# AN ITERATIVE METHOD WITH CONVERGENCE RATE CHOSEN A PRIORI*

MARIAN BREZINA†, CAROLINE HEBERTON‡, JAN MANDEL§ AND PETR VANĚK¶

**Abstract.** With increasing demand for large-scale three-dimensional simulations, iterative methods emerge as methods of choice. In practical situations, the convergence rate of an iterative method may suffer from practices such as violation of local quasi-uniformity of the mesh, certain implementations of the boundary conditions and of multi-point constraints. These phenomena are typically of local nature, e.g, may occur only along a part the boundary or interface between two parts of the model with different physical interpretation. The algebraic multigrid based on the concept of smoothed aggregation coarse space studied previously in [9, 3, 7] has proven to be efficient in solving these problems. However, the effect of the prolongator smoothing is global, and performing more smoothing to deal with a few local difficulties would be unnecessarily expensive. We will recall the framework used in [7] and extend it by adaptively enriching the coarse space locally in order to reduce the influence of the above mentioned convergence impeding phenomena. The resulting two-level method is, in principle, capable of yielding any convergence rate chosen a priori, up to becoming a direct solver in the extreme case.

**Key words.** Algebraic variational multigrid, unstructured meshes, robust convergence, adaptive coarse space enriching, smoothed aggregations.

**AMS(MOS) subject classifications.** 65N55, 65F10

## 1. Introduction.

Solving large scale systems of linear equations arising from industrial finite element models poses two sorts of difficulties: There are problems related to the underlying differential equations, and those caused by the problematic ways often used in engineering practice to discretize the equations.

When designing an iterative solver, resolution of the first sort of difficulty is often the easier to achieve. This is because the physics the equations model is known in advance. For example, in a multilevel context, based on the class of problems to be handled, one can decide which functions should be resolved exactly by a coarse space, the necessary smoothness of coarse space basis functions, etc. This allows one to carry out an efficient iterative method, assuming a reasonable finite element discretization.

In contrast, difficulties tied to finite element discretization vary model by model, and can be of many different sorts. These include strongly irregular meshes, insensitive ways of implementing essential boundary conditions, joining several parts of a model by multi-point constraints, etc. This paper focuses on treating these phenomena.

* This paper was presented at the 1998 Copper Mountain Conference on Iterative Methods. This research was supported by NSF grant ECS-9725504, DOE grant DE-FG03-94ER25217, and Czech academic grant VS 97156.

† Department of Applied Mathematics, University of Colorado at Boulder, Boulder CO 80309-0526, mbrezina@math.cudenver.edu

‡ Department of Mathematics, University of Colorado at Denver, Denver, CO 80217-3364, cheber@math.cudenver.edu

§ Department of Mathematics, University of Colorado at Denver, Denver, CO 80217-3364, jmandel@math.cudenver.edu

¶ Department of Mathematics, University of Colorado at Denver, Denver, CO 80217-3364, pvanek@tiger.cudenver.edu

Of course, preventing the impact of such difficulties on convergence can only be done at a certain computational expense, and assuming that "anything could happen anywhere" would result in a very expensive solver. Fortunately, these troubles are typically of local nature, occuring on small clusters of "bad" elements, along a part of the boundary, or on the interface between two parts of the model.

The above reasons call for designing an adaptive method able to detect problematic regions and prevent them from deteriorating convergence by a corresponding "local" modification of the iterator.

In [10] we proposed and analyzed a variational two-level method, which we modify here to implement such an adaptive principle. The components of the method are a *tentative prolongator* based on a generalized aggregation principle and a *prolongator smoother* enforcing the smoothness of a coarse space by multiplying the range of the tentative prolongator (the tentative coarse space). The assumptions of the abstract convergence theorem are a weak approximation property imposed on the tentative coarse space and a smoothing property measuring the "strength" of the prolongator smoother. We choose the prolongator smoother to be an appropriate polynomial in the stiffness matrix $A$. The "strength" of the resulting prolongator smoother is then determined by its degree. The smoothing property imposed on the prolongator smoother is a simple algebraic condition which is "problem independent", the condition number of the resulting preconditioned system decreases with the square of the degree of the prolongator smoother for any stiffness matrix $A$ and the tentative prolongator $\hat{P}$.

