CONTINUOUS FRÉCHET DIFFERENTIABILITY WITH RESPECT TO LIPSCHITZ DOMAIN AND A STABILITY ESTIMATE FOR DIRECT ACOUSTIC SCATTERING PROBLEMS

RABIA DJELLOULI*, CHARBEL FARHAT†, JAN MANDEL‡, AND PETR VANĚK§

June 16, 1998

Abstract. We consider direct acoustic scattering problems with either a sound-soft or sound-hard obstacle, or lossy boundary conditions, and establish continuous Fréchet differentiability with respect to the shape of the scatterer of the scattered field and its corresponding far-field pattern. Our proof is based on the Implicit Function Theorem, and assumes that the boundary of the scatterer as well as the deformation are only Lipschitz continuous. From continuous Fréchet differentiability, we deduce a stability estimate governing the variation of the far-field pattern with respect to the shape of the scatterer. We illustrate this estimate with numerical results obtained for a two-dimensional high-frequency acoustic scattering problem.

Key words. Scattering, acoustics, Fréchet differentiability, stability, domain derivative, Lipschitz boundary, Lipschitz continuous transformation, Implicit Function Theorem, unbounded domain

1. Introduction.

The determination of the shape of an obstacle from its effects on known acoustic or electromagnetic waves is an important problem in many technologies such as sonar, radar, geophysical exploration, medical imaging and nondestructive testing. However, this inverse scattering problem is difficult to solve, especially from a numerical viewpoint, because it is ill-posed and nonlinear. Furthermore, any attempt to study this inverse problem requires the knowledge of the theory for the corresponding direct scattering problem.

In the framework of an inverse problem, it is especially interesting to analyze the dependence of the solution of the direct scattering problem on the domain of the scatterer. In this paper, we establish a relevant stability estimate for the far-field pattern in the case of acoustic scattering problems [3, 16], by using differentiation with respect to the domain of the obstacle and applying the Implicit Function Theorem. Similar approach has already been proposed by Simon [18] for a bounded domain. Here, we generalize the Implicit Function Theorem technique to the case of an unbounded domain. This generalization is made possible by a suitable choice of norms that incorporate the Sommerfeld radiation condition. In addition, we assume that the boundary of the scatterer is only Lipschitz continuous, and prove that the scattered field and its corresponding far-field pattern are *continuously* Fréchet differentiable with respect to Lipschitz continuous deformation of the the shape of the scatter. These results appear to be new. Previously published works dealt only with the Fréchet differentiability and required stronger assumptions on the regularity of the shape of the scatterer [6, 8, 14]. Ramm [15] has previously established a stability

*Department of Aerospace Engineering Sciences, and Center for Aerospace Structures, University of Colorado at Boulder, Boulder, CO 80309-0429

†Department of Aerospace Engineering Sciences, and Center for Aerospace Structures, University of Colorado at Boulder, Boulder, CO 80309-0429

‡Department of Mathematics, University of Colorado at Denver, Denver, CO 80217-3364, and Department of Aerospace Engineering Sciences, University of Colorado at Boulder, Boulder, CO 80309-0429

§Department of Mathematics, University of Colorado at Denver, Denver, CO 80217-3364, and Department of Aerospace Engineering Sciences, University of Colorado at Boulder, Boulder, CO 80309-0429

estimate similar to ours, for the particular case when the scatterer is approximated by polyhedra, which is useful for the solution of scattering problems by the boundary element method. Here, our estimate is independent of the target numerical solution method; it is a property of the boundary value problem.

Auxiliary results about domains with Lipschitz boundary may be of independent interest.

The remainder of this paper is organized as follows. In Section 2, we first specify the nomenclature and assumptions used in this work, formulate the acoustic scattering problem, and announce the main results. Section 3 contains the proof of continuous Fréchet differentiability of the scattered field with respect to the domain. In Section 4, we formulate and prove the results for the far-field pattern. We illustrate in Section 5 our stability result with several numerical simulations, and state our conclusions in Section 6. Appendix A contains auxiliary propositions on Fréchet differentiability of Nemytski operators [7], and on Lipschitz continuous transformations and change of variable in boundary integral for domains with Lipschitz boundary.

2. Problem statement and main result.

2.1. Nomenclature and assumptions. Throughout this paper, we adopt the following nomenclature and assumptions:

- Ω is a bounded domain of $\mathbb{R}^n$ representing an impenetrable obstacle.
- $\Omega^e = \mathbb{R}^n \setminus \overline{\Omega}$ is the homogeneous isotropic medium in $\mathbb{R}^n$ where the obstacle is embedded.
- Γ is the boundary of Ω^e, and is assumed to be Lipschitz continuous (cf., Section A.2 below).
- $|\cdot|$ is the Euclidean norm in $\mathbb{R}^n$, $n \geq 1$.
- x is a point of $\mathbb{R}^n$, understood as a column vector, and $r = |x|$ is the distance from an origin point to x.
- $S^1 = \{x \in \mathbb{R}^n \mid |x| = 1\}$ is the unit sphere in $\mathbb{R}^n$.
- ∇ is the gradient operator in $\mathbb{R}^n$. The gradient of a scalar function is a row vector field. The gradient of a column vector field is a matrix.
- Δ is the Laplace operator in $\mathbb{R}^n$.
- ν is the outward normal to Γ and $\frac{\partial}{\partial \nu}$ is the normal derivative operator.
- k is a positive number representing the wavenumber of the incident wave.
- $L^p(E)$ are the standard Lebesgue spaces, and $H^p(E)$ are Sobolev spaces [1].
- $\mathcal{C}^m(E)$ is the space of functions with continuous derivatives up to order m on E, with the maximum norm of all derivatives.
- $\mathcal{D}(E)$ is the space of infinitely differentiable functions with compact support in E.
- $\mathcal{C}^{0,1}(E)$ is the set of all Lipschitz continuous functions on $E \subset \mathbb{R}^n$, equipped with the norm $\|\phi\|_{C^{0,1}(E)} = \|\phi\|_{\mathcal{C}^0(E)} + |\phi|_{\mathrm{Lip}(E)}$, where $|\phi|_{\mathrm{Lip}} = \sup_{x_1 \neq x_2} |\phi(x_1) - \phi(x_2)|/|x_1 - x_2|$.
- $H^{\mathrm{div}}(\mathbb{R}^n) = \{w \in L^2(\mathbb{R}^n) \mid \operatorname{div} w \in L^2(\mathbb{R}^n)\}$
- $I : \mathbb{R}^n \to \mathbb{R}^n$ is the identity mapping, or the unit matrix
- $\theta : \mathbb{R}^n \to \mathbb{R}^n$ denotes an admissible perturbation in $(\mathcal{C}^{0,1}(\mathbb{R}^n))^n$, small enough so that $I + \theta$ is bijective in $\mathbb{R}^n$, e.g., $\|\theta\|_{(\mathcal{C}^{0,1}(\mathbb{R}^n))^n} < 1$.
- $\Omega^e_\theta = (I + \theta)\Omega^e$ is an admissible perturbed configuration of the reference domain Ω^e; note that $\Omega^e_0 = \Omega^e$, Γ_θ is the boundary of Ω^e_θ, and ν_θ is the outward normal of Ω_θ.
- $\mathcal{F}'(f)h$ is the derivative of $\mathcal{F}$ at f in the direction h, and, for $\mathcal{F} = \mathcal{F}(f, g)$,

$\mathcal{F}'_f(a,b)h$ is the derivative with respect to the argument f at (a,b), in the direction h.
- Tr_Γ is the trace operator on Γ.
- For normed spaces, the symbol $\hookrightarrow$ denotes continuous injection.

2.2. Formulation of the problem. The scattering of time-harmonic acoustic waves by an impenetrable obstacle embedded in a homogeneous medium can be formulated as the exterior boundary value problem (BVP)

$$\Delta u_\theta + k^2 u_\theta = 0 \text{ in } \Omega_\theta^e \subset \mathbb{R}^n, \quad n = 2 \text{ or } 3, \tag{2.1}$$

$$\left(a + b\frac{\partial}{\partial \nu_\theta}\right)\left(u_\theta + e^{ikx\cdot d}\right) = 0 \text{ on } \Gamma_\theta, \tag{2.2}$$

$$\lim_{r\to\infty} r^{\frac{n-1}{2}}\left(\frac{\partial u_\theta}{\partial r} - iku_\theta\right) = 0, \tag{2.3}$$

where u_θ is the acoustic scattered field, $d \in S^1$ is the normalized direction of the incident wave, and a and b are constants, not both zero. The boundary condition (2.2) allows a concise representation of all of the Dirichlet, Neumann, and lossy boundary conditions that are usually encountered in acoustic scattering problems [3, 9, 16, 23].

