ON PETROV-GALERKIN FORMULATIONS FOR THE LINEAR HYPERBOLIC EQUATION

L. P. Franca
Department of Mathematics
University of Colorado at Denver
P.O. Box 173364, Campus Box 170
Denver, CO 80217-3364, USA

A. Russo
Istituto di Analisi Numerica del CNR
via Abbiategrasso 209
27100 Pavia, ITALY

Abstract

We consider conforming Petrov-Galerkin formulations for the advective and advective-diffusive equations. For the linear hyperbolic equation, the continuous formulation is set up using different spaces and the discretization follows with different "bubble" enrichments for the test and trial spaces. Boundary conditions for residual-free bubbles are modified to accommodate with the first order equation case and regular bubbles are used to enrich the other space. Using piecewise linears with these enrichments, the final formulations are shown to be equivalent to the SUPG method, provided the data is assumed to be piecewise constant. Generalization to include diffusion is also presented.

1 Introduction

The linear hyperbolic equation is simply the "pure" advective equation, where we drop the diffusion term from an advection-diffusion equation. The immediate consequence is that the equation is first order and therefore suitable boundary conditions have to be considered to make the problem well posed. The variational spaces for trial and test functions in this case differ from the advective-diffusive equation in that only $L_2(\Omega)$ functions are required plus $L_2(\Omega)$ for the streamline derivative. The other space consists of $L_2(\Omega)$ functions.

In this paper we build up on our study of Galerkin methods using residual-free bubbles [3–11] to explore the applicability of those discretizations in the limit of vanishing diffusivity. It turns out that the Galerkin method is not the appropriate starting point in the limit case, if we wish to reproduce the SUPG method, after eliminating the bubbles. Instead, we consider the trial space enriched with regular bubbles (say the MINI bubble [1], a cubic polynomial on a triangle) that have zero value on the boundary of the element. For the test space, we enrich it with *adjoint* residual-free bubbles. Here, these bubbles are zero only in part of the boundary, as we elaborate in the paper.

In the next section we describe the differential equation and the variational formulation with respective continuous spaces. Discretization using a conforming Petrov-Galerkin is described in the following section. Then we establish a relationship with the SUPG method by eliminating the bubbles degrees-of-freedom. In the subsequent section the model includes diffusion and, with a similarly defined Petrov-Galerkin formulation, it is shown that the method can also be reduced to a SUPG method.

2 The Linear Hyperbolic Model

Herein, we will consider discretizations of the linear hyperbolic model, defined by the following set of equations:

$$\begin{cases} \boldsymbol{a} \cdot \nabla u = f & \text{in } \Omega\,, \\ \qquad u = g & \text{on } \partial\Omega^-\,. \end{cases} \tag{1}$$

where $\boldsymbol{a}$ is the velocity field, f is a given source function, g is a prescribed Dirichlet boundary condition, and $\partial\Omega^-$ is the inflow boundary defined by:

$$\partial\Omega^- = \{x \in \partial\Omega : \boldsymbol{a}(x) \cdot \boldsymbol{n}(x) < 0\}. \tag{2}$$

For the sake of simplicity, we assume the domain Ω to be a polygon, and the velocity field $\boldsymbol{a}$ to be piecewise-constant.

To set up an appropriate variational framework for this model, let us consider the following spaces:

$$\begin{cases} X = \{u \in L_2(\Omega) \mid \boldsymbol{a} \cdot \nabla u \in L_2(\Omega) \text{ and } u = 0 \text{ on } \partial\Omega^-\}, \\ Y = L_2(\Omega) \end{cases} \tag{3}$$

with corresponding norms:

$$\begin{cases} ||u||_X = \sqrt{||u||^2 + ||\boldsymbol{a} \cdot \nabla u||^2}, \\ ||v||_Y = ||v|| = ||v||_{L_2(\Omega)} = \sqrt{\int_\Omega v^2}\,. \end{cases} \tag{4}$$

We define $X^{(g)}$ as

$$X^{(g)} = \{u \in L_2(\Omega) \mid \boldsymbol{a} \cdot \nabla u \in L_2(\Omega) \text{ and } u = g \text{ on } \partial\Omega^-\}. \tag{5}$$

We may now write the variational formulation corresponding to (1) as:

$$\begin{cases} \text{find } u \in X^{(g)} \text{ such that} \\ (\boldsymbol{a} \cdot \nabla u, v) = (f, v) \quad \forall v \in Y \end{cases} \tag{6}$$

where $(\cdot\,,\,\cdot)$ stands for the usual $L_2(\Omega)$ inner product.

