Constrained Interval Arithmetic

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Abstract: *This paper presents an approach to solving the long-standing dependency problem in interval arithmetic. An extension to interval arithmetic, called here constrained interval arithmetic, is developed. Unlike interval arithmetic, constrained interval arithmetic has an additive inverse, a multiplicative inverse and satisfies the distributive law. This means that the algebraic structure of constrained interval arithmetic is different than that of interval arithmetic. The applicability of constrained interval arithmetic is explored.*

1. Introduction: It is well-known in the interval analysis literature that interval arithmetic overestimates the resultant width of the interval when dependencies are present. This overestimation can be arbitrarily large (see, for example, (Neumaier 1990, pages 16-19)).

Example 1 *Consider* $y = f(x) = x(1-x)$, $x \in [0,1]$. *The implementation of interval arithmetic yields* $y = [0,1] \times (1 - [0,1]) = [0,1] \times [0,1] = [0,1]$. *The actual range is* $f(x) \in [0, \frac{1}{4}]$

The reason for this overestimation is that each of the two occurrences of the variable x is treated as an independent variable. The traditional solution to this problem is to subdivide the intervals or to reformulate the expression and subdivide the intervals; for example, centered forms are such reformulations (see (Ratschek and Rokne 1988)). In interval arithmetic, tighter upper and lower bounds of an expression that possesses dependencies require subdivision of intervals, albeit in conjunction with "preconditioning" of the expression such as putting the expression in centered form.

Example 2 *Consider* $y = f(x) = x(1-x)$, $x \in [0,1] = [0, \frac{1}{2}] \cup [\frac{1}{2}, 1]$. *In this case we have*

$$\begin{aligned} y &= \{[0,\frac{1}{2}] \times (1 - [0,\frac{1}{2}])\} \cup \{[\frac{1}{2},1] \times (1 - [\frac{1}{2},1])\} \\ &= \{[0,\frac{1}{2}] \times [\frac{1}{2},1]\} \cup \{[\frac{1}{2},1] \times [0,\frac{1}{2}]\} \end{aligned}$$

$$
\begin{aligned}
&= [0, \frac{1}{2}] \cup [0, \frac{1}{2}] \\
&= [0, \frac{1}{2}].
\end{aligned}
$$

It is known that applying interval arithmetic to the union of intervals of decreasing width yield tighter bounds on the result. Of course, in n-dimensional problems, "intervals" are rectangular parallelepipeds (boxes) and as the diameters of these boxes go to zero, the union of the result approaches the correct bound for the expression. Theorems proving convergence to the exact bound of the expression and the rates associated with subdividing intervals can be found in (Moore 1966, 1979), (Ratschek and Rokne 1988) and (Neumaier 1990). What is proposed here is to redefine interval numbers in such a way that dependencies are kept. The ensuing arithmetic will be called constrained interval arithmetic. This new arithmetic is an extension of interval arithmetic in the sense that if there are no dependencies present; that is, each interval in the arithmetic expression is independent, the usual rules for interval arithmetic hold (see page 11 of (Moore 1979)) and when dependencies are present, the correct result is obtained.

2. Constrained Interval Arithmetic: We begin by redefining an interval number as a function of three parameters rather than two as is the case with interval arithmetic.

Definition 3 *An interval number $[\underline{x}, \bar{x}]$ (or interval for short) is a number of the form*

$$X^I(\underline{x}, \bar{x}, \alpha_x) = \{x | x = (1-\alpha_x)\underline{x} + \alpha_x \bar{x},\ 0 \le \alpha_x \le 1\}. \qquad ((1))$$

Three parameters are necessary to deal with dependencies. Strictly speaking, in (1), since the numbers, $\underline{x}$ and $\bar{x}$ are known (inputs), they are parameters whereas α_x is varying and hence a variable that is constrained between 0 and 1, hence the name "constrained interval arithmetic." That is, in (1) x is a function of α_x with $\underline{x}$ and $\bar{x}$ given parameters of this function. We drop the capitalization and superscript on interval variables since the context will be clear. We focus only on interval numbers with non-zero width; that is, where $\underline{x} < \bar{x}$, since zero width (real) numbers do not exhibit the dependency problem. The algebraic operations are defined as usual,

$$
\begin{aligned}
z &= x \circ y = \{z \mid z = x \circ y, \text{ for all } \underline{x} \le x \le \bar{x} \text{ and } \underline{y} \le y \le \bar{y}\} = [\underline{z}, \bar{z}], (2) \\
\text{with } z &= x \circ x = \{z \mid z = x \circ x, \text{ for all } \underline{x} \le x \le \bar{x}\} = [\underline{z}, \bar{z}], \\
\text{where } \underline{z} &= \min\ z \text{ and } \bar{z} = \max\ z \text{ and } \circ \in \{+, -, \times, \div\}.
\end{aligned}
$$