We will also study the far-field pattern $u_{\theta,\infty}$ of a solution u_θ of (2.1)–(2.3), defined on the unit sphere S^1 from the asymptotic behavior of the scattered field u_θ,

$$u_\theta(x) = \frac{e^{ikr}}{r^{(n-1)/2}}\left(u_{\theta,\infty}\big(\frac{x}{r}\big) + O\big(\frac{1}{r}\big)\right), \quad r \to +\infty, \tag{2.4}$$

cf., e.g. [3].

The direct scattering problem (2.1)–(2.3) has been extensively investigated, and a considerable amount of results pertaining to the existence, uniqueness, regularity and asymptotic behavior of the solution u_θ can be found in [3, 16, 23, 9, 4, 12, 17], among others.

2.3. Announcement of the main results. We first use the Implicit Function Theorem to prove that the mapping $\theta \mapsto u_\theta$ is continuously Fréchet differentiable on a neighborhood of zero from $(\mathcal{C}^{0,1}(\mathbb{R}^n))^n$ to $\mathcal{C}^2(D^e)$, for any domain D such that $\overline{\Omega} \subset D$ (Theorem 3.3 and Corollary 3.4). This will allow us to show (Theorem 4.2) that the mapping $\theta \mapsto u_{\theta,\infty}$ is continuously Fréchet differentiable on a neighborhood of zero from $(\mathcal{C}^{0,1}(\mathbb{R}^n))^n$ to $\mathcal{C}^m(S^1)$, for any m, and, consequently, $\|u_{\theta,\infty} - u_{0,\infty}\|_{\mathcal{C}^m(S^1)} \leq C\|\theta\|_{(\mathcal{C}^{0,1}(\mathbb{R}^n))^n}$, for small θ (Theorem 4.3).

3. Continuous differentiability of the scattered field. In this section, we establish the continuous Fréchet differentiability with respect to the domain of the given obstacle of the scattered field and its far-field pattern, by exploiting the Implicit Function Theorem (cf., e.g., [2]). The main idea is to reformulate the BVP (2.1)–(2.3) so that the scattered field is a zero of a nonlinear operator in Banach spaces, which do not change with θ.

3.1. Variational setting and transported solution. We now rewrite the boundary value problem (2.1)–(2.3) as an equivalent operator equation of the form $\mathcal{F}(\theta, v) = 0$.

Let D_1 and D_2 be two bounded domains with smooth boundaries and such that

$$\overline{\Omega} \subset D_1 \subset \overline{D}_1 \subset D_2. \tag{3.1}$$

Define the following Banach spaces and norms:

$$
\begin{aligned}
X &= \{\theta \in \mathcal{C}^{0,1}(\mathbb{R}^n) \mid \operatorname{supp}\theta \subset \overline{D}_1\}, \\
\|\theta\|_X &= \|\theta\|_{\mathcal{C}^{0,1}(\mathbb{R}^n)}, \\
Y &= \{w \mid \|w\|_Y < +\infty\}, \\
\|w\|_Y &= \|w\|_{H^1(D_2\cap\Omega^e)} + \|w\|_{\mathcal{C}^2(D_2^e)} + \left\| r^{\frac{n+1}{2}}\left(\frac{\partial w}{\partial r} - ikw\right)\right\|_{\mathcal{C}^0(D_2^e)} \\
Z &= \{(z_1, z_2) \mid z_1 \in H^{-1}(D_2 \cap \Omega^e) \cap \mathcal{C}^0(D_2^e),\ z_2 \in H^s(\Gamma)\}, \\
\|((z_1, z_2)\|_Z &= \|z_1\|_{H^{-1}(D_2\cap\Omega^e)} + \|z_1\|_{\mathcal{C}^0(D_2^e)} + \|z_2\|_{H^s(\Gamma)},
\end{aligned}
$$

where $s = 1/2$ in the case of sound soft scatterer, i.e., $b = 0$ in (2.2), and $s = -1/2$ otherwise.

For a solution u_θ of (2.1)–(2.3), define the *transported solution* v_θ on Ω^e by

$$
v_\theta = u_\theta \circ (I + \theta), \tag{3.2}
$$

Since $u_\theta(x) = v_\theta(x)$ for all x such that $\theta(x) = 0$, and θ has compact support, it follows that v_θ satisfies the Sommerfeld radiation condition (2.3).

LEMMA 3.1. *The BVP (2.1)–(2.3) is equivalent to the operator equation $\mathcal{F}(\theta, v_\theta) = 0$, where v_θ is the transported solution (3.2), $\mathcal{F} : X \times Y \to Z$ is continuous on a neighborhood of $(0, v_0)$ from $X \times Y$ to Z, and its Fréchet derivatives $\mathcal{F}'_\theta$ and $\mathcal{F}'_v$ exist and are continuous on a neighborhood of $(0, v_0)$ in $X \times Y$.*

Proof. Assume that $|\theta|_{\mathrm{Lip}}$ is sufficiently small so that $(I + \theta)^{-1}$ exists and Ω_θ has Lipschitz boundary, cf., Lemma A.3.

Case $b = 0$. The weak form of (2.1)–(2.2) is

$$
\int_{\Omega_\theta^e} (\nabla u_\theta \nabla \phi - k^2 u\phi)\, d\mu = 0, \qquad \forall \phi \in \mathcal{D}(\Omega_\theta^e), \tag{3.3}
$$

$$
\mathrm{Tr}_{\Gamma_\theta}(u_\theta - e^{ikx\cdot d}) = 0. \tag{3.4}
$$

Denote $P_\theta = I + \nabla\theta$. Using the substitution (3.2), the chain rule $\nabla v_\theta = \nabla u_\theta P_\theta$, and change of variable (Lemma A.6), equations (3.3) and (3.4) become

$$
\int_{\Omega^e} \left(\nabla v_\theta P_\theta^{-1} P_\theta^{-T} \cdot \nabla \psi - k^2 v_\theta \psi\right) |\det P_\theta|\, d\mu = 0, \qquad \forall \psi \in \mathcal{D}(\Omega^e), \tag{3.5}
$$

$$
\mathrm{Tr}_\Gamma(v_\theta - e^{ik(x+\theta(x))\cdot d}) = 0. \tag{3.6}
$$

Now (3.5) is equivalent to $\mathcal{F}_1(\theta, v_\theta) = 0$, where $\mathcal{F}_1(\theta, v) \in \mathcal{D}'(\Omega^e)$ is the functional

$$
\mathcal{F}_1(\theta, v) : \psi \mapsto \int_{\Omega^e} \left(\nabla v P_\theta^{-1} P_\theta^{-T} \cdot \nabla \psi - k^2 v\psi\right) |\det P_\theta|\, d\mu. \tag{3.7}
$$

Because θ is Lipschitz continuous, we have $\nabla\theta \in (L^\infty(\Omega^e))^{n\times n}$. Hence, $\mathcal{F}_1(\theta, v)$ is a bounded linear functional on $H_0^1(\Omega^e \cap D_2)$; i.e., $\mathcal{F}(\theta, v) \in H^{-1}(\Omega^e \cap D_2)$. If $\operatorname{supp}\psi \subset D_2^e$ and $v \in \mathcal{C}^2(D_2^e)$, then $\theta = 0$ on $\operatorname{supp}\psi$, and so $\mathcal{F}_1(\theta, v)\psi = \int_{\Omega^e}(-\Delta v + k^2 v)\psi\, d\mu$, consequently the distribution $\mathcal{F}(\theta, v) \in \mathcal{C}^0(D_2^e)$. Therefore,

$$
\mathcal{F}_1 : X \times (H^1(D_2 \cap \Omega^e) \cap \mathcal{C}^2(D_2^e)) \to H^{-1}(D_2 \cap \Omega^e) \cap \mathcal{C}^0(D_2^e).
$$

The continuous Fréchet differentiability of $\mathcal{F}_1$ follows from the fact that the mappings $\nabla\theta \mapsto |\det P_\theta|$ and $\nabla\theta \mapsto P_\theta^{-1} P_\theta^{-T} |\det P_\theta|$ are continuously

Fréchet differentiable on a neighborhood of zero from $(L^\infty(\mathbb{R}^n))^n$ to $L^\infty(\mathbb{R}^n)$ and $(L^\infty(\mathbb{R}^n))^{n\times n}$, respectively, by Lemma A.1, and that the integrand in (3.7) is the sum of trilinear forms of v, $P_\theta^{-1}P_\theta^{-T}|\det P_\theta|$, ψ, and of v, $|\det P_\theta|$, ψ, which are bounded on $H^1(D_2\cap\Omega^e)\times(L^\infty(D_2))^{n\times n}\times H_0^1(D_2\cap\Omega^e)$, and on $H^1(D_2\cap\Omega^e)\times L^\infty(D_2)\times H_0^1(D_2\cap\Omega^e)$, respectively.

