.3118	2.3122	2.3122	2.3122
$h=1/4$	1.2034	1.2044	1.2049	1.2050	1.2050	1.2050
$h=1/8$	0.60877	0.60891	0.60897	0.60898	0.60898	0.60898
$h=1/16$	0.30529	0.30531	0.30531	0.30532	0.30532	0.30532
$h=1/32$	0.15276	0.15276	0.15276	0.15276	0.15276	0.15276

TABLE 1
Error for Case I, exact solution $u=(x+y)x(2-x)y(2-y)$.

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