Remark 1*In the sequel we take $-x$ to be $[-\bar{x}, -\underline{x}]$. Standard texts on interval analysis contain the justification.*

Since all the algebraic operations are continuous as long as division by zero is disallowed, the minimization and maximizations are well-defined, attained and the resultant z is in turn an interval. The extension to division by zero and infinite width intervals is well-known (see (Kahan 1968) or (Kearfott 1996 page 9)). Section 2 develops the arithmetic operations for constrained interval arithmetic using definition (1) and (2). There will be three cases we consider, (a) $x \circ y$, (b) $x \circ x$, and (c) $x \circ (-x)$. Distinct variables will mean that the operations are independent as in (a). The operation with the variable as in (b) and (c) denote the dependency of the variable. Of course, for arbitrary expressions, multiple dependencies occur so that a constrained optimization problem ensues from constrained interval arithmetic by the addition of the variable $0 \leq \alpha_x \leq 1$. However, the ensuing constrained optimization is structured and adds one simple bound constraint per independent variable. That is, evaluation of an arbitrary expression using constrained interval arithmetic requires the exchange of each variable in the expression by a one variable and two known parameters. This new expression is then optimized (to obtain the global maximum and the global minimum) subject to simple bound constraints. Applications of constrained interval arithmetic to algebraic expressions are discussed in section 3.

2.1 Addition and Subtraction: The computations for the maximum and minimum bounds associated with interval number addition and subtraction take into account the constraint on the α parameter $0 \leq \alpha \leq 1$ of three parameter interval number as follows.

2.1.1 Independent addition: $z = x + y$

$$\text{opt z} = \max / \min_{\alpha} \{(1-\alpha_x)\underline{x} + \alpha_x\bar{x} + (1-\alpha_y)\underline{y} + \alpha_y\bar{y}\},$$

subject to the constraints $0 \leq \alpha_x \leq 1$, and $0 \leq \alpha_y \leq 1$.

The associated Lagrangian for this problem is,

$$L(\alpha_x, \alpha_y, \overrightarrow{\lambda}) = (1-\alpha_x)\underline{x}+\alpha_x\bar{x}+(1-\alpha_y)\underline{y}+\alpha_y\bar{y}-\lambda_1\alpha_x+\lambda_2(\alpha_x-1)-\lambda_3\alpha_y+\lambda_4(\alpha_y-1).$$

The solution to the Lagrangian problem is the solution set associated with the following set of equations:

$$\begin{aligned}\frac{\partial L}{\partial \alpha_x} &= -\underline{x} + \bar{x} - \lambda_1 + \lambda_2 = 0 \\ \frac{\partial L}{\partial \alpha_y} &= -\underline{y} + \bar{y} - \lambda_3 + \lambda_4 = 0,\end{aligned}$$

together with the complementarity slackness conditions

$$\begin{aligned}
-\lambda_1\alpha_x &= 0, \lambda_1 \geq 0,\\
\lambda_2(\alpha_x - 1) &= 0, \lambda_2 \geq 0,\\
-\lambda_3\alpha_y &= 0, \lambda_3 \geq 0,\\
\lambda_4(\alpha_y - 1) &= 0, \lambda_4 \geq 0.
\end{aligned}$$

Case 1: $\lambda_1 = 0$, $\lambda_2 = 0 \Longrightarrow -\underline{x} + \bar{x} = 0$. Since we have assumed that $\underline{x} < \overline{x}$, case 1 does not hold.
Case 2: $\lambda_1 > 0$, $\lambda_2 = 0 \Longrightarrow \alpha_x = 0$
Case 3: $\lambda_1 = 0$, $\lambda_2 > 0 \Longrightarrow \alpha_x = 1$
Case 4: $\lambda_1 > 0$, $\lambda_2 > 0 \Longrightarrow \alpha_x = 0$ and $\alpha_x = 1$ which cannot be.
The cases associated with λ_3 and λ_4 are done in the same way. That is,
Case 5: $\lambda_3 = 0$, $\lambda_4 = 0 \Longrightarrow -\underline{y} + \bar{y} = 0$. Since we have assumed that $\underline{y} < \bar{y}$, case 1 does not hold.
Case 6: $\lambda_3 > 0$, $\lambda_4 = 0 \Longrightarrow \alpha_y = 0$
Case 7: $\lambda_3 = 0$, $\lambda_4 > 0 \Longrightarrow \alpha_y = 1$
Case 8: $\lambda_3 > 0$, $\lambda_4 > 0 \Longrightarrow \alpha_y = 0$ and $\alpha_y = 1$ which cannot be.
The four solutions for cases 2, 3, 6, and 7 correspond to the usual definition of interval addition.