Equation (3.6) can be written as $\mathcal{F}_2(\theta, v_\theta)=0$, where

$$\mathcal{F}_2(\theta,v)=\mathrm{Tr}_\Gamma(v-e^{ik(x+\theta(x))\cdot d}).$$

Since the trace operator $\theta\mapsto\mathrm{Tr}_\Gamma\,\theta$ is linear and bounded from $(\mathcal{C}^{0,1}(\mathbb{R}^n))^n$ to $(\mathcal{C}^{0,1}(\Gamma))^n$, it follows from Lemma A.2 that the mapping

$$\theta\mapsto(x\mapsto e^{ik(x+\theta(x))\cdot d})$$

is continuously Fréchet differentiable from $(\mathcal{C}^{0,1}(\mathbb{R}^n))^n$ to $\mathcal{C}^{0,1}(\Gamma)$. Since $\mathcal{C}^{0,1}(\Gamma)\hookrightarrow H^{1/2}(\Gamma)$, cf., [5, 13], it follows that $\mathcal{F}_2$ is continuously Fréchet differentiable with respect to θ on a neighborhood of zero from $(\mathcal{C}^{0,1}(\mathbb{R}^n))^n$ to $H^{1/2}(\Gamma)$. It remains to note that the trace operator is linear and continuous from $H^1(\Omega^e\cap D_2)$ to $H^{1/2}(\Gamma)$.

Case $b\neq0$. Integrating (2.1) by parts on Ω_θ^e and substituting for the normal derivative from the boundary condition (2.2),

$$\frac{\partial u_\theta}{\partial\nu_\theta}=-\frac{a}{b}(u_\theta+e^{ikx\cdot d})-\frac{\partial}{\partial\nu_\theta}e^{ikx\cdot d}=-\frac{a}{b}u_\theta-(\frac{a}{b}+ikd\cdot\nu_\theta)e^{ikx\cdot d},$$

we obtain the weak form

$$\int_{\Omega_\theta^e}(\nabla u_\theta\cdot\nabla\psi-k^2u_\theta\psi)\,d\mu+\int_{\Gamma_\theta}\Big(\frac{a}{b}u_\theta\psi+(\frac{a}{b}+ikd\cdot\nu_\theta)e^{ikx\cdot d}\psi\Big)\,d\mu_{\Gamma_\theta}=0,$$

for all $\psi\in\mathcal{D}(\mathbb{R}^n\setminus\Omega_\theta)$. Transporting the solution to Ω^e by (3.2) and transforming the integrals using Lemmas A.6 and A.7, we get

$$\begin{aligned}&\int_{\Omega^e}(\nabla v_\theta P_\theta^{-1}P_\theta^{-T}\cdot\nabla\psi-k^2v_\theta\psi)|\det P_\theta|\,d\mu+\\&\qquad\int_\Gamma\Big(\frac{a}{b}v_\theta+(\frac{a}{b}+ikd\cdot\nu_\theta(x+\theta(x))e^{ik(x+\theta(x))\cdot d}\Big)\,J_{T\theta}\psi\,d\mu_\Gamma(x)=0,\end{aligned}\tag{3.8}$$

for all $\psi\in\mathcal{D}(\mathbb{R}^n\setminus\Omega)$.

For test functions $\psi\in\mathcal{D}(\Omega^e)$, we obtain the same $\mathcal{F}_1$ as for $b=0$, cf., (3.7). Now we derive $\mathcal{F}_2$. By density, (3.8) holds for all $\psi\in H^1(\Omega^e)$. Let D be a bounded domain such that $\overline{\Omega}\subset D$. There exists an extension operator $\mathcal{E}$ from Γ, which is linear and bounded from $H^{1/2}(\Gamma)$ to $H^1(\Omega^e)$, and such that $\operatorname{supp}\mathcal{E}\phi\subset\overline{D}$, for all $\phi\in H^{1/2}(\Gamma)$, cf., [13]. Let $\phi\in H^{1/2}(\Gamma)$. From (3.8) with $\psi=\mathcal{E}\phi$, we obtain that v_θ satisfies $\mathcal{F}_2(\theta,v_\theta)=0$, where $\mathcal{F}_2(\theta,v):H^{1/2}(\Gamma)\to\mathbb{C}$ is defined by

$$\begin{aligned}\mathcal{F}_2(\theta,v):\phi\mapsto&\int_{\Omega^e}(\nabla vP_\theta^{-1}P_\theta^{-T}\cdot\nabla\mathcal{E}\phi-k^2v\mathcal{E}\phi)|\det P_\theta|\,d\mu+\\&\int_\Gamma\Big(\frac{a}{b}v+(\frac{a}{b}+ikd\cdot\nu_\theta(x+\theta(x))e^{ik(x+\theta(x))\cdot d}\Big)\,J_{T\theta}\phi\,d\mu_\Gamma(x).\end{aligned}\tag{3.9}$$

Hence, $\mathcal{F}_2(\theta,v)\in H^{-1/2}(\Gamma)$ for $v\in Y$. The Fréchet differentiability of the first integral (over Ω^e) in (3.9) follows by the same argument as for $\mathcal{F}_1$ above and using the linearity and continuity of the extension operator $\mathcal{E}$. To prove Fréchet differentiability of the second integral in (3.9), note that the integral is of the form $\int_\Gamma K(\theta,v)\psi\,d\mu_\Gamma$, where, by Lemmas A.1, A.5, and A.7, the integrand $K(\theta,v)$ is continuously Fréchet differentiable for θ in a neighborhood of zero and all v, from $(\mathcal{C}^{0,1}(\mathbb{R}^n))^n\times L^2(\Gamma)$ to $L^2(\Gamma)$. The proof is concluded by a use of the trace theorem. ☐

3.2. Continuous Fréchet differentiability of the scattered field. We need to summarize some well-known results about the solution of the Helmholtz problem [17, 23, 9, 4, 12].

LEMMA 3.2. *Let $\Omega \subset R^n$, $n = 2$ or $n = 3$, be a domain with Lipschitz boundary, $z_1 \in H^{-1}_{\text{loc}}(\Omega^e)$ with a compact support, and $z_2 \in H^s(\Gamma)$, where $s = 1/2$ for the case $b = 0$ and $s = -1/2$ otherwise. Then there exists a unique solution $w \in H^1_{\text{loc}}(\Omega^e)$ of the boundary value problem $\Delta w + k^2 w = z_1$ in Ω^e, $(a + b\partial/\partial\nu)w = z_2$ on Γ, $\lim_{r\to\infty} r^{(n-1)/2}(\partial w/\partial r - ikw) = 0$. In addition, $\|w\|_{\mathcal{C}^2(D^e)} < +\infty$ for any domain D such that $\overline{\Omega} \cup \operatorname{supp} z_1 \subset D$.*

We are now ready to use the Implicit Function Theorem.

THEOREM 3.3. *The mapping $\theta \mapsto v_\theta$, where v_θ is the transported solution (3.2) of the BVP (2.1)–(2.3), is continuously Fréchet differentiable at $\theta = 0$, from X to Y.*

Proof. Let $\mathcal{F}$ be the function constructed in Lemma 3.1. Since $\mathcal{F}$ is affine in v,

$$\mathcal{F}'_v(0, v) : w \mapsto \left(-\Delta w - k^2 w, \operatorname{Tr}_\Gamma \left(a + b\frac{\partial}{\partial\nu}\right) w\right)$$

is a bounded linear operator from Y to Z, which does not depend on v. In the case when $b \neq 0$, the trace operator is understood in the sense of (3.9). Define

$$\tilde{Z} = \{(z_1, z_2) \in Z \mid \operatorname{supp} z_1 \subset D_2\} = H^{-1}(D_2 \cap \Omega^e) \times H^s(\Gamma),$$
$$\tilde{Y} = \mathcal{F}'_v(0, v)^{-1}\tilde{Z}.$$

Since $\tilde{Z}$ is a closed subspace of Z, and, by Lemma 3.2, $\tilde{Z} \subset \mathcal{F}'_v(0, v)Y$, it follows that $\mathcal{F}'_v(0, v)$ is a isomorphism of the Banach spaces $\tilde{Y}$ and $\tilde{Z}$.

Let θ be in a suitable neighborhood of zero in X and $v \in \tilde{Y}$, From the definition of $\tilde{Y}$, the definition of $\mathcal{F}_1$, cf., (3.7), and from the fact that $\operatorname{supp}\theta \subset \overline{D}_1$, it follows that $\Delta v + k^2 v = 0$ in D_2^e. Hence, $\mathcal{F}(\theta, v) \in \tilde{Z}$.