Applying the prolongator smoother has a global effect on the smoothed coarse space since it changes all of its basis functions. Increasing the degree of the prolongator smoother is therefore suitable for compensating global convergence problems tied to the underlying differential equation.

On the other hand, locally enriching the tentative prolongator allows for elimination of convergence problems stemming from local irregularities of the finite element discretization. Here we use coarse space enrichment by the local eigenmodes in combination with smoothing the tentative prolongator. This results in an efficient iterative method for standard ($h$-version) finite elements. Similar local enriching of the coarse space was proposed in [4, 5] for the $p$-version finite elements, where, based on solution of local eigenvalue problems associated with parts of the problem, polynomial functions were added to the coarse space.

The structure of the tentative prolongators is derived from a system of non-overlapping subdomains $\Omega_i$ covering the computational domain. For each of the subdomains $\Omega_i$ we create one *supercolumn* consisting of vectors active only on $\Omega_i$. The weak approximation property can be localized subdomainwise as a set of generalized eigenvalue problems, each of them associated with one subdomain. To satisfy the the weak approximation property globally with a given constant $C$, it is sufficient to create supercolumns consisting of generalized eigenvectors corresponding to eigenvalues smaller than $C$. This construction establishes the desirable adaptive effect: on the subdomains where the problem is "good" we end up with supercolumns consisting of only a few eigenvectors, typically only with the local kernels of $A$ (see [7]). The more problematic regions are

characterized by a larger number of small generalized eigenvalues, and our algorithm locally enriches the coarse space by creating larger supercolumns consisting of the corresponding eigenvectors.

The paper is organized as follows: In the next section, we recall the two-level variational method based on the concept of smoothed aggregations, and discuss its differences from the more classical application of the multigrid theory. Section 3 discusses ways of adaptively improving the approximation property of the tentative prolongator prior to smoothing. This section provides motivation for our simple prototype of a possible implementation of the adaptive principle presented in Section 4.

**2. Smoothed aggregation multigrid.** We now recall some concepts and the theoretical framework previously studied in [10]. Choosing a prolongator $P : \mathbb{R}^m \to \mathbb{R}^n$ and pre- and post-smoothers $\mathcal{S}_S, \mathcal{S}'_S : \mathbb{R}^n \to \mathbb{R}^n$, the classic two-level variational method for solving a system of algebraic equations

$$Ax = b, \tag{1}$$

with an $n \times n$ symmetric positive definite matrix $A$ can be written as follows:

ALGORITHM 1. *For the given initial guess* $x \in \mathbb{R}^n$,

*repeat*

1. $x \leftarrow \mathcal{S}_S(x, b)$,
2. *solve the* $m \times m$ *coarse level problem* $P^T A P v = P^T(Ax - b)$,
3. $x \leftarrow x - Pv$,
4. $x \leftarrow \mathcal{S}_{S'}(x, b)$

*until convergence;*

The performance and efficiency of multilevel methods depends critically on the appropriate choice of the prolongation operator $P$.

The method of [8] proposes using $P = S\hat{P}$, where the $n \times m$ matrix $\hat{P}$ is the so-called *tentative prolongator*, and $S$ is the $n \times n$ *prolongation smoother*, to be specified at the end of this section. For practical reasons, we want $\hat{P}$ to be easily constructed, possibly by purely algebraic means. The prolongation smoothing was originally introduced in [6] to allow low-energy error components not in Range $(\hat{P})$ to be represented by the coarse space. We note beforehand that the typical choice for $S$ is a polynomial in $A$. This choice of components makes it possible to view Algorithm 1 as a method for solving problem with the smoothed matrix

$$A_S = S^2 A \tag{2}$$

and the simple prolongation operator $\hat{P}$, which is useful for analysis.

Following [10], set

$$S' = I - \frac{\omega}{\bar{\varrho}(A_S)} A_S, \quad \omega \in (0, 2), \tag{3}$$

where $\bar{\varrho}(A_S)$ denotes an upper bound on the spectral radius $\varrho(A_S)$, and define the pre-smoother $x \leftarrow \mathcal{S}_S(x, b)$ and the post-smoother $x \leftarrow \mathcal{S}_{S'}(x, b)$ to be linear iterative

methods consistent with (1) such that their error propagation operators are matrices $S$ and $S'$, respectively.