2.1.2 Dependent addition 1: $z = x + x$

$$\begin{aligned}
\text{opt z} &= \max / \min_{\alpha}\{(1-\alpha_x)\underline{x} + \alpha_x\bar{x} + (1-\alpha_x)\underline{x} + \alpha_x\bar{x}\}\\
&= \max / \min_{\alpha}\{2(1-\alpha_x)\underline{x} + 2\alpha_x\bar{x}\}\\
\text{subject to } 0 &\leq \alpha_x \leq 1\\
&= [2\underline{x}, 2\overline{x}].
\end{aligned}$$

2.1.3 Dependent addition 2: $z = x + (-x)$

$$\begin{aligned}
\text{opt z} &= \max / \min_{\alpha}\{(1-\alpha_x)\underline{x} + \alpha_x\bar{x} - ((1-\alpha_x)\underline{x} + \alpha_x\bar{x})\}\\
&= \max / \min_{\alpha}\{0\}\\
\text{subject to } 0 &\leq \alpha_x \leq 1\\
&= [0, 0].
\end{aligned}$$

Thus, there is an additive inverse for interval number as defined by (1).

2.1.4 Independent subtraction: $z = x - y$

$$\begin{aligned}
\text{opt z} &= \max / \min_{\alpha}\{(1-\alpha_x)\underline{x} + \alpha_x\bar{x} - [(1-\alpha_y)\underline{y} + \alpha_y\bar{y}]\},\\
\text{subject to } 0 &\leq \alpha_x \leq 1, \text{ and } 0 \leq \alpha_y \leq 1.
\end{aligned}$$

The associated Lagrangian for this problem is,

$$L(\alpha_x, \alpha_y, \overrightarrow{\lambda}) = (1-\alpha_x)\underline{x} + \alpha_x\bar{x} - (1-\alpha_y)\underline{y} - \alpha_y\bar{y} - \lambda_1\alpha_x + \lambda_2(\alpha_x - 1) - \lambda_3\alpha_y + \lambda_4(\alpha_y - 1).$$

The solution to the Lagrangian problem is the solution set associated with the following set of equations:

$$\begin{aligned} \frac{\partial L}{\partial \alpha_x} &= -\underline{x} + \bar{x} - \lambda_1 + \lambda_2 = 0 \\ \frac{\partial L}{\partial \alpha_y} &= \underline{y} - \bar{y} - \lambda_3 + \lambda_4 = 0, \end{aligned}$$

together with the complementarity slackness conditions

$$\begin{aligned} -\lambda_1 \alpha_x &= 0, \lambda_1 \geq 0, \\ \lambda_2(\alpha_x - 1) &= 0, \lambda_2 \geq 0, \\ -\lambda_3 \alpha_y &= 0, \lambda_3 \geq 0, \\ \lambda_4(\alpha_y - 1) &= 0, \lambda_4 \geq 0. \end{aligned}$$

Case 1: $\lambda_1 = 0$, $\lambda_2 = 0 \Longrightarrow -\underline{x} + \bar{x} = 0$. Since we have assumed that $\underline{x} < \overline{x}$, case 1 does not hold.
Case 2: $\lambda_1 > 0$, $\lambda_2 = 0 \Longrightarrow \alpha_x = 0$
Case 3: $\lambda_1 = 0$, $\lambda_2 > 0 \Longrightarrow \alpha_x = 1$
Case 4: $\lambda_1 > 0$, $\lambda_2 > 0 \Longrightarrow \alpha_x = 0$ and $\alpha_x = 1$ which cannot be.
The cases associated with λ_3 and λ_4 are done in the same way. That is,
Case 5: $\lambda_3 = 0$, $\lambda_4 = 0 \Longrightarrow \underline{y} - \bar{y} = 0$. Since we have assumed that $\underline{y} < \bar{y}$, case 1 does not hold.
Case 6: $\lambda_3 > 0$, $\lambda_4 = 0 \Longrightarrow \alpha_y = 0$
Case 7: $\lambda_3 = 0$, $\lambda_4 > 0 \Longrightarrow \alpha_y = 1$
Case 8: $\lambda_3 > 0$, $\lambda_4 > 0 \Longrightarrow \alpha_y = 0$ and $\alpha_y = 1$ which cannot be.
The four solutions for cases 2, 3, 6, and 7 correspond to the usual definition of interval subtraction.