Remark: The continuous problem is posed in spaces that are different for trial and test functions. This naturally leads to the question of the appropriate framework for discretization. While in the presence of a small diffusion, test and trial spaces can be posed as subspaces of $H^1(\Omega)$ and the Galerkin method can be used (see, e.g.,[2]), here the distinct nature of the spaces for the continuous problem seems to indicate that the Petrov Galerkin method should be considered as an alternative for discretizations.

3 A Petrov-Galerkin Method

Let $\mathscr{T}_h = \{K\}$ be a partition of Ω into triangles, with $h = \max_K \operatorname{diam}(K)$, such that $\boldsymbol{a}$ and f are piecewise-constant with respect to $\mathscr{T}_h$. For any element $K \in \mathscr{T}_h$, we select a bubble b_K, i.e. an element of $H_0^1(K)$, such that $\int_K b_K \neq 0$. We define:

$$B_h = \bigoplus_{K \in \mathscr{T}_h} \operatorname{span}\{b_K\}. \tag{7}$$

In our case, the usual cubic bubble employed in the MINI-element can be used. Then, for each element K, we define the adjoint residual-free bubble, which in this case is the solution to:

$$\begin{cases} -\boldsymbol{a}\cdot\nabla\widetilde{b}_K = 1 & \text{in } K\,, \\ \qquad\;\; \widetilde{b}_K = 0 & \text{on } \partial K^+\,. \end{cases} \tag{8}$$

where the element outflow boundary ∂K^+ is defined as $\partial K^+ = \{x \in \partial K \mid \boldsymbol{a}(x)\cdot\boldsymbol{n}(x) > 0\}$. In Figure 1 we plot a typical $\widetilde{b}_K$. Let us define

$$ARFB_h = \bigoplus_{K \in \mathscr{T}_h} \operatorname{span}\{\widetilde{b}_K\}. \tag{9}$$

We may now construct the subspaces to be used in the Petrov-Galerkin approximation. We start by introducing

$$V_1 = \{v \in C^0(\bar{\Omega}) \mid v_{|K} \text{ is linear}\} \tag{10}$$

and

$$\begin{aligned} V_1^{(0)} &= \{v \in V_1 \mid v = 0 \text{ on } \partial\Omega^-\} \\ V_1^{(g)} &= \{v \in V_1 \mid v = g \text{ on } \partial\Omega^-\}\,. \end{aligned} \tag{11}$$

Then we take $X_h^{(g)} \subset X^{(g)}$ and $Y_h \subset Y$ as

$$X_h^{(g)} = V_1^{(g)} \oplus B_h, \quad Y_h = V_1^{(0)} \oplus ARFB_h. \tag{12}$$

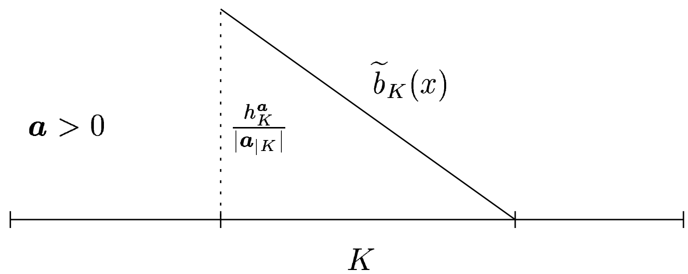

Figure 1: The adjoint residual-free bubble in 1D

In other words, we take as the trial space: continuous, piecewise linear plus bubbles; and as the test space: continuous, piecewise linear plus adjoint residual-free bubbles. Note that the functions in the test space Y_h are generally discontinuous. Thus, our Petrov-Galerkin approximation of (6) is:

$$\left\{ \begin{array}{l} \text{find } u_h \in X_h^{(g)} \text{ such that} \\ (\boldsymbol{a} \cdot \nabla u_h, v_h) = (f, v_h) \quad \forall v_h \in Y_h \end{array} \right. \tag{13}$$