A two-level method using this framework with favorable convergence and computational complexity properties was analyzed in [10]. One distinguishing feature of the theory found there is that it proves convergence of the two-level method independently of the coarse grid size, provided sufficently powerful prolongation smoother is used, and assuming only an $l^2$ approximation property of the tentative prolongator $\hat{P}$:

ASSUMPTION 2.1. *There exists a positive constant $C_{\text{apx}}$ such that:*

1. *There is a linear mapping $Q$ of $\mathbb{R}^n$ onto* Range $(\hat{P})$ *such that*

$$\|(I-Q)u\|^2 \leq \frac{C_{\text{apx}}^2}{\bar{\varrho}(A_S)}\|u\|_A^2 \quad \forall u \in \mathbb{R}^n, \tag{4}$$

   *where $\bar{\varrho}(A_S)$ denotes an upper bound on $\varrho(A_S)$.*
2. *The prolongator smoother $S$ is symmetric, commutes with $A$, and satisfies $\varrho(S) \leq 1$.*

REMARK 2.2. *If the constant $C_{\text{apx}}$ in (4) is independent of the fine and coarse grid sizes $m$, $n$, the convergence rate estimate we obtain will be also independent of $m, n$.*

These assumptions can be interpreted similarly to the regularity-free multilevel theories which require two kinds of properties on coarse spaces: a weak approximation condition similar to (4), and smoothness of the coarse spaces, which is usually formulated as the energetic stability of the coarse space interpolator $Q$ ([1, 11, 12]). We would like to point out that even though we will ultimately use prolongator $P = S\hat{P}$, the assumption (4) regards the approximation property of $\hat{P}$, not $P$.

The following theorem shows that the convergence rate of Algorithm 1 with $P = S\hat{P}$ satisfying Assumption 2.1 is independent of dimensions of the coarse and fine spaces, provided $C_{\text{apx}}$ is made so.

THEOREM 2.3. *([10]) Let $e_i$ denote the error after $i$ iterations given by Algorithm 1 with $P = S\hat{P}$, and $e_i^S = Se_i$ the error smoothed by the prolongator smoother. Then it holds that*

$$\|e_{i+1}\|_{A_S}^2 \leq (1-C_3)\|e_i\|_{A_S}^2, \quad \textit{and} \quad \|e_i^S\|_A^2 \leq (1-C_3)^i\|e_0\|_A^2, \tag{5}$$

*where $C_3 = \frac{(C_{\text{apx}})^{-2}\omega(2-\omega)}{1+(C_{\text{apx}})^{-2}\omega(2-\omega)} > 0$, and $\omega$ is the damping parameter from* (3).

As noted above, the most practical choice of the prolongator smoother $S$ is a polynomial in $A$. For the given degree $\deg(S)$, the optimal $S$ is a scaled Chebyshev polynomial in $A$:

LEMMA 2.4. *For any $L > 0$ and any integer $n > 0$, there is a unique polynomial $p$ of degree $n$ such that $\max\{p^2(\lambda)\lambda | 0 \leq \lambda \leq L\}$ is minimal under the constraint $p(0) = 1$. The polynomial $p$ is given by*

$$p(\lambda) = \left(1 - \frac{\lambda}{r_1}\right) \ldots \left(1 - \frac{\lambda}{r_n}\right), \tag{6}$$

*where the roots $r_k$ of $p$ are*

$$r_k = \frac{L}{2}\Big(1 - \cos\frac{2k\pi}{2n+1}\Big), \quad k = 1, \ldots, n. \tag{7}$$

*In addition, the polynomial $p$ satisfies*

$$\max_{0\le\lambda\le L} p^2(\lambda)\lambda = \frac{L}{(2n+1)^2} \tag{8}$$

*and*

$$\max_{0\le\lambda\le L} |p(\lambda)| = 1. \tag{9}$$

*Proof.* From minimax properties of Chebyshev polynomials, if $p^2(\lambda)\lambda$ can be written as the linearly transformed Chebyshev polynomial of order $2n+1$,

$$p^2(\lambda)\lambda = q(\lambda) = \frac{c}{2}(1 - T_{2n+1}(1 - 2\lambda/L)), \quad c = \max_{0\le\lambda\le L} p^2(\lambda)\lambda, \tag{10}$$

then $p$ is the sought polynomial. The zeros of $q(\lambda)$ are the points $\lambda$ where $T_{2n+1}(1 - 2\lambda/L) = 1$, that is, $1 - 2\lambda/L = \cos\frac{2k\pi}{2n+1}$. The value $k = 0$ gives the simple root $\lambda = 0$ of $q$, while $k = 1, \ldots, n$ yield double roots of $q$, given by (7). This proves that $p$ is indeed the polynomial (6).