2.1.5 Dependent subtraction 1: $z = x - x = x + (-x) = [0, 0]$ according to 2.1.3 above.

2.1.6 Dependent subtraction 2: $z = x - (-x) = x + x = [2\underline{x}, 2\overline{x}]$ according to 2.1.2 above.

2.2 Multiplication: As in addition and subtraction, the computations of the maximum and minimum bounds associated with multiplication must take into account the constraint of the α parameter of three parameter interval number.

2.2.1 Independent multiplication: $z = xy$

$$\begin{aligned} \text{opt z} &= \max / \min_{\alpha}\{((1-\alpha_x)\underline{x} + \alpha_x \bar{x})((1-\alpha_y)\underline{y} + \alpha_y \bar{y})\} \\ &= \max / \min_{\alpha}\{(1-\alpha_x)(1-\alpha_y)\underline{x}\underline{y} + (1-\alpha_x)\alpha_y \underline{x}\bar{y} + \alpha_x(1-\alpha_y)\bar{x}\underline{y} + \alpha_x \alpha_y \bar{x}\bar{y}\} \end{aligned}$$

subject to $0 \leq \alpha_x \leq 1$, and $0 \leq \alpha_y \leq 1$.

The associated Lagrangian for this problem is,

$$\begin{aligned} L(\alpha_x, \alpha_y, \overrightarrow{\lambda}) &= (1-\alpha_x)(1-\alpha_y)\underline{xy} + (1-\alpha_x)\alpha_y\underline{x}\bar{y} + \alpha_x(1-\alpha_y)\bar{x}\underline{y} + \alpha_x\alpha_y\bar{x}\bar{y} \\ &\quad -\lambda_1\alpha_x + \lambda_2(\alpha_x - 1) - \lambda_3\alpha_y + \lambda_4(\alpha_y - 1). \end{aligned}$$

The solution to the Lagrangian problem is the solution set associated with the following set of equations:

$$\begin{aligned} \frac{\partial L}{\partial \alpha_x} &= -(1-\alpha_y)\underline{xy} - \alpha_y\underline{x}\bar{y} + (1-\alpha_y)\bar{x}\underline{y} + \alpha_y\bar{x}\bar{y} - \lambda_1 + \lambda_2 = 0 \\ \frac{\partial L}{\partial \alpha_y} &= -(1-\alpha_x)\underline{xy} + (1-\alpha_x)\underline{x}\bar{y} - \alpha_x\bar{x}\underline{y} + \alpha_x\bar{x}\bar{y} - \lambda_3 + \lambda_4 = 0 \end{aligned}$$

together with the complementarity slackness conditions

$$\begin{aligned} -\lambda_1\alpha_x &= 0, \lambda_1 \geq 0, \\ \lambda_2(\alpha_x - 1) &= 0, \lambda_2 \geq 0, \\ -\lambda_3\alpha_y &= 0, \lambda_3 \geq 0, \\ \lambda_4(\alpha_y - 1) &= 0, \lambda_4 \geq 0. \end{aligned}$$

Case 1: $\lambda_1 = 0,\ \lambda_2 = 0$

$$\begin{aligned} -(1-\alpha_y)\underline{xy} - \alpha_y\underline{x}\bar{y} + (1-\alpha_y)\bar{x}\underline{y} + \alpha_y\bar{x}\bar{y} &= \\ -\underline{xy} + \alpha_y\underline{xy} - \alpha_y\underline{x}\bar{y} + \bar{x}\underline{y} - \alpha_y\bar{x}\underline{y} + \alpha_y\bar{x}\bar{y} &= 0 \\ \alpha_y(\underline{xy} - \underline{x}\bar{y} + \bar{x}\underline{y} - \bar{x}\underline{y} + \bar{x}\bar{y}) &= \underline{xy} - \bar{x}\underline{y} \\ \alpha_y &= \frac{\underline{y}}{\underline{y} - \bar{y}} \end{aligned}$$