Any $u_h \in X_h^{(g)}$ can be written in a unique way as $u_h = u_1 + u_b$ with $u_b \in B_h$ and $u_b = \sum_K c_K b_K$, and any $v_h \in Y_h$ can be written as $v_h = v_1 + v_{\tilde{b}}$ with $v_{\tilde{b}} \in ARFB_h$ and $v_{\tilde{b}} = \sum_K \tilde{c}_K \widetilde{b}_K$.

4 Relationship with SUPG

The aim of our analysis is to show that after the bubbles have been eliminated, our Petrov-Galerkin approximation gives the SUPG method for problem (1).

We start by taking as test function the adjoint residual-free bubble, i.e., $v_h = \widetilde{b}_K$ on element K and zero elsewhere. From (13), we have:

$$(\boldsymbol{a} \cdot \nabla (u_1 + c_K b_K), \widetilde{b}_K)_K = (f, \widetilde{b}_K)_K \tag{14}$$

and

$$c_K \, (\boldsymbol{a} \cdot \nabla b_K, \widetilde{b}_K)_K = (f - \boldsymbol{a} \cdot \nabla u_1, \widetilde{b}_K)_K. \tag{15}$$

By integration by parts, we have

$$(\boldsymbol{a} \cdot \nabla b_K, \widetilde{b}_K)_K = -(b_K, \boldsymbol{a} \cdot \nabla \widetilde{b}_K)_K + \int_{\partial K} b_K \widetilde{b}_K \boldsymbol{a} \cdot \boldsymbol{n} \tag{16}$$

and the last term is zero because $b_K = 0$ on ∂K. By definition of $\widetilde{b}_K$, we have

$$\boldsymbol{a} \cdot \nabla \widetilde{b}_K = -1, \tag{17}$$

and combining with (15) and (16) above yields

$$c_K = (f - \boldsymbol{a} \cdot \nabla u_1)_{|K} \frac{\int_K \widetilde{b}_K}{\int_K b_K} \tag{18}$$

(note that by $\int_K b_K \neq 0$ by assumption). The argument is repeated for each element K to get all c_K's. Taking now $v_h = v_1$ in (13), we get:

$$(\boldsymbol{a} \cdot \nabla(u_1 + u_b), v_1) = (f, v_1) \tag{19}$$

or

$$(\boldsymbol{a} \cdot \nabla(u_1 + \sum_K c_K b_K), v_1) = (f, v_1). \tag{20}$$

Thus

$$(\boldsymbol{a} \cdot \nabla u_1, v_1) + \sum_K c_K \, (\boldsymbol{a} \cdot \nabla b_K, v_1)_K = (f, v_1). \tag{21}$$

By integration by parts

$$(\boldsymbol{a} \cdot \nabla b_K, v_1)_K = -(\boldsymbol{a} \cdot \nabla v_1, b_K)_K + \int_{\partial K} v_1 b_K \boldsymbol{a} \cdot \boldsymbol{n} \tag{22}$$

where the last integral is zero again because $b_K = 0$ on ∂K. Then we have

$$(\boldsymbol{a} \cdot \nabla b_K, v_1)_K = -(\boldsymbol{a} \cdot \nabla v_1)_{|K} \int_K b_K. \tag{23}$$

Hence (21) becomes

$$(\boldsymbol{a} \cdot \nabla u_1, v_1) + \sum_K c_K \left[-(\boldsymbol{a} \cdot \nabla v_1)_{|K} \int_K b_K \right] = (f, v_1) \tag{24}$$