We can conclude that X, $\tilde{Y}$, and $\tilde{Z}$ are Banach spaces, $\mathcal{F}(0, v_0) = 0$, $\mathcal{F} = \mathcal{F}(\theta, v)$ is continuous on a neighborhood of $(0, v_0)$ in $X \times \tilde{Y}$ to $\tilde{Z}$, the Fréchet derivatives $\mathcal{F}'_\theta$ and $\mathcal{F}'_v$ exist and are continuous on a neighborhood of $(0, v_0)$, and $\mathcal{F}'_v(0, v_0)$ has bounded inverse defined on $\tilde{Y}$. From the Implicit Function Theorem (see, e.g., [2]), it follows that for every θ in a neighborhood of zero in X, there is a unique v_θ in a neighborhood of v_0 in Y such that $\mathcal{F}(\theta, v_\theta) = 0$, and the Fréchet derivative of the mapping $\theta \mapsto v_\theta$ exists and is continuous on a neighborhood of zero in X. ◻

COROLLARY 3.4. *For any domain $D \supset \overline{\Omega}$, the mapping $\theta \mapsto u_\theta$, where u_θ is the solution of (2.1)–(2.3), is continuously Fréchet differentiable on a neighborhood of zero from $(\mathcal{C}^{0,1}(\mathbb{R}^n))^n$ to $\mathcal{C}^2(D^e)$.*

Proof. For a given D, choose D_1 and D_2 such that $D \supset \overline{D}_2$ and (3.1) holds. There is $\phi \in \mathcal{D}(D_1)$ such that $\phi = 1$ on $\overline{\Omega}$. The mapping $\theta \mapsto \phi\theta$ is linear and bounded from $(\mathcal{C}^{0,1}(\mathbb{R}^n))^n$ to X, and $\Omega_{\phi\theta} = \Omega_\theta$, since $\theta = \phi\theta$ on $\overline{\Omega}$. ◻

4. Continuous Fréchet differentiability of the far-field pattern. We first need a smoothing property of the far field pattern operator.

LEMMA 4.1. *Let D be a bounded domain with a smooth boundary Γ_1 and $w \in H^1_{\text{loc}}(D^e)$ be a solution of the Helmholtz equation $\Delta w + k^2 w = 0$ in D^e, satisfying the Sommerfeld radiation condition $\lim_{r\to\infty} r^{(n-1)/2}(\partial w/\partial r - ikw) = 0$. Denote by $w_1 = \operatorname{Tr}_{\Gamma_1} w$ the trace of w on Γ_1 and by w_∞ the far-field pattern (2.4) of w, and let $m > 0$ be an integer and $s \in \mathbb{R}$. Then there exists a constant $C = C(\Gamma_1, m, s)$ such that $\|w_\infty\|_{\mathcal{C}^m(S^1)} \leq C\|w_1\|_{H^s(\Gamma_1)}$.*

Proof. The proof we present here relies on an integral representation approach, and therefore depends on the value of the wavenumber k. For simplicity, first suppose that k^2 is not an eigenvalue of the operator $-\Delta$ in D with a Dirichlet boundary condition on Γ_1, i.e., k^2 is not an internal resonance value of D.

It is well known, e.g. [3, 4], that there exists a single layer potential q such that $Jq = w$ a.e. in D^e, where J is the operator

$$Jq(x) = \int_{\Gamma_1} q(y)G_n(x,y)d\mu_{\Gamma_1}(y), \quad G_n(x,y) = \begin{cases} \frac{1}{4\pi}\frac{e^{ik|x-y|}}{|x-y|}, & n=3, \\ \frac{i}{4}H_0^{(1)}(k|x-y|), & n=2, \end{cases}$$

where $H_0^{(1)}$ is the Hankel function of the first kind and zero order. The far field patters can be written in the terms of the single layer potential as

$$w_\infty(x) = K_n \int_{\Gamma_1} q(y)e^{-ikx\cdot y}\, d\mu_{\Gamma_1}(y), \qquad x \in S^1, \tag{4.1}$$

where $K_3 = 1/4\pi$ and $K_2 = e^{i\pi/4}/\sqrt{8\pi k}$. Under the assumption specified above for k, the single-layer operator J is an isomorphism of $H^{s-1}(\Gamma_1)$ to $H^s(\Gamma_1)$, for all real s, cf., [4]. Hence,

$$\|q\|_{H^{s-1}(\Gamma_1)} \leq C_1\|w_1\|_{H^s(\Gamma_1)},$$

and the proof is concluded by noting that the kernel $e^{-ikx\cdot y}$ is smooth.

When k^2 is an eigenvalue of the interior Dirichlet problem, the Lemma can be proved using a similar reasoning after replacing the single-layer representation of the far-field pattern by either a double-layer operator or the Brakhage-Werner integral representation [3, 4] of the far-field pattern. □

We are now ready to establish the continuous Fréchet differentiability of the far-field pattern.

THEOREM 4.2. *For every integer $m \geq 0$, the mapping $\theta \mapsto u_{\theta,\infty}$ is continuously Fréchet differentiable at $\theta = 0$ from $(\mathcal{C}^{0,1}(\mathbb{R}^n))^n$ to $\mathcal{C}^m(S^1)$.*

Proof. Let D be a domain with a smooth boundary such that $\overline{\Omega} \subset D$, and k^2 is not an eigenvalue of the interior Dirichlet problem on D. Denote by Γ_1 the boundary of D. Since the trace operator is continuous from Y to $\mathcal{C}^0(\Gamma_1)$ and $\mathcal{C}^0(\Gamma_1) \hookrightarrow L^2(\Gamma_1)$, use of Lemma 4.1 and Corollary 3.4 concludes the proof. □

The continuity estimate now follows by a standard argument.

THEOREM 4.3. *For every integer m, there exist positive constants ϵ, C, depending on k, d, m, and Γ, such that*

$$\|u_{\theta,\infty} - u_{0,\infty}\|_{\mathcal{C}^m(S^1)} \leq C\|\theta\|_{(\mathcal{C}^{0,1}(\mathbb{R}^n))^n},$$

for all $\|\theta\|_{(\mathcal{C}^{0,1}(\mathbb{R}^n))^n} < \epsilon$.

Proof. Denote by $\mathcal{T}$ the mapping $\mathcal{T} : \theta \mapsto u_{\theta,\infty}$. From Theorem 4.2, the Fréchet derivative $\mathcal{T}'$ exists and is continuous from $(\mathcal{C}^{0,1}(\mathbb{R}^n))^n$ to $\mathcal{C}^m(S^1)$ on a neighborhood of zero. Hence, there is C and $\varepsilon > 0$ such that $\|\mathcal{T}'(\theta)\tau\|_{\mathcal{C}^m(S^1)} \leq C\|\tau\|_{(\mathcal{C}^{0,1}(\mathbb{R}^n))^n}$ for all $\|\theta\|_{(\mathcal{C}^{0,1}(\mathbb{R}^n))^n} < \varepsilon$ and all $\tau \in (\mathcal{C}^{0,1}(\mathbb{R}^n))^n$. Let $\|\theta\|_{(\mathcal{C}^{0,1}(\mathbb{R}^n))^n} < \varepsilon$. Then, cf., e.g., [2],

$$\mathcal{T}(\theta) - \mathcal{T}(0) = \int_0^1 \mathcal{T}'(t\theta)\theta\, dt,$$

which gives $\|\mathcal{T}(\theta) - \mathcal{T}(0)\|_{\mathcal{C}^m(S^1)} \leq C\|\theta\|_{(\mathcal{C}^{0,1}(\mathbb{R}^n))^n}$. □

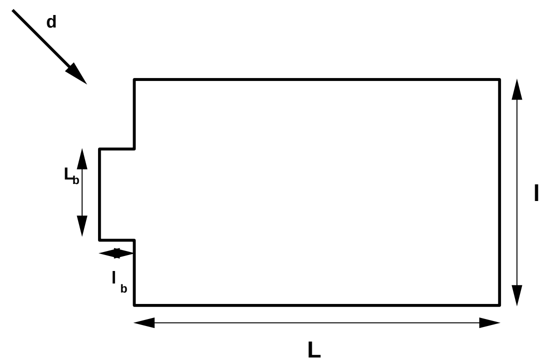

FIG. 5.1. *The scatterer*

5. Numerical illustration. In order to illustrate the stability estimate proved in this paper, we investigate the sensitivity of the solution of a two-dimensional direct acoustic scattering problem to local changes in the shape of the scatterer.

5.1. Application problem. We consider a rectangular obstacle of length L and width l with a rectangular "bump" of length L_b and width $l_b = L_b/2$ centered on its left vertical edge (Fig. 1). Our objective is to determine numerically, for a given incident wave in a direction d, the relative variations of the scattering amplitude due to perturbations in the size of the bump.