Case 1.1: $\underline{y} < \bar{y} < 0$. In this case, we have:

$$\begin{aligned} \underline{y} - \bar{y} &< 0 \text{ so that} \\ \alpha_y &= \frac{\underline{y}}{\underline{y} - \bar{y}} > 1 \text{ which violates the constraint for } \alpha \text{ and case 1.1 does not hold} \end{aligned}$$

Case 1.2: $0 < \underline{y} < \bar{y}$

$\alpha_y = \dfrac{\underline{y}}{\underline{y} - \bar{y}} < 0$ which violates the constraint for α and case 1.2 does not hold

Case 1.3: $\underline{y} \leq 0 < \bar{y}$ or $\underline{y} < 0 \leq \bar{y}$. Substituting for α_y

$$(1-\alpha_y)\underline{y} + \alpha_y\bar{y} = (1 - \frac{\underline{y}}{\underline{y} - \bar{y}})\underline{y} + \frac{\underline{y}}{\underline{y} - \bar{y}}\bar{y} = 0$$

Case 2: $\lambda_1 > 0,\ \lambda_2 = 0 \Longrightarrow \alpha_x = 0$

Case 3: $\lambda_1 = 0,\ \lambda_2 > 0 \Longrightarrow \alpha_x = 1$
Case 4: $\lambda_1 > 0,\ \lambda_2 > 0 \Longrightarrow \alpha_x = 0$ and $\alpha_x = 1$ which cannot be.
The cases associated with λ_3 and λ_4 are done similarly. That is,
Case 5: $\lambda_3 = 0,\ \lambda_4 = 0$, and as before, this will only occur if $\underline{x} \leq 0 < \bar{x}$ or $\underline{x} < 0 \leq \bar{x}$

$$\alpha_x = \frac{\underline{x}}{\underline{x} - \bar{x}} \text{ and } (1-\alpha_x)\underline{x} + \alpha_x \bar{x} = 0$$

Case 6: $\lambda_3 > 0,\ \lambda_4 = 0 \Longrightarrow \alpha_y = 0$
Case 7: $\lambda_3 = 0,\ \lambda_4 > 0 \Longrightarrow \alpha_y = 1$
Case 8: $\lambda_3 > 0,\ \lambda_4 > 0 \Longrightarrow \alpha_y = 0$ and $\alpha_y = 1$ which cannot be.
If either case 1 or case 5 holds, $z = 0$. However, if case 1 or case 5 hold the respective interval span zero so that the maximum will not be zero unless one of the four endpoints is zero. Therefore, all cases produce

$$\text{opt z} = \max / \min \ \{\underline{x}\underline{y}, \underline{x}\bar{y}, \bar{x}\underline{y}, \bar{x}\bar{y}\}$$

which is the usual definition for interval multiplication.

2.2.2 Dependent multiplication 1: $z = xx = x^2$

$$\begin{aligned} \text{opt z} &= \max / \min_{\alpha} [(1-\alpha_x)\underline{x} + \alpha_x \bar{x}][(1-\alpha_x)\underline{x} + \alpha_x \bar{x}] \\ &= \max / \min_{\alpha} \{(1 - 2\alpha_x + \alpha_x^2)\underline{x}^2 + 2(1-\alpha_x)\alpha_x \underline{x}\bar{x} + \alpha_x^2 \bar{x}^2\} \\ \text{subject to } 0 &\leq \alpha_x \leq 1. \end{aligned}$$

The associated Lagrangian for this problem is,

$$L(\alpha_x, \overrightarrow{\lambda}) = (1 - 2\alpha_x + \alpha_x^2)\underline{x}^2 + (2\alpha_x - 2\alpha_x^2)\underline{x}\bar{x} + \alpha_x^2 \bar{x}^2 - \lambda_1 \alpha_x + \lambda_2(\alpha_x - 1)$$

The solution to the Lagrangian problem is the solution set associated with the following set of equations:

$$\frac{\partial L}{\partial \alpha_x} = 2(\alpha_x - 1)\underline{x}^2 + 2(1 - 2\alpha_x)\underline{x}\bar{x} + 2\alpha_x \bar{x}^2 - \lambda_1 + \lambda_2 = 0$$

together with the complementarity slackness conditions,

$$\begin{aligned} -\lambda_1 \alpha_x &= 0, \lambda_1 \geq 0, \\ \lambda_2(\alpha_x - 1) &= 0, \lambda_2 \geq 0. \end{aligned}$$