By recalling the formula for c_K we have:

$$(\boldsymbol{a} \cdot \nabla u_1, v_1) + \sum_K (\boldsymbol{a} \cdot \nabla u_1 - f)_{|K} \frac{\int_K \widetilde{b}_K}{\int_K b_K} \, (\boldsymbol{a} \cdot \nabla v_1)_{|K} \int_K b_K = (f, v_1) \tag{25}$$

and by simplifying $\int_K b_K$, we end up with the following equation:

$$(\boldsymbol{a}\cdot\nabla u_1, v_1) + \sum_K \left[\frac{1}{|K|}\int_K \widetilde{b}_K\right] \int_K (\boldsymbol{a}\cdot\nabla u_1 - f)\,\boldsymbol{a}\cdot\nabla v_1 = (f, v_1) \tag{26}$$

which coincides with the SUPG method for problem (1). The value of the stabilization parameter $\frac{1}{|K|}\int_K \widetilde{b}_K$ can be readily obtained by noting that the adjoint residual-free bubble $\widetilde{b}_K$ is a pyramid in 2D, of height $h_K^{\boldsymbol{a}}/|\boldsymbol{a}_{|K}|$, $h_K^{\boldsymbol{a}}$ being the length of the longest segment parallel to $\boldsymbol{a}_{|K}$ and contained in K. The location of the top of the pyramid depends on the number of inflow edges in K: if there are two inflow edges, then the top is located on the common vertex, while if there is only one inflow edge, the top is located on the intersection of this edge with the line parallel to $\boldsymbol{a}_{|K}$ passing through the opposite vertex (see fig. 2). For both cases,

$$\int_K \widetilde{b}_K = \frac{1}{3}\,\frac{h_K^{\boldsymbol{a}}}{|\boldsymbol{a}_{|K}|}\,|K| \tag{27}$$

so that the stabilization parameter provided by our Petrov-Galerkin approach is given by

$$\frac{1}{3}\,\frac{h_K^{\boldsymbol{a}}}{|\boldsymbol{a}_{|K}|} \tag{28}$$

The comparison of this value with the one used in the SUPG method is discussed for instance in [5].

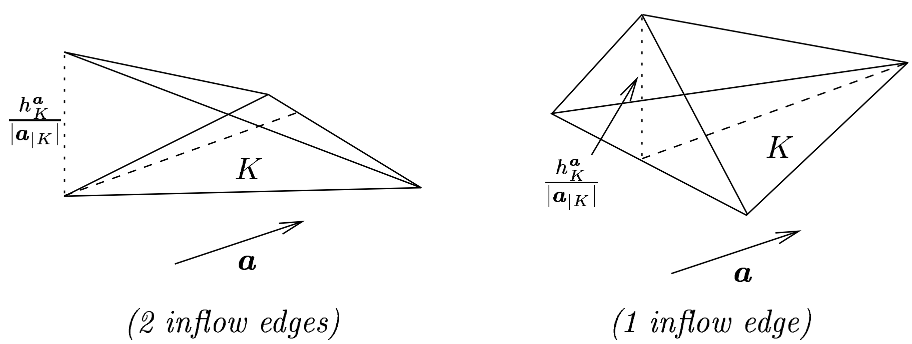

Figure 2: The adjoint residual-free bubble in 2D

5 Including diffusion

In this section, we show that a form identical Petrov-Galerkin method, for the advective-diffusive equation, yields the same equivalence as the one established for the limiting case, in which diffusion is absent. This provides a framework that works in the presence or absence of diffusion, framework which is not possible with the Galerkin method enriched with residual-free bubbles, as defined herein.

Recall the advective-diffusive model given by:

$$\begin{cases} \boldsymbol{a}\cdot\nabla u-\nu\triangle u=f & \text{in } \Omega\,, \\ u=g & \text{on } \partial\Omega\,. \end{cases} \tag{29}$$

where $\nu>0$ is the diffusivity parameter, herein, assumed to be constant. For this model the appropriate spaces are

$$\begin{cases} X^{(g),\nu}=\{u\in H^1(\Omega)\mid u=g,\ \text{on } \partial\Omega\} \\ Y^\nu=H_0^1(\Omega)\,. \end{cases} \tag{30}$$

and the variational formulation of the continuous problem is written as:

$$\begin{cases} \text{find } u\in X^{(g),\nu} \text{ such that} \\ (\boldsymbol{a}\cdot\nabla u,v)+(\nu\nabla u,\nabla v)=(f,v) \quad \forall v\in Y^\nu \end{cases} \tag{31}$$

This formulation is amenable to a Galerkin method enriched with residual-free bubbles as described for instance in [2]. However, our object is to propose a Petrov-Galerkin method with a choice of spaces similar to the one for the limiting case described in the previous Sections, so that a single formulation can be used for both cases. Here we need to change the building block spaces as follows.