To prove (8), we determine the quantity $c$ in (10) from the condition that $p(0) = 1$. This condition implies that the linear term of $p^2(\lambda)\lambda$ is one, hence from (10), $1 = (p^2(\lambda)\lambda)'(0) = (c/L)T'_{2n+1}(1)$. Therefore,

$$c = L/T'_{2n+1}(1) = L/(2n+1)^2. \tag{11}$$

It remains to prove (9). First, (9) is equivalent to $p^2(\lambda) \le 1$ for all $\lambda$, $0 \le \lambda \le L$. Using (10) and (11), this is equivalent to

$$\frac{L(1 - T_{2n+1}(1 - 2\lambda/L))}{2\lambda T'_{2n+1}(1)} \le 1, \text{ for all } \lambda,\ 0 \le \lambda \le L. \tag{12}$$

Using the substitution $1 - 2\lambda/L = x$, (12) becomes by a simple manipulation

$$T_{2n+1}(x) \ge 1 + T'_{2n+1}(1)(x - 1), \text{ for all } x,\ -1 \le x \le 1,$$

which is the well known fact that the graph of a Chebyshev polynomial lies above its tangent at $x = 1$. □

In view of the previous lemma, we choose

$$S = \Big(I - \frac{1}{r_1}A\Big)\ldots\Big(I - \frac{1}{r_{\deg(S)}}A\Big), \tag{13}$$

where

$$r_k = \frac{\varrho(A)}{2}\Big(1 - \cos\frac{2k\pi}{2\deg(S)+1}\Big), \quad k = 1, \ldots, \deg(S). \tag{14}$$

For this choice of $S$ Lemma 2.4 guarantees the estimate

$$\varrho(A_S) \leq \frac{\varrho(A)}{(2\deg(S)+1)^2}, \qquad cf.\ (2). \tag{15}$$

REMARK 2.5. **(Comparison to classical approach).** *Under Assumption 2.1, Theorem 2.3 yields a convergence rate estimate in the form*

$$1 - f^S(C_{\text{apx}}), \quad f^S(x) = \frac{1}{1+x^2} \tag{16}$$

*To compare our result to the classical framework, consider Algorithm 1 with the unsmoothed prolongator $P = \hat{P}$, and pre- and post-smoothers $I - \frac{1}{\bar{\varrho}(A)}A$, where $\bar{\varrho}(A) \geq \varrho(A)$ is a known bound. Then a trivial modification of the two-level theory found in Brandt* [2] *proves the A-norm convergence rate bound*

$$1 - f^B(C_B), \quad f^B(x) = \frac{1}{\max\{x^2, 1\}} \tag{17}$$

*under the assumption that*

$$\|(I-Q)u\|^2 \leq \frac{C_B^2}{\bar{\varrho}(A)}\|u\|_A^2 \quad \forall u \in \mathbb{R}^n. \tag{18}$$

*We note that $\frac{1}{2}f^B(x) \leq f^S(x) \leq f^B(x) \quad \forall x \in [0,\infty)$, and the functions $f^S(x), f^B(x)$ are asymptotically equivalent as $x \to \infty$.*

*Also, adding more pre– or post–smoothing may not improve convergence in the classical theory unless additional regularity is assumed (see* [2]*).*

*In order to see the advantage of the current theoretical framework, we compare inequalities* (4) *and* (18) *to obtain the relationship*

$$C_{\text{apx}} = C_B \frac{\bar{\varrho}(A_S)}{\bar{\varrho}(A)},$$

*which can be made significantly smaller than $C_B$ by selecting $S$ so that $\bar{\varrho}(A_S)$ is small. This will result in a smaller convergence rate estimate* (16)*. We will have control over the reduction by using prolongation smoother $S$ such that (see* (15)*) $\varrho(A_S) \leq \frac{\varrho(A)}{(2\deg(S)+1)^2}$. This means that increasing the prolongator smoother degree does not lose its effect and always yields an improvement of the convergence rate. Figure 1 offers comparison of the corresponding convegence rate estimates, demonstrating the improvement achieved by smoothing the prolongator.*

In view of Theorem 2.3 and Remark 2.5, we will strive to decrease the constant $C_{\text{apx}}$ in inequality (4) to obtain a better convergence rate estimate.