5.2. Discretization method and iterative solver. The Helmholtz equation poses a special challenge to usual discretization methods (i.e. finite elements, boundary elements) because it behaves like an elliptic problem on scales less or equal to the wavelength (the low frequency case), and like a hyperbolic problem on scales much larger than the wavelength (the high frequency case). The discussion of which discretization method is best suitable for the solution of an acoustic scattering problem is beyond the scope of this paper. However, we note that we are interested in high frequency applications, and that for such problems, all approximation methods lead to a large-scale system of linear equations that often requires the usage of an iterative solver. The specific formulation of the Helmholtz problem adopted here and its discretization lead to a symmetric system of equations that is positive definite for any frequency. Hence, this approach, which is described below for two-dimensional problems, is interesting because it allows the use of an efficient algebraic multigrid solver [21].

Let $D \subset \mathbb{R}^2$ be a sufficiently large domain with a boundary denoted by Γ_1, and $\Omega \subset D$ be an obstacle with a boundary denoted by Γ. For discretization purposes, the BVP is reformulated as seeking a complex pressure $u \in H^1(D \setminus \overline{\Omega})$ and its gradient $u_g \in H^{div}(D \setminus \overline{\Omega})$, that minimize the convex functional

$$F(u, u_g) = \int_{D\setminus\Omega} |\nabla u - u_g|^2 \, d\mu + \int_{D\setminus\Omega} |\nabla \cdot u_g + k^2 u|^2 \, d\mu + \int_{\Gamma_1} |u_g \cdot r - iku|^2 \, d\mu_{\Gamma_1},$$

subject to the Dirichlet boundary condition $u(x) = -e^{ikx\cdot d}$ on Γ. The first two integrals above define a first-order least square formulation of the Helmholtz problem,

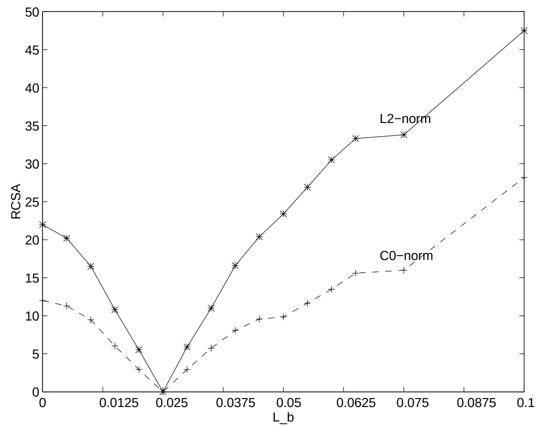

FIG. 5.2. *Incident wave in the direction* $d = (1, 0)$

cf., [10]. The third integral enforces the "radiation condition"

$$\frac{\partial u}{\partial r} = iku \quad \text{on } \Gamma_1.$$

The computational domain $D \setminus \Omega$ is decomposed into quadrilateral elements, and both u and u_g are approximated by Q1 finite elements. Note that the above integrals contain both first-order and zero-order derivatives. For this reason, and because both u and u_g are approximated by Q1 finite elements, standard reduced integration is employed to avoid locking [11]. See [20, 21] for more details on the discretization.

The formulation and discretization method described above require three complex unknowns per grid point, and lead to a sparse system of linear complex equations whose associated matrix is Hermitian. By separating the real and imaginary parts of the unknowns, six real unknowns can be introduced at each grid point, and the previous system of complex equations can be reformulated as a real system of equations whose associated matrix is symmetric positive definite. In this paper, this real system of equations is solved iteratively by the algebraic multigrid algorithm presented in [20, 21, 19].

5.3. Results and analysis. For all computations reported herein, the dimensions of the obstacle described in Section 5.1 and depicted graphically in Fig. 1 are set to $L = 0.305$ and $l = 0.300$. Three incident waves with a fixed wavenumber $k = 125.6637$ are considered. The first incident wave is aligned with the horizontal axis of the obstacle and impacts it from the side where the bump is located ($d = (1, 0)$). The second one is also aligned with the horizontal axis of the scatterer but impacts it from the smooth back side ($d = (-1, 0)$). The third incident wave impacts the obstacle in the direction $d = (\sqrt{2}/2, \sqrt{2}/2)$. Note that $k \times l \approx 37.7$, which, for submarine applications, can be considered as a high frequency case.

The computational domain is chosen as $D = [-1, 1] \times [-1, 1]$, and is discretized by 640000 Q1 elements — that is, 800 Q1 elements along each side. Hence, the characteristic mesh size is $h = 2/800 = 0.0025$, the number of elements per wave length is $(2 \times \pi)/(125.6637 \times 0.0025) \approx 20$, and the total number of real unknowns is $801 \times 801 \times 6 = 3849606$.

Initially, the dimensions of the bump are set to $L_b^0 = 0.025$ and $l_b^0 = L_b^0/2 = 0.0125$. Subsequently, the size of this bump is perturbed by varying its length between

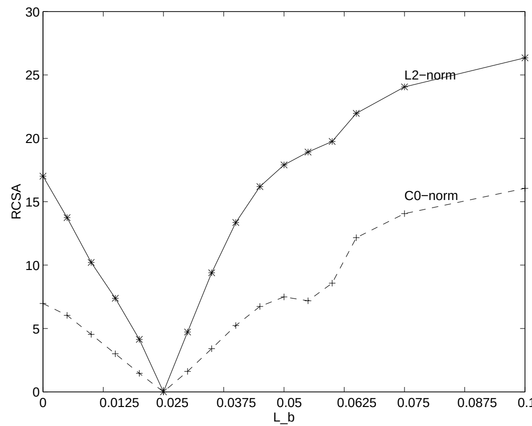

FIG. 5.3. *Incident wave in the direction* $d = (-1, 0)$

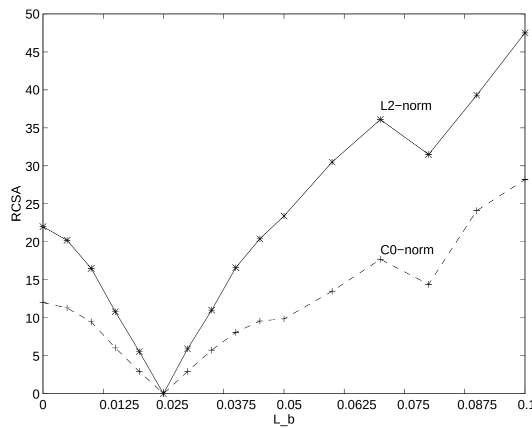

FIG. 5.4. *Incident wave in the direction* $d = (\sqrt{2}/2, \sqrt{2}/2)$

$0 \leq L_b \leq 0.1$, while maintaining its length to width ratio constant $L_b/l_b = 2$. Note that $L_b = 0$ corresponds to removing the bump, and $L_b = 0.1$ to increasing its initial area by a factor equal to 16. However, it should be noted that even in the latter case, the size of the perturbation 0.005 is only about 5% of the size of the obstacle 0.0918125. In each case, the BVP is solved for each of the three incident waves described above, and the relative change in the scattering amplitude (RCSA) with respect to the configuration with the initial bump size, $RCSA = \|u_\infty(L_b) - u_\infty(L_b^0)\|/\|u_\infty(L_b^0)\|$, is recorded for both $\mathcal{C}^0(S^1)$ and $L^2(S^1)$ norms. All computations are performed on a 64-processor Origin 2000 Silicon Graphics system with 4 Gigabytes of real memory. The obtained numerical results are reported in Figs. 5.2–5.4.

Let $\lambda = 2 \times \Pi/k \approx 0.05$ be the common wavelength to all three incident waves. The following observations are noteworthy

- the continuous dependence of the scattering amplitude on the shape of the scatterer is well illustrated for all three incident waves using both norms.
- for $0 \leq (L_b - L_b^0) < 0.025 = \lambda/2$, and $-\lambda/2 = -0.025 < (L_b - L_b^0) \leq 0$, the scattering amplitude is shown to depend *linearly* on the size of the

FIG. 5.5. *Effect of the direction of the incident wave*

perturbation for both norms.

- if one sets the critical relative change in the scattering amplitude (RCSA) to 10% — that is, the relative change in the scattering amplitude beyond which two scatterers are declared to be two different obstacles — the results reported in Figs. 2–4 suggest that the maximum relative perturbation of the obstacle area for which the RCSA is not reached — or the minimum relative perturbation of the obstacle area for which the RCSA is exceeded is equal to 0.05% when the RCSA is measured the $L^2(S^1)$ norm, and is equal to 6.6 % when it is measured with the the $\mathcal{C}^0(S^1)$ norm.