The space B_h^ν is defined as before, i.e. we set $B_h^\nu=B_h$, while the space $ARFB_h^\nu$ is defined as $\oplus_K\text{span}\{\widetilde{b}_K\}$, where now $\widetilde{b}_K$ solves in each element K the following elliptic boundary-value problem:

$$\begin{cases} -\boldsymbol{a}\cdot\nabla\widetilde{b}_K-\nu\triangle\widetilde{b}_K=1 & \text{in } K\,, \\ \widetilde{b}_K=0 & \text{on } \partial K\,. \end{cases} \tag{32}$$

We then set:

$$\begin{aligned} V_1^{(0),\nu}&=\{v_h\in V_1\mid v_h=0 \text{ on } \partial\Omega\} \\ V_1^{(g),\nu}&=\{v_h\in V_1\mid v_h=g \text{ on } \partial\Omega\}. \end{aligned} \tag{33}$$

where V_1 is defined in (10). Then we define $X_h^{(g),\nu}\subset X^{(g),\nu}$ and $Y_h^\nu\subset Y^\nu$ as

$$X_h^{(g),\nu}=V_1^{(g),\nu}\oplus B_h^\nu,\quad Y_h^\nu=V_1^{(0),\nu}\oplus ARFB_h^\nu \tag{34}$$

and our Petrov-Galerkin method reads:

$$\begin{cases} \text{find } u_h\in X_h^{(g),\nu} \text{ such that} \\ (\boldsymbol{a}\cdot\nabla u_h,v_h)+(\nu\nabla u_h,\nabla v_h)=(f,v_h) \quad \forall v_h\in Y_h^\nu \end{cases} \tag{35}$$

We now relate this formulation to the SUPG method as in the previous section. First we take $v_h = \widetilde{b}_K$ in an element K and zero elsewhere. From (35), and by writing u_h as

$$u_h = u_1 + u_b, \tag{36}$$

where $u_1 \in V_1^{(g),\nu}$, $u_b = \sum_K c_K b_K \in B_h^\nu$, we have

$$(\boldsymbol{a} \cdot \nabla(u_1 + c_K b_K), \widetilde{b}_K)_K + \nu(\nabla(u_1 + c_K b_K), \nabla\widetilde{b}_K)_K = (f, \widetilde{b}_K)_K. \tag{37}$$

Noting that

$$(\nabla u_1, \nabla\widetilde{b}_K)_K = (-\Delta u_1, \widetilde{b}_K)_K + \int_{\partial K} \frac{\partial u_1}{\partial n} \widetilde{b}_K = 0 \tag{38}$$

since u_1 is linear in K and $\widetilde{b}_K$ is zero on ∂K, we have:

$$c_K \left[(\boldsymbol{a} \cdot \nabla b_K, \widetilde{b}_K)_K + \nu(\nabla b_K, \nabla\widetilde{b}_K)_K \right] = (f - \boldsymbol{a} \cdot \nabla u_1)_{|K} \int_K \widetilde{b}_K. \tag{39}$$

Integrating by parts the expression in square brackets, we have:

$$c_K \int_K \left[-\boldsymbol{a} \cdot \nabla\widetilde{b}_K - \nu \triangle \widetilde{b}_K \right] b_K = (f - \boldsymbol{a} \cdot \nabla u_1)_{|K} \int_K \widetilde{b}_K \tag{40}$$

and since by (32) $-\boldsymbol{a} \cdot \nabla\widetilde{b}_K - \nu \triangle \widetilde{b}_K = 1$, we have

$$c_K = (f - \boldsymbol{a} \cdot \nabla u_1)_{|K} \frac{\int_K \widetilde{b}_K}{\int_K b_K} \tag{41}$$

which is form-identical to the expression (18) of c_K for the purely hyperbolic case. The second step is to take $v_h = v_1 \in V_1^{(0),\nu}$ in (35):

$$(\boldsymbol{a} \cdot \nabla(u_1 + u_b), v_1) + \nu(\nabla(u_1 + u_b), \nabla v_1) = (f, v_1). \tag{42}$$