There are two ways to achieve this: The first possibility is to use a more powerful prolongation smoother $S$, forcing $\varrho(A_S) \ll \varrho(A)$. With our choice (15) of the prolongation smoother, this amounts to using a higher degree of $S$. Another possibility is, for $S$ fixed, to select a better tentative prolongator $\hat{P}$ so that (4) is satisfied with a smaller constant $C_{\text{apx}}$.

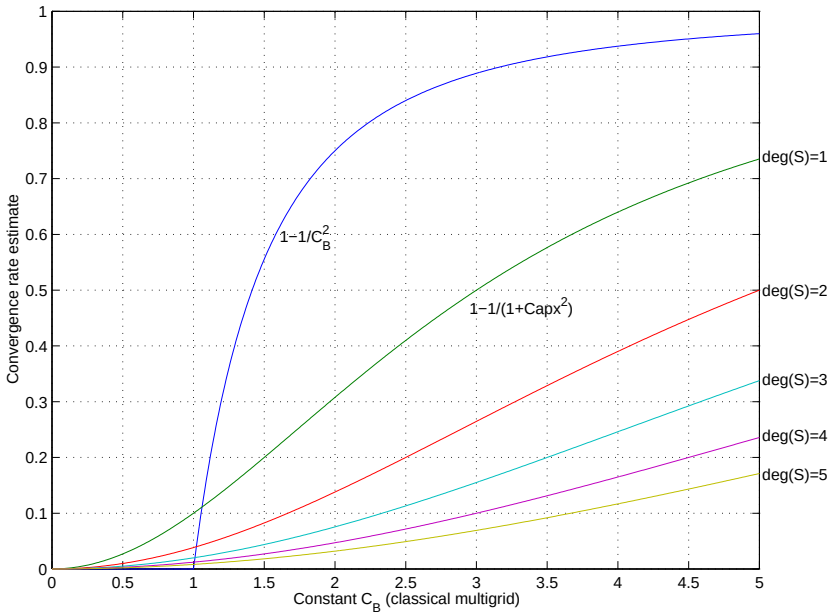


FIG. 1. *Comparison of the convergence rate estimates for the classical multigrid versus the smoothed-prolongator algorithm with smoother degree* $\deg(S)$*, in dependence on the constant* $C_B$*.*

The implications of different prolongator smoothing procedures have been previously studied in [7], so we refer the reader there for details. In this paper, we will investigate the improvements of $\hat{P}$.

We note that the idea of enriching the coarse space was already recognized in [9] in the treatment of problems of the fourth order. However, the coarse-space enriching was done heuristically, without the adaptivity we are striving to achieve. Adaptive enriching of the coarse space in the context of $p$-version finite elements was proposed in Mandel [4, 5].

**3. The Adaptive Principle.** In this section, we explain how to construct a tentative prolongator $\hat{P}$ for which the weak approximation property

$$\forall u \in \mathbb{R}^n \; \exists v \in \mathbb{R}^m : \quad \|u - \hat{P}v\| \leq \bar{C}_{\mathrm{req}} \|u\|_A$$

is satisfied with an a priori specified constant $\bar{C}_{\mathrm{req}}$. Establishing this inequality, together with the knowledge of the bound $\bar{\varrho}(A_S)$, guarantees the convergence rate given by Theorem 2.3 with constant $C_{\mathrm{apx}} = \bar{C}_{\mathrm{req}} \bar{\varrho}(A_S)$.