In order to further investigate the effect of the direction of the incident wave on the RCSA for a fixed perturbation of the bump size, we set $L_b = 2 \times L_b^0 = 0.05$ which doubles the initial area of the bump, and vary the angle α of the incident wave ($d = (\cos\alpha, \sin\alpha)^T$ between 0^{deg} and 180^{deg}. For this problem, the results shown in Fig. 5.5 show that the variation of the direction of the incident wave between 40^{deg} and 140^{deg} has little effect on the RCSA. However, for $0^{\text{deg}} \leq \alpha \leq 40^{\text{deg}}$ and $140^{\text{deg}} \leq \alpha \leq 180^{\text{deg}}$, the RCSA varies by up to 50%.

Further computational results can be found in [20].

6. Conclusions. In this paper, we have established the continuous dependence of the scattering amplitude on the shape of the scatterer, by proving first that both the scattered field and the far-field pattern are differentiable with respect to the domain of the obstacle. We have also illustrated our stability result with the investigation at a high frequency of the sensitivity of the scattering amplitude to changes in the shape of a given scatterer. We have shown numerically that for perturbation sizes smaller than half the incident wave length, the scattering amplitude depends linearly on the size of the perturbation. We have also shown that for a fixed perturbation of the shape of the scatterer, the direction of the incident wave has little effect on the perturbation of the scattering amplitude.

Acknowledgements. The authors acknowledge the support by the Office of Naval Research under Grant N-00014-95-1-0663, and thank the Center for Computational Science at the Navy Research Laboratory for providing them the CPU time needed for performing this research.

Appendix A. Fréchet derivatives and Lipschitz domains.

The statements here are formulated in $\mathbb{R}^n$, $n \geq 1$, since they are of a more general nature.

A.1. Fréchet derivatives of Nemytski operators. We will need the following auxiliary results on Fréchet differentiability of operators defined by composition, known as Nemytski operators [7]. The proofs are simple exercises in Nonlinear Functional Analysis and they are included only for completeness.

LEMMA A.1. *Let M be a measurable set and $F = F(x,z) : M \times \mathbb{R}^m \to \mathbb{R}^n$ be bounded and differentiable, with bounded derivative, all uniformly with respect to $x \in M$ and z in a neighborhood of zero in $\mathbb{R}^m$. Then the Nemytski operator*

$$\mathcal{F} : f \mapsto \big(x \mapsto F(x, f(x))\big)$$

is Fréchet differentiable on a neighborhood of zero from $(L^\infty(M))^m$ to $(L^\infty(M))^n$, and

$$\mathcal{F}'(f) : h \mapsto \big(x \mapsto \nabla_z F(x, f(x))h(x)\big).$$

If, in addition, the derivative $\nabla_z F(x,z)$ is continuous function of z uniformly with respect to $x \in M$ and z on a neighborhood of zero in $\mathbb{R}^m$, then $\mathcal{F}'(f)$ is a continuously Fréchet differentiable function of f in a neighborhood of zero from $(L^\infty(M))^m$ to the uniform operator topology from $(L^\infty(M))^m$ to $(L^\infty(M))^n$.

Proof. Write the first assumption as

$$\begin{gathered} |F(x,z)| \le C, \quad |\nabla_z F(x,z)| \le C, \\ |F(x, z+tw) - F(x,z) - t\nabla_z F(x,z)w| \le \xi(t)|w|, \end{gathered} \tag{A.1}$$

for all $x \in M$, $z \in \mathbb{R}^n$, $|z| \le c$, $|w| < c$, $0 < t < 1$, where $c > 0$ and $\xi(t) \to 0$ as $t \to 0$. Let $f, h \in L^\infty(M)^m$, $\|f\|_{L^\infty(M)^m} \le c/2$, $0 < t < c/2$, $\|h\|_{L^\infty(M)^m} \le 1$. From (A.1), $|F(x, f(x))| \le C$ a.e. on M, and

$$|F(x, f(x)+th(x)) - F(x, f(x)) - t\nabla_z F(x, f(x))h(x)| \le \xi(t),$$

a.e. on M. Hence, $\mathcal{F}(f) \in L^\infty(M)^n$, $\|\mathcal{F}'(f)(h)\|_{L^\infty(M)^n} \le C\|h\|_{L^\infty(M)^m}$, and

$$\|\mathcal{F}(f+th) - \mathcal{F}(f) - t\mathcal{F}'(f)h\|_{L^\infty(M)^n} \le \xi(t)\|h\|_{L^\infty(M)^m}. \tag{A.2}$$

To prove that the derivative is continuous, write the second assumption as

$$|\nabla_z F(x,z) - \nabla_z F(x,\bar z)| \le \eta(|z-\bar z|),$$

for all $|z| \le c$, $|\bar z| \le c$, where $\eta(\tau) \to 0$ as $\tau \to 0$, and η is nonincreasing. Then

$$|\nabla_z F(x, f(x))h(x) - \nabla_z F(x, \bar f(x))h(x)| \le \eta(|f(x) - \bar f(x)|)|h(x)|$$

a.e. on M, hence

$$\|\mathcal{F}'(f)h - F'(\bar f)h\|_{L^\infty(M)} \le \eta(\|f - \bar f\|^n_{L^\infty(M)})\|h\|_{L^\infty(M)^m}, \tag{A.3}$$

which shows that the Fréchet derivative is continuous. □

LEMMA A.2. *Let $F \in \mathcal{C}^2(-\delta,\delta)$, $\delta > 0$, and J be an interval. Then the mapping $\mathcal{F} : f \mapsto F \circ f$ is continuously Fréchet differentiable on a neighborhood of zero from $C^{0,1}(J)$ to $C^{0,1}(J)$.*

Proof. Let $\|f\|_{\mathcal{C}^{0,1}(J)} < \delta/2$. In view of Lemma A.1, cf., (A.2), to prove the existence of the Fréchet derivative, we only need to show that

$$\left|\frac{d}{dx}\big(F(f(x)+th(x)) - F(f(x)) - tF'(f(x))h(x)\big)\right| \le \xi(t)\|h\|_{\mathcal{C}^{0,1}(J)}, \tag{A.4}$$

a.e. in J and for all $\|f\|_{\mathcal{C}^{0,1}(J)} \le \delta/2$ and $\|h\|_{\mathcal{C}^{0,1}(J)} \le \delta/2$, with $\xi(t) \to 0$ as $t \to 0$. Since F' is continuously differentiable, we have

$$F'(f(x)+th(x)) = F'(f(x)) + tF''(f(x))h(x) + t\varepsilon(t,x), \tag{A.5}$$

where $\sup_J |\varepsilon(t,x)| \to 0$ as $t \to 0$, because $f(x), h(x) \in [-\delta/2, \delta/2]$. Since f and h are Lipschitz continuous, f' and h' exist a.e. in J. Using (A.5) and the chain rule at the points where f' and h' exist, we obtain by a direct computation that

$$\begin{aligned} &\frac{d}{dx}\big(F(f(x)+th(x)) - F(f(x)) - tF'(f(x))h(x)\big) \\ &\quad = t\varepsilon(t,x)f'(x) + t^2F''(f(x))h'(x)h(x) + t^2\varepsilon(t,x)h'(x), \end{aligned}$$

which proves (A.4). Again, in view of Lemma A.1, cf., (A.3), to prove that the Fréchet derivative is continuous, it is enough to show that for all $\|f\|_{\mathcal{C}^{0,1}(J)} < \delta/2$, $h \in \mathcal{C}^{0,1}(J)$, it holds that

$$\left\| \frac{d}{dx}(\mathcal{F}'(f)h - \mathcal{F}'(\bar{f})h) \right\|_{L^\infty(J)} = \eta(\|f-\bar{f}\|_{\mathcal{C}^{0,1}(J)})\|h\|_{\mathcal{C}^{0,1}(J)}, \tag{A.6}$$

where $\eta(t) \to 0$ as $t \to 0$. But

$$\frac{d}{dx}(\mathcal{F}'(f)h)(x) = (F'(f(x))h(x))' = F''(f(x))f'(x)h(x) + F'(f(x))h'(x)$$

a.e. in J, and (A.6) follows from the uniform continuity of F' and F'' on the interval $[-\delta/2, \delta/2]$. □

A.2. Lipschitz transformation of Lipschitz domain. We will need some properties of the transformation of domains with Lipschitz boundary and the associated boundary integral by a Lipschitz continuous mapping. These results do not appear to be standard and they may be of independent interest.