Case 1: $\lambda_1 = 0,\ \lambda_2 = 0$

$$\begin{aligned} 2\alpha_x \underline{x}^2 - \underline{x}^2 + 2\underline{x}\bar{x} - 4\alpha_x \underline{x}\bar{x} + 2\alpha_x \bar{x}^2 &= 0 \text{ so that} \\ \alpha_x(2\underline{x}^2 - 4\underline{x}\bar{x} + 2\bar{x}^2) &= \underline{x}^2 - 2\underline{x}\bar{x} \\ 2\alpha_x(\underline{x} - \bar{x})^2 &= \underline{x}(\underline{x} - \bar{x}) \\ \alpha_x &= \frac{\underline{x}}{\underline{x} - \bar{x}}. \end{aligned}$$

As before, this case can only occur if $\underline{x} \leq 0 < \bar{x}$ or $\underline{x} < 0 \leq \bar{x}$ and results in $z = 0$.
Case 2: $\lambda_1 > 0,\ \lambda_2 = 0 \Longrightarrow \alpha_x = 0$
Case 3: $\lambda_1 = 0,\ \lambda_2 > 0 \Longrightarrow \alpha_x = 1$
Case 4: $\lambda_1 > 0,\ \lambda_2 > 0 \Longrightarrow \alpha_x = 0$ and $\alpha_x = 1$ which cannot be. Therefore,

$$\begin{aligned} \underline{z} &= \min\ \{0, \underline{x}^2, \bar{x}^2\} \\ \bar{z} &= \max\ \{0, \underline{x}^2, \bar{x}^2\} \end{aligned}$$

where zero occurs only if the interval number spans zero which is the standard (extended) interval definition for squaring of an interval number.

2.2.3 Dependent multiplication 2: $z = x(-x) = -xx = -x^2$.

2.3 Division As in addition and subtraction, the computations for the maximum and minimum bounds associated with division must take into account the constraint of the α parameter of three parameter interval number. Moreover, we assume that the denominator contains no zero.

2.3.1 Independent multiplication: $z = x \div y, 0 \notin [\underline{y}, \bar{y}]$

$$\begin{aligned} \text{opt z} &= \max / \min_{\alpha} \{ \frac{(1-\alpha_x)\underline{x} + \alpha_x \bar{x}}{(1-\alpha_y)\underline{y} + \alpha_y \bar{y}} \} \\ \text{subject to } 0 &\leq \alpha_x \leq 1, \text{ and } 0 \leq \alpha_y \leq 1. \end{aligned}$$

The associated Lagrangian is given by:

$$L(\alpha_x, \alpha_y, \overrightarrow{\lambda}) = \frac{(1-\alpha_x)\underline{x} + \alpha_x \bar{x}}{(1-\alpha_y)\underline{y} + \alpha_y \bar{y}} - \lambda_1 \alpha_x + \lambda_2(\alpha_x - 1) - \lambda_3 \alpha_y + \lambda_4(\alpha_y - 1).$$

The solution to the Lagrangian problem is the solution set associated with the following set of equations:

$$\begin{aligned} \frac{\partial L}{\partial \alpha_x} &= \frac{-\underline{x} + \bar{x}}{(1-\alpha_y)\underline{y} + \alpha_y \bar{y}} - \lambda_1 + \lambda_2 = 0 \\ \frac{\partial L}{\partial \alpha_y} &= \frac{(1-\alpha_x)\underline{x} + \alpha_x \bar{x}}{((1-\alpha_y)\underline{y} + \alpha_y \bar{y})^2}(-\underline{y} + \bar{y}) - \lambda_3 + \lambda_4 = 0, \end{aligned}$$

together with the complementarity slackness conditions

$$\begin{aligned} -\lambda_1 \alpha_x &= 0, \lambda_1 \geq 0, \\ \lambda_2(\alpha_x - 1) &= 0, \lambda_2 \geq 0, \\ -\lambda_3 \alpha_y &= 0, \lambda_3 \geq 0, \\ \lambda_4(\alpha_y - 1) &= 0, \lambda_4 \geq 0. \end{aligned}$$