Let the computational domain $\Omega$ be decomposed as

$$\Omega = \bigcup_{i=1}^{J} \bar{\Omega}_i, \quad \Omega_i \cap \Omega_j = \emptyset \text{ if } i \neq j,$$

where, each $\bar{\Omega}_i$ is a connected cluster of elements. Let $n_i$ be the number of degrees of freedom in $\bar{\Omega}_i$. We denote by $A_i$ the $n_i \times n_i$ local stiffness matrix corresponding to $\bar{\Omega}_i$, and $N_i$ denotes a zero-one matrix mapping the degrees of freedom associated with

$\bar{\Omega}_i$ into the set of global degrees of freedom. That is, the stiffness matrix $A$ can be assembled from the local stiffness matrices $A_i$ corresponding to the elements in $\bar{\Omega}_i$:

$$A = \sum_{i=1}^{J} N_i A_i N_i^T. \tag{19}$$

We will utilize a system of aggregates of nodes $\{\mathcal{A}_i\}_{i=1}^{J}$ forming a disjoint covering of the set of all nodes, such that all the nodes in $\mathcal{A}_i$ lie in $\bar{\Omega}_i$. Such $\{\mathcal{A}_i\}$ can be obtained from the sets of nodes on $\bar{\Omega}_i$, by distributing any nodes shared by more than one $\bar{\Omega}_i$ to exactly one of the respective $\mathcal{A}_i$ at will.

For each subdomain $\bar{\Omega}_i$ with number of degrees of freedom $n_i$ we define a zero-one diagonal matrix $I_i$ of dimension $n_i$ by

$$(I_i)_{jj} = \begin{cases} 1 & \text{if the degree of freedom } j \text{ corresponds to a node in } \mathcal{A}_i \\ 0 & \text{otherwise .} \end{cases} \tag{20}$$

We will define for each subdomain $\bar{\Omega}_i$ a set of vectors $\{w_j^{(i)}\}_{j=1}^{m_i}$, $w_j^{(i)} \in$ Range $(I_i)$. Each vector $w_j^{(i)}$ will imply one coarse level degree of freedom $x_j^{(i)}$. The number $m_i$ can then be interpreted as the number of degrees of freedom for coarse level "node" $i$. The specific choice of $w_j^{(i)}$ will be addressed below.

For convenience of notation, $W_i$ will denote $n_i \times m_i$ matrix consisting of columns $\{w_j^{(i)}\}_{j=1}^{m_i}$, and $x^{(i)}$ the vector with entries $x_j^{(i)}$, $\quad j = 1, \ldots, m_i$ (i.e., the vector of coarse-space degrees of freedom associated with the aggregate $\mathcal{A}_i$). Then, we organize coarse-level degrees of freedom as a vector

$$x = \begin{pmatrix} x^{(1)} \\ \vdots \\ x^{(J)} \end{pmatrix} \in \mathbb{R}^m, \quad \text{where} \quad m = \sum_{i=1}^{J} m_i.$$

We now define the prolongator $\hat{P} : \mathbb{R}^m \mapsto \mathbb{R}^n$ by

$$\hat{P}x = \sum_{i=1}^{J} N_i W_i x^{(i)}. \tag{21}$$

That is, each column of matrix $\hat{P}$ will consist of the appropriate vector $w_j^{(i)}$ extended by zero outside $\mathcal{A}_i$. Note that because the entries in vectors $w_j^{(i)}$ corresponding to degrees of freedom not in $\mathcal{A}_i$ are zero, (block) columns $\hat{P}_{*,i}$ of $\hat{P}$ corresponding to different aggregates will have disjoint sparsity structure (Figure 2 depicts the sparsity structure; for simplicity, Figure 2 assumes that the degrees of freedom are numbered consecutively aggregate-by-aggregate).

Having introduced the notation, we can provide the motivation for this section. The theory recalled in Section 2 guarantees that the rate of convergence of Algorithm 1 under the assumption that

$$\|(I - Q)u\|^2_{l^2(\bar{\Omega})} \leq \frac{C^2_{\text{apx}}}{\varrho(S^2 A)} \|u\|^2_A \quad \forall u \in \mathbb{R}^n \tag{22}$$

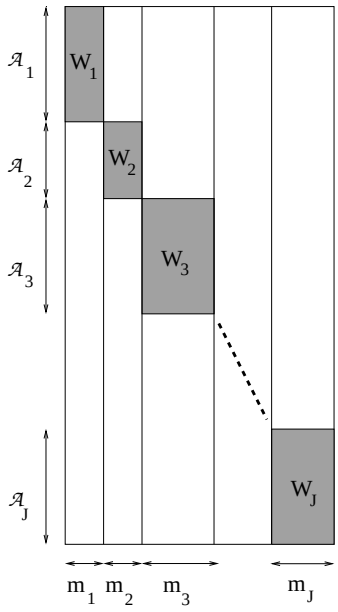


FIG. 2. *The sparsity structure of tentative prolongator $\hat{P}$. The (block) column $\hat{P}_{*,i}$ is nonzero only for entries corresponding to the degrees of freedom in aggregate $\mathcal{A}_i$.*

is bounded by $1-\frac{C_{\rm apx}^{-2}}{1+C_{\rm apx}^{-2}}$, where $Q$ is a linear operator such that Range $(Q)$ = Range $(\hat{P})$.