Since $(\nabla u_b, \nabla v_1) = \sum_K c_K (\nabla b_K, \nabla v_1)_K = 0$, we have the following equation:

$$(\boldsymbol{a} \cdot \nabla u_1, v_1) + (\boldsymbol{a} \cdot \nabla u_b, v_1) + \nu(\nabla u_1, \nabla v_1) = (f, v_1). \tag{43}$$

The additional term due to the bubble stabilization is then given by

$$\begin{aligned} (\boldsymbol{a} \cdot \nabla u_b, v_1) = \sum_K c_K (\boldsymbol{a} \cdot \nabla b_K, v_1)_K = -\sum_K c_K (b_K, \boldsymbol{a} \cdot \nabla v_1)_K = \\ -\sum_K c_K (\boldsymbol{a} \cdot \nabla v_1)_{|K} \int_K b_K = \sum_K (\boldsymbol{a} \cdot \nabla u_1 - f)_{|K} \frac{\int_K \widetilde{b}_K}{\int_K b_K} (\boldsymbol{a} \cdot \nabla v_1)_{|K} \int_K b_K \end{aligned} \tag{44}$$

and by simplifying the term $\int_K b_K$ we obtain

$$(\boldsymbol{a} \cdot \nabla u_b, v_1) = \sum_K \left[\frac{1}{|K|} \int_K \widetilde{b}_K \right] \int_K (\boldsymbol{a} \cdot \nabla u_1 - f)\, \boldsymbol{a} \cdot \nabla v_1 \tag{45}$$

which is form-identical to (26) and to the expression for usual residual-free bubble correction discussed, for instance, in [3].

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References

[1] D. N. Arnold, F. Brezzi, and M. Fortin, *A stable finite element for the Stokes equations*, Calcolo, 23 (1984), pp. 337–344.

[2] F. Brezzi, M. Bristeau, L. Franca, M. Mallet, and G. Rogé, *A relationship between stabilized finite element methods and the Galerkin method with bubble functions*, Comput. Methods Appl. Mech. Engrg., 96 (1992), pp. 117–129.

[3] F. Brezzi, L. P. Franca, T. J. R. Hughes, and A. Russo, $b = \int g$, Comput. Methods Appl. Mech. Engrg., 145 (1997), pp. 329–339.

[4] F. Brezzi, L. P. Franca, and A. Russo, *Further considerations on residual-free bubbles for advective-diffusive equations.* To appear in Comput. Methods Appl. Mech. Engrg.

[5] F. Brezzi and A. Russo, *Choosing bubbles for advection-diffusion problems*, Math. Models Meth. Appl. Sci., 4 (1994), pp. 571–587.

[6] L. P. Franca, C. Farhat, A. P. Macedo, and M. Lesoinne, *Residual-free bubbles for the Helmholtz equation*, Int. J. Num. Meth. Eng., 40 (1997), pp. 4003–4009.

[7] L. P. Franca and A. Russo, *Approximation of the Stokes problem by residual-free macro bubbles*, East-West J. Appl. Math., 4 (1996), pp. 265–278.

[8] ——, *Deriving upwinding, mass lumping and selective reduced integration by residual-free bubbles*, Appl. Math. Letters, 9 (1996), pp. 83–88.

[9] ——, *Mass lumping emanating from residual-free bubbles*, Comput. Methods Appl. Mech. Engrg., 142 (1997), pp. 353–360.

[10] ——, *Unlocking with residual-free bubbles*, Comput. Methods Appl. Mech. Engrg., 142 (1997), pp. 361–364.

[11] T. J. R. Hughes, *Multiscale phenomena: Green's functions, the Dirichlet-to-Neumann formulation, subgrid scale models, bubbles and the origin of stabilized methods*, Comput. Methods Appl. Mech. Engrg., 127 (1995), pp. 387–401.