Case 1: $\lambda_1 = 0,\ \lambda_2 = 0$. For this case we have,

$$\frac{-\underline{x} + \bar{x}}{(1-\alpha_y)\underline{y} + \alpha_y \bar{y}} = 0$$

which cannot be since we're assuming that $\underline{x} < \bar{x}$.
Case 2: $\lambda_1 > 0,\ \lambda_2 = 0 \Longrightarrow \alpha_x = 0$
Case 3: $\lambda_1 = 0,\ \lambda_2 > 0 \Longrightarrow \alpha_x = 1$
Case 4: $\lambda_1 > 0,\ \lambda_2 > 0 \Longrightarrow \alpha_x = 0 \Longrightarrow \alpha_x = 0$ and $\alpha_x = 1$ which cannot be.
The cases associated with λ_3 and λ_4 are done similarly. That is,
Case 5: $\lambda_3 = 0,\ \lambda_4 = 0$. For this case we have,

$$\begin{aligned}\frac{(1-\alpha_x)\underline{x}+\alpha_x\bar{x}}{((1-\alpha_y)\underline{y}+\alpha_y\bar{y})^2}(-\underline{y}+\bar{y}) &= 0 \text{ which means that}\\ (1-\alpha_x)\underline{x}+\alpha_x\bar{x} &= 0 \text{ since } \underline{y} < \bar{y} \text{ and}\\ \alpha_x &= -\frac{\underline{x}}{\bar{x}-\underline{x}}\end{aligned}$$

Case 5.1: $\underline{x} \leq 0 < \bar{x}$ then

$$z = \frac{(1-\alpha_x)\underline{x}+\alpha_x\bar{x}}{(1-\alpha_y)\underline{y}+\alpha_y\bar{y}} = \frac{(1-(-\frac{\underline{x}}{\bar{x}-\underline{x}}))\underline{x}+(-\frac{\underline{x}}{\bar{x}-\underline{x}})\bar{x}}{(1-\alpha_y)\underline{y}+\alpha_y\bar{y}} = 0$$

Case 5.2:$\underline{x} > 0$ and $\bar{x} < 0$ cannot occur since this would make $\alpha_x < 0$ and $\alpha_x > 1$ respectively.
Case 6: $\lambda_3 > 0,\ \lambda_4 = 0 \Longrightarrow \alpha_y = 0$
Case 7: $\lambda_3 = 0,\ \lambda_4 > 0 \Longrightarrow \alpha_y = 1$
Case 8: $\lambda_3 > 0,\ \lambda_4 > 0 \Longrightarrow \alpha_y = 0$ and $\alpha_y = 1$ which cannot be.

Remark 2*If $z = 0$, then either $\underline{x} = 0$ or $\underline{x} < 0 < \bar{x}$. If $\underline{x} = 0$, the $z = 0$ is included in $\underline{x}/\underline{y}$, $\underline{x}/\bar{y}$. If $\underline{x} < 0 < \bar{x}$, then the minimum and maximum z cannot be zero but determined by case 2, 3, 6, or 7. These cases define the extremes of the intervals. Therefore, cases 1, 4, 5 and 8 do not apply and the minimum and maximum of z are determined by the extremes of $\underline{x}$, $\bar{x}$, $\underline{y}$, and $\bar{y}$. Therefore, for division of independent variable we have the same rules as in standard interval arithmetic.*

$$[\underline{z},\bar{z}] = [\min\{\frac{\underline{x}}{\underline{y}},\frac{\underline{x}}{\bar{y}},\frac{\bar{x},}{\underline{y}},\frac{\bar{x},}{\bar{y}}\},\max\{\frac{\underline{x}}{\underline{y}},\frac{\underline{x}}{\bar{y}},\frac{\bar{x},}{\underline{y}},\frac{\bar{x},}{\bar{y}}\}]$$

2.3.2 Dependent multiplication 1: $z = x \div x,\ 0 \notin x$.

$$z = \frac{(1-\alpha_x)\underline{x}+\alpha_x\bar{x}}{(1-\alpha_x)\underline{x}+\alpha_x\bar{x}} = 1$$

2.3.2 Dependent multiplication 2: $z = x \div (-x),\ 0 \notin x$.

$$z = \frac{(1-\alpha_x)\underline{x}+\alpha_x\bar{x}}{-((1-\alpha_x)\underline{x}+\alpha_x\bar{x})} = -1$$

3. Algebraic Structure of Constrained Interval Arithmetic and Its Use in Expressions: Constrained interval arithmetic is an extension of interval arithmetic so that it possesses the same properties as interval arithmetic (see Moore 1966, 1979). In addition, constrained interval arithmetic has an additive, a multiplicative inverse and the distributive law holds. That the distributive holds for constrained interval arithmetic can be seen as follows. The left side of the distributive law is,

$$\begin{aligned} \text{opt } x(y+z) &= \max/\min_{\alpha}\{((1-\alpha_x)\underline{x}+\alpha_x\bar{x})((1-\alpha_y)\underline{y}+\alpha_y\bar{y}+(1-\alpha_z)\underline{z}+\alpha_z\bar{z})\} \\ \text{subject to } 0 &\le \alpha_x,\alpha_y,\alpha_z \le 1. \end{aligned}$$