Recall [13, 1, 5] that a domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is defined to have Lipschitz boundary if there exists a finite collection of open sets $\mathcal{O}^0, \ldots, \mathcal{O}^m$ such that

$$\Omega \subset \bigcup_{j=0}^{m} \mathcal{O}^j, \qquad \Gamma \subset \bigcup_{j=1}^{m} \mathcal{O}^j, \tag{A.7}$$

local orthogonal systems of coordinates $x = (\tilde{x}^j, x_n^j)$, $\tilde{x}^j \in \mathbb{R}^{n-1}$, $x_n^j \in \mathbb{R}$, and Lipschitz continuous functions $f^j(\tilde{x}^j)$, defined on the projection $\tilde{\mathcal{O}}^j$ of $\mathcal{O}^j$ on the $\tilde{x}^j$ plane, such that, for all j,

$$\Omega \cap \mathcal{O}^j = \{(\tilde{x}^j, x_n^j) \in \mathcal{O}^j \mid x_n^j < f^j(\tilde{x}^j)\}. \tag{A.8}$$

LEMMA A.3. *Let $\Omega \subset \mathbb{R}^n$ be a domain with Lipschitz boundary. Then there exists $c = c(\Omega) > 0$ such that $\Omega_\theta = (I+\theta)\Omega$ has Lipschitz boundary for all $\theta : \mathbb{R}^n \to \mathbb{R}^n$ with $|\theta|_{\mathrm{Lip}} < c$.*

Proof. Let $|\theta|_{\mathrm{Lip}} < 1$. It is a simple exercise involving the Banach contraction principle to show that there exists $\psi \in (\mathcal{C}^{0,1}(\mathbb{R}^n))^n$ such that

$$(I+\theta)^{-1} = I + \psi, \qquad |\psi|_{\mathrm{Lip}} \leq \frac{|\theta|_{\mathrm{Lip}}}{1-|\theta|_{\mathrm{Lip}}}. \tag{A.9}$$

Let $\mathcal{O}^j$ and f^j be as in (A.7) and (A.8). Since $I+\theta$ has continuous inverse, $\mathcal{O}_\theta^j = (I+\theta)\mathcal{O}^j$ are open sets, and (A.7) holds with Ω_θ, $\mathcal{O}_\theta^j$, and Γ_θ in place of Ω, $\mathcal{O}^j$, and Γ, respectively. We choose the same systems of local orthogonal coordinates for $y = (I+\theta)x$ as for x; hence,

$$\tilde{\mathcal{O}}_\theta^j = \tilde{\mathcal{O}}^j. \tag{A.10}$$

Write

$$y = (\tilde{y}^j, y_n^j),\ x = \psi(\tilde{y}^j, y_n^j) = (\tilde{\psi}^j(\tilde{y}^j, y_n^j), \psi_n^j(\tilde{y}^j, y_n^j)).$$

The proof will be completed if we show that

$$\Omega_\theta \cap \mathcal{O}_\theta^j = \{(\tilde{y}^j, y_n^j) \in \mathcal{O}_\theta^j \mid y_n^j < g_\theta^j(\tilde{y}^j)\} \tag{A.11}$$

for some Lipschitz continuous functions g^j_θ. Let $y = (\tilde{y}^j, y^j_n) \in \mathcal{O}^j_\theta$, $x = \psi(y)$. Then

$$\begin{aligned}(\tilde{y}^j, y^j_n) \in \Omega_\theta &\iff x^j_n < f^j(\tilde{x}^j)\\ &\iff y^j_n + \psi^j_n(\tilde{y}^j, y^j_n) < f^j(\tilde{y}^j + \tilde{\psi}^j(\tilde{y}^j, y^j_n))\\ &\iff y^j_n < L^j_\theta(\tilde{y}^j, y^j_n),\end{aligned}$$

where

$$L^j_\theta(\tilde{y}^j, y^j_n) = f^j(\tilde{y}^j + \tilde{\psi}^j(\tilde{y}^j, y^j_n)) - \psi^j_n(\tilde{y}^j, y^j_n). \tag{A.12}$$

We can estimate

$$|L^j_\theta|_{\mathrm{Lip}} \leq (|f^j|_{\mathrm{Lip}} + 1)|\psi|_{\mathrm{Lip}} \leq (|f^j|_{\mathrm{Lip}} + 1)\frac{|\theta|_{\mathrm{Lip}}}{1 - |\theta|_{\mathrm{Lip}}} < 1, \tag{A.13}$$

cf., (A.9), if $|\theta|_{\mathrm{Lip}}$ is sufficiently small.

Let $\tilde{x}^j = \tilde{y}^j \in \tilde{\mathcal{O}}^j$ be fixed. From (A.13), it follows that $y^j_n - L^j(\tilde{y}^j, y^j_n)$ is a strictly increasing continuous function of y^j_n. Since there is x^j_n such that $(\tilde{x}^j, x^j_n) \in \Omega \cap \mathcal{O}^j$, hence $x^j_n < f^j(\tilde{x}^j)$, there exists an y^j_n such that

$$(\tilde{y}^j, y^j_n) \in \mathcal{O}^j_\theta, \qquad y^j_n < L^j(\tilde{y}^j, y^j_n).$$

Similarly, there is an y^j_n such that

$$(\tilde{y}^j, y^j_n) \in \mathcal{O}^j_\theta, \qquad y^j_n > L^j(\tilde{y}^j, y^j_n).$$

Consequently, there exists a unique y^{j*}_n such that $y^{j*}_n = L(\tilde{y}, y^{j*}_n)$. It follows that for every y^j_n, $(\tilde{y}^j, y^j_n) \in \Omega^j_\theta$ is equivalent to $y^j_n < y^{j*}_n$. If we define $g^j_\theta(\tilde{y}^j) = y^{j*}_n$,, then (A.11) holds. □

Remark A.4. *Note that in general, Ω_θ need not have Lipschitz boundary when Ω does and $|\theta|_{\mathrm{Lip}} < 1$. E.g., if*

$$\Omega = \left\{(x_1, x_2) \in \mathbb{R}^2 \mid 0 < x_1 < 1, -\frac{x_1}{4} < x_2 < \frac{x_1}{4}\right\}, \quad \theta(x_1, x_2) = \left(0, \frac{x_1}{3}\sin\frac{1}{x_1}\right),$$

then there is no function that would describe Ω_θ on a neighborhood of zero as in (A.8).

When Ω has Lipschitz boundary, one can define the Lebesgue boundary measure by

$$\mu_\Gamma(E) = \int_{\tilde{E}} (1 + |\nabla f^j|^2)^{1/2}\, d\tilde{x}, \qquad E \subset \Gamma \cap \mathcal{O}^j,$$

where $\tilde{E}$ is the projection of E on the $\tilde{x}^j$ plane. The Lebesgue boundary integral then equals to

$$\int_\Gamma h\, d\mu_\Gamma = \sum_{j=1}^{m} \int_{\tilde{\mathcal{O}}^j} h(\tilde{x}^j, f^j_n(\tilde{x}^j))\phi^j(\tilde{x}^j, f^j_n(\tilde{x}^j))(1 + |\nabla f^j(\tilde{x}^j)|^2)^{1/2}\, d\tilde{x}^j, \tag{A.14}$$

where $\{\phi^j\}$ is a decomposition of unity,

$$\phi^j \in \mathcal{D}(\mathcal{O}^j), \quad \sum_{j=0}^{m} \phi^j = 1 \text{ on } \overline{\Omega},$$

cf., e.g. [13].

Since a Lipschitz function is differentiable a.e., the normal vector ν exists a.e. on Γ, and it follows from Lemma A.3 that the normal ν_θ of Γ_θ exists a.e. on Γ_θ if θ is in a neighborhood of zero in $(\mathcal{C}^{0,1}(\mathbb{R}^n))^n$.

LEMMA A.5. *Let $\Omega \subset \mathbb{R}^n$ be a domain with Lipschitz boundary. Then the mapping $\theta \mapsto \nu_\theta \circ (I+\theta)$ is continuously Fréchet differentiable on a neighborhood of zero from $(\mathcal{C}^{0,1}(\mathbb{R}^n))^n$ to $L^\infty(\Gamma)^n$.*

Proof. Using the notation of the proof of Lemma A.3, we have the local parametric descriptions of Γ_θ,

$$\Gamma_\theta \cap \mathcal{O}^j = \{p^j_\theta(\tilde{x}^j) \mid \tilde{x}^j \in \tilde{\mathcal{O}}^j\}, \quad p^j_\theta(\tilde{x}^j) = (I+\theta)(\tilde{x}^j, f^j(\tilde{x}^j)), \tag{A.15}$$

cf., (A.10). The mapping $\theta \mapsto p^j_\theta$ is linear and bounded from $(\mathcal{C}^{0,1}(\mathbb{R}^n))^n$ to $\mathcal{C}^{0,1}(\tilde{\mathcal{O}}^j)$, hence the mapping $\theta \mapsto \nabla p^j_\theta$ is linear and bounded from $(\mathcal{C}^{0,1}(\mathbb{R}^n))^n$ to $L^\infty(\tilde{\mathcal{O}}^j)^{n-1}$. But $\nu_\theta \circ (I+\theta)(x) = F(\nabla_{\tilde{x}}\, p^j_\theta(x))$, with

$$F(v) = \frac{N(v)}{|N(v)|}, \quad N(v) = \det \begin{pmatrix} \mathbf{e}_1 & \dots & \mathbf{e_n} \\ v_{11} & \dots & v_{1n} \\ & \vdots & \\ v_{n-1,1} & \dots & v_{n-1,n} \end{pmatrix}, \quad v_{lm} = \frac{\partial (p^j_\theta(x))_m}{\partial \tilde{x}^j_l},$$

where $\mathbf{e_l}$ is the l-th canonical unit vector in $\mathbb{R}^n$. It remains to apply Lemma A.1. □

We will state a result about Lipschitz continuous change of variable for reference, cf., [22, Corollary 2].