Inequality (22) is equivalent to existence of $v \in \mathbb{R}^m$ such that

$$\|u - \hat{P}v\|^2_{l^2(\bar{\Omega})} \leq \frac{C^2_{\rm apx}}{\varrho(S^2A)} \|u\|^2_A \quad \forall u \in \mathbb{R}^n. \tag{23}$$

Using (19) and the sparsity structure of $\hat{P}$, we can write for any $u \in \mathbb{R}^n$,

$$\|u - \hat{P}v\|^2_{l^2(\bar{\Omega})} = \sum_{i=1}^{J} \|u - \hat{P}_{*,i}v_i\|^2_{l^2(\mathcal{A}_i)}$$

and

$$\|u\|^2_A = \sum_{i=1}^{J} \langle A_i N_i^T u, N_i^T u \rangle.$$

Here and below, $\|\cdot\|_{l^2(\mathcal{A}_i)}$ denotes the restriction of Euclidean norm to the set of degrees of freedom corresponding to the nodes in aggregate $\mathcal{A}_i$. We can now consider the local estimates

$$\|u - \hat{P}_{*,i}v_i\|^2_{l^2(\mathcal{A}_i)} \leq \frac{C^2_{\rm apx}}{\varrho(S^2A)} \langle A_i N_i^T u, N_i^T u \rangle. \tag{24}$$

As these estimates are uniform with respect to $i$, summing (24) over $i = 1, \ldots, J$ yields (23). Of course, the satisfaction of (24) will depend on the range of $P_{*,i}$ in other words, on the columns of $\hat{P}$ corresponding to individual aggregates.

We will design a procedure for enriching Range $(\hat{P})$ as necessary to satisfy (24). Let us first explore the most straightforward possibility.

Let $Q_i$ be an $l^2$-projector such that Range $(Q_i)$ = Range $(P_{*,i})$. Then for each $i$ we can find constant $\bar{C}_i$ such that

$$\inf_{v_i \in \mathbb{R}^{m_i}} \|u - \hat{P}_{*,i} v_i\|_{l^2(\mathcal{A}_i)} = \langle (I - Q_i)u, u \rangle_{l^2(\mathcal{A}_i)} \leq \bar{C}_i^2 \|u\|_{A_i}^2 \quad \forall u \in \mathbb{R}^n. \tag{25}$$

This can be reformulated as a generalized eigenvalue problem for $I_i(I - Q_i)I_i$ preconditioned by $A_i$ (see Lemma 3.1 below). The constant $\bar{C}_i$ will be the upper bound for the largest finite characteristic number of this generalized eigenvalue problem (characteristic number = 1/eigenvalue).

Now for each $i$, the constant $\bar{C}_i$ in (25) can be decreased by finding the largest eigenvalue and adding the corresponding eigenvector to the range of $Q_i$. This amounts to adding new columns corresponding to aggregate $i$ to the tentative prolongator $\hat{P}$. The procedure can be repeated until (25) is eventually satisfied with the given constant $\bar{C}_{\text{req}}$ for any $i$.

However, as solving the generalized eigenvalue problem is undesirably expensive, we will provide a cheaper alternative in Lemma 3.2 below.