The right side of the distributive law is,

$$\begin{aligned} \text{opt } \{xy+xz\} &= \max/\min_{\alpha}\{((1-\alpha_x)\underline{x}+\alpha_x\bar{x})((1-\alpha_y)\underline{y}+\alpha_y\bar{y})+ \\ &\quad ((1-\alpha_x)\underline{x}+\alpha_x\bar{x})((1-\alpha_z)\underline{z}+\alpha_z\bar{z})\} \\ \text{subject to } 0 &\le \alpha_x,\alpha_y,\alpha_z \le 1. \end{aligned}$$

The left and right sides of the distributive law are the same minima and maxima since the values inside the parentheses are real numbers for each α. Therefore, the algebraic structure of constrained interval arithmetic is similar to real arithmetic.

The use of constrained interval arithmetic to evaluate arbitrary expressions requires the computing of the global optimum of a constrained optimization problem. Each distinct independent variable is exchanged for a new variable α as a variable with two fixed parameters and an associated simple bound constraint inequality. The associated optimizations are optimizations with simple bound constraints. To illustrate, consider our original example except computed with constrained interval arithmetic rather than interval arithmetic.

Let $y = f(x) = x(1-x)$ $x = [0,1]$.

$$\begin{aligned} \text{opt } y &= \max/\min_{\alpha}\{((1-\alpha_x)0+\alpha_x 1)(1-((1-\alpha_x)0+\alpha_x 1)\} \\ \text{subject to } 0 &\le \alpha_x \le 1. \end{aligned}$$

The associated Lagrangian is

$$L(\alpha_x, 0, 1, \overrightarrow{\lambda}) = \alpha_x(1-\alpha_x) - \lambda_1\alpha_x + \lambda_2(\alpha_x - 1).$$

The solution to the Lagrangian problem is the solution set associated with the following set of equations:

$$\frac{\partial L}{\partial \alpha_x} = 1 - 2\alpha_x - \lambda_1 + \lambda_2 = 0$$

together with the complementarity slackness conditions

$$\begin{aligned} -\lambda_1 \alpha_x &= 0, \lambda_1 \geq 0, \\ \lambda_2(\alpha_x - 1) &= 0, \lambda_2 \geq 0. \end{aligned}$$

Case 1: $\lambda_1 = 0$, $\lambda_2 = 0$. For this case we have,

$$\begin{aligned} 1 - 2\alpha_x &= 0 \\ \alpha_x &= \frac{1}{2} \end{aligned}$$

Case 2: $\lambda_1 > 0$, $\lambda_2 = 0 \Longrightarrow \alpha_x = 0$
Case 3: $\lambda_1 = 0$, $\lambda_2 > 0 \Longrightarrow \alpha_x = 1$
Case 4: $\lambda_1 > 0$, $\lambda_2 > 0 \Longrightarrow \alpha_x = 0 \Longrightarrow \alpha_x = 0$ and $\alpha_x = 1$ which cannot be.
Therefore,

$$[\underline{y}, \bar{y}] = [\max\{0, \frac{1}{4}\}, \min\{0, \frac{1}{4}\}] = [0, \frac{1}{4}].$$

The complexity of evaluating with constrained interval arithmetic may be higher. However, note that in above the constrained optimization problem case 4 can always be skipped. In addition, cases 2 and 3 are always the endpoint solutions that interval analysis produces. It is only case 1 that needs to be analyzed. Thus, constrained interval arithmetic is more costly with the additional complexity associated with case 1 for each independent variable.

Bibliography

Alefeld, G, & Herzberger, J. 1983. *Introduction to Interval Computations.* Academic Press, New York.

Kahan, H. 1968. *A More Complete Interval Arithmetic.* Lecture Notes for a Summer Course at University of Michigan.

Kearfott, R. Baker. 1996. *Rigorous Global Search: Continuous Problem.* Kluwer Academic Publishers, Boston.

Moore, R. E. 1966. *Interval Analysis.* Prentice-Hall, Englewood Cliffs, N.J.

Moore, R. E. 1979. *Methods and Applications of Interval Analysis.* SIAM, Philadelphia.

Neumaier, A. 1990. *Interval Methods for Systems of Equations.* Cambridge Press, Cambridge.

Ratschek, H. and J. Rokne. 1988. *New Computer Methods for Global Optimization.* Horwood, Chichester, England.