LEMMA A.6. *Let $M \subset \mathbb{R}^n$ be open, $T : M \to \mathbb{R}^n$ be Lipschitz continuous and $\det \nabla T > 0$ a.e. on M. Then, for any $f \in L^\infty(M)$, it holds that $f \circ T^{-1} \in L^\infty(T(M))$, and $\int_{T(M)} f \circ T^{-1} \det \nabla T \, d\mu = \int_M f \, d\mu$.*

We need a similar result for Lipschitz continuous change of variable in boundary integral. For the case when the change of variable is continuously differentiable, cf. [18, Lemma 4.1].

LEMMA A.7. *Let $\Omega \subset \mathbb{R}^n$ have Lipschitz boundary. Then there is a constant $C > 0$ such that if $\|\theta\|_{\mathrm{Lip}} < C$, then*

$$\int_{\Gamma_\theta} w \, d\mu_{\Gamma_\theta} = \int_\Gamma w \circ (I+\theta) J_{T\theta} \, d\mu_\Gamma \tag{A.16}$$

for all $w \in L^1(\Gamma_\theta)$, where the tangential Jacobian $J_{T\theta} \in L^\infty(\Gamma)$. The mapping $\theta \mapsto J_{T\theta}$ is continuously Fréchet differentiable on a neighborhood of zero from $(\mathcal{C}^{0,1}(\mathbb{R}^n))^n$ to $L^\infty(\Gamma)$.

Proof. We again use the notation of the proof of Lemma A.3. Due to (A.14), we only need to consider the case when $\operatorname{supp} w \subset \mathcal{O}^j_\theta \cap \Gamma_\theta$. From the parametric description of the boundary (A.15), it follows that

$$\int_{\mathcal{O}^j \cap \Gamma} w \circ (I+\theta) \, d\mu_\Gamma = \int_{\tilde{\mathcal{O}}^j} w \circ (I+\theta)(\tilde{x}^j, f^j(\tilde{x}^j)) |\nabla_{\tilde{x}^j}(\tilde{x}^j, f^j(\tilde{x}^j))| \, d\tilde{x}^j,$$

$$\int_{\mathcal{O}^j_\theta \cap \Gamma_\theta} w \, d\mu_{\Gamma_\theta} = \int_{\tilde{\mathcal{O}}^j} w \circ (I+\theta)(\tilde{x}^j, f^j(\tilde{x}^j)) |\nabla_{\tilde{x}^j}((I+\theta)(\tilde{x}^j, f^j(\tilde{x}^j)))| \, d\tilde{x}^j.$$

Comparing these two equations, we obtain (A.16) with

$$J_{T\theta}(\tilde{x}^j) = \frac{\left|\nabla_{\tilde{x}^j}\left((I+\theta)(\tilde{x}^j, f^j(\tilde{x}^j))\right)\right|}{|\nabla_{\tilde{x}^j}(\tilde{x}^j, f^j(\tilde{x}^j))|}.$$

It remains to note that the mapping

$$\theta \mapsto \left(\tilde{x}^j \mapsto \nabla_{\tilde{x}^j}(I+\theta)(\tilde{x}^j, f^j(\tilde{x}^j))\right)$$

is linear and bounded from $(\mathcal{C}^{0,1}(\mathbb{R}^n))^n$ to $(L^\infty(\tilde{\mathcal{O}}^j))^n$, and to use Lemma A.1. □

REMARK A.8. *The proof of Lemma A.3 was more complicated because we needed to prove that Ω_θ lies locally on one side of Γ_θ; in the proofs of Lemmas A.5 and A.7, we could simply operate with the parametric description of Γ_θ by Lipschitz continuous functions.*

REFERENCES

[1] R. A. ADAMS, *Sobolev spaces*, Academic press, New York, 1975.

[2] M. S. BERGER, *Nonlinearity and Functional Analysis*, Academic Press, New York, 1977.

[3] D. COLTON AND R. KRESS, *Inverse Acoustic and Electromagnetic Scattering Theory*, Springer Verlag, Berlin, 1992.

[4] J. GIROIRE, *Étude de quelques problèmes aux limites exterieurs et résolution par equations integrales.* Thèse de Doctorat d'État, Université Pierre et Marie Curie, Paris, 1987.

[5] P. GRISVARD, *Elliptic Problems in Nonsmooth Domains*, Pitman, Boston, 1985.

[6] F. HETTLICH, *Fréchet-differentiability of boundary integral operators in inverse acoustic scattering*, Inverse Problems, 10 (1995), pp. 371–382.

[7] V. KHATSKEVICH AND D. SHOIYKHET, *Differentiable Operators and Nonlinear Equations*, vol. 66 of Operator Theory, Advances and Applications, Birkhäuser, Basel, 1994.

[8] A. KIRSH, *The domain derivative and two applications in inverse scattering theory*, Inverse Problems, 9 (1993), pp. 81–86.

[9] P. LAX AND R. S. PHILLIPS, *Scattering Theory*, Academic Press, New York, 1989.

[10] B. LEE, T. MANTEUFFEL, S. MCCORMICK, AND J. RUGE, *Multilevel first-order system least squares (FOSLS) for Helmholtz equation.* Presented at 2nd International Conf. on Approx. and Num. Meths. for the Solution of the Maxwell Equations, Washington, D.C. ftp://amath.colorado.edu/pub/multigrid/helm.ps, 1993.

[11] D. S. MALKUS AND T. J. R. HUGHES, *Reduced and selective integration techniques: A unification of concepts*, Comput. Meths. Appl. Mech. Engrg., 15 (1978), pp. 63–81.

[12] M. MITREA, *Bouldary value problems and Hardy spaces associated to the Helmholtz equation*, J. Math. Anal. Appl., 202 (1996), pp. 819–842.

[13] J. NEČAS, *Les méthodes directes en théorie des équations elliptiques*, Academia, Prague, 1967.

[14] R. POTTHAST, *Frechet-differentiability of boundary integral operators in inverse acoustic scattering*, Inverse Problems, 10 (1994), pp. 431–447.

[15] A. G. RAMM, *Continuous dependence of scattering amplitude on the surface of an obstacle*, Math. Meth. Appl. Sci., 18 (1985), pp. 121–126.

[16] ———, *Scattering by obstacles*, D. Reidel Publ. Co., Dotrecht, Holland, 1986.

[17] J. SANCHEZ HUBERT AND E. SANCHEZ-PALENCIA, *Vibration and Coupling of Continuous Systems*, Springer-Verlag, Berlin, 1989.

[18] J. SIMON, *Differentiation with respect to the domain in boundary value problems*, Num. Funct. Anal. Opt., 2 (1980), pp. 649–687.

[19] P. VANĚK, J. MANDEL, AND M. BREZINA, *Algebraic multigrid based on smoothed aggregation for second and fourth order problems*, Computing, 56 (1996), pp. 179–196.

[20] P. VANĚK, J. MANDEL, AND M. BREZINA, *Solving a two-dimensional Helmholtz problem by algebraic multigrid*, UCD/CCM Report 110, Center for Computational Mathematics, University of Colorado at Denver, October 1997. http://www-math.cudenver.edu/ccmreports/rep110.ps.gz.

[21] ———, *Two-level algebraic multigrid for the Helmholtz problem*, Contemporary Mathematics, 218 (1998), pp. 349–356. Proceedings of the 10th International Conference on Domain Decomposition, J. Mandel, C. Farhat, and X.-C. Cai, editors, AMS, Providence, RI, in print.

[22] D. VARBERG, *Change of variables in multiple integrals*, Amer. Math. Monthly, 78 (1971), pp. 42–45.

[23] C. H. WILCOX, *Scattering theory for the D'Alembert equation in exterior domains*, vol. 442 of Lecture Notes in Mathematics, Springer-Verlag, Berlin, 1975.