Let $Q_i = W_i(W_i^T W_i)^+ W_i^T$, i.e., $Q_i$ is the $\mathbb{R}^{n_i}$–orthogonal projector onto the Range $(W_i)$, and define

$$Q = \sum_{i=1}^{J} N_i Q_i N_i^T. \tag{26}$$

LEMMA 3.1. *Suppose there exists a constant $\bar{C}_{\text{req}}$ such that for each local stiffness matrix $A_i$, $i = 1, \ldots, J$ and every $v \in \mathbb{R}^{n_i}$,*

$$\langle I_i(I - Q_i)I_i v, v \rangle_{\mathbb{R}^{n_i}} \leq \bar{C}_{\text{req}} \langle A_i v, v \rangle.$$

*Then*

$$\|(I - Q)u\|^2 \leq \bar{C}_{\text{req}} \langle Au, u \rangle \quad \text{for all } u \in \mathbb{R}^n.$$

*Proof.* As $Q_i = Q_i I_i$, we have for any $u \in \mathbb{R}^n$,

$$\begin{aligned} \|(I - Q)u\|^2 &= \sum_{i=1}^{J} \|(I - Q)u\|_{l^2(\mathcal{A}_i)}^2 \\ &= \sum_{i=1}^{J} \|I_i(I - Q_i){N_i}^T u\|_{\mathbb{R}^{n_i}}^2 \\ &= \sum_{i=1}^{J} \|I_i(I - Q_i)I_i {N_i}^T u\|_{\mathbb{R}^{n_i}}^2. \end{aligned}$$

Obviously, $I_i(I - Q_i)I_i$ is an orthogonal projector, so, using $\sum_{i=1}^{J} \langle N_i A_i N_i{}^T u, u\rangle = \langle Au, u\rangle$, we obtain

$$\begin{aligned} \|(I-Q)u\|^2 &= \sum_{i=1}^{J} \langle I_i(I-Q_i)I_i N_i{}^T u, N_i{}^T u\rangle_{\mathbb{R}^{n_i}} \\ &\leq \sum_{i=1}^{J} \bar{C}_{\text{req}} \langle A_i N_i{}^T u, N_i{}^T u\rangle_{\mathbb{R}^{n_i}} \\ &\leq \bar{C}_{\text{req}}\ \langle Au, u\rangle, \end{aligned}$$

which was to be proved. □

The following lemma suggests a selection of vectors $w_j^{(i)}$ that is computationally more feasible than the choice of the eigenvectors of the generalized eigenvalue problem suggested above.

LEMMA 3.2. *Let $w_j^{(i)} = I_i \hat{w}_j^{(i)}$, where $\hat{w}_j^{(i)}$ are the eigenvectors of matrix $A_i$ corresponding to the eigenvalues of $A_i$ that are smaller than $\Lambda > 0$. Then*

$$\langle I_i(I-Q_i)I_i u, u\rangle_{\mathbb{R}^{n_i}} \leq \frac{1}{\Lambda}\|u\|^2_{A_i} \quad \textit{for all } u \in \mathbb{R}^{n_i}.$$

*Proof.* Let $W_i$ be the matrix formed by columns $w_j^{(i)}$, and $\hat{W}_i$ be the matrix formed by columns $\hat{w}_j^{(i)}$, i.e., $W_i = I_i \hat{W}_i$. We recall the projector

$$Q_i = W_i (W_i^T W_i)^+ W_i^T.$$

Let us define the $l^2(\mathbb{R}^{n_i})$–orthogonal projector onto Range $(\hat{W}_i)$,

$$\hat{Q}_i = \hat{W}_i (\hat{W}_i^T \hat{W}_i)^{-1} \hat{W}_i^T,$$

and note that from the spectral decomposition of $A_i$, $\Lambda\langle (I-\hat{Q}_i)u, u\rangle_{\mathbb{R}^{n_i}} \leq \langle A_i u, u\rangle_{\mathbb{R}^{n_i}}$. Since $I_i Q_i = Q_i$ and Range $(I_i\hat{Q}_i) \subset$ Range $(W_i)$, we have

$$\begin{aligned} \langle I_i(I-Q_i)I_i u, u\rangle_{\mathbb{R}^{n_i}} &= \|I_i(I-Q_i)I_i u\|^2_{\mathbb{R}^{n_i}} = \|(I-Q_i)I_i u\|^2_{\mathbb{R}^{n_i}} \\ &= \inf_{w \in \text{ Range } (W_i)} \|I_i u - w\|^2_{\mathbb{R}^{n_i}} \leq \|I_i(I-\hat{Q}_i)u\|^2_{\mathbb{R}^{n_i}} \\ &\leq \|(I-\hat{Q}_i)u\|^2_{\mathbb{R}^{n_i}} \leq \frac{1}{\Lambda}\langle A_i u, u\rangle_{\mathbb{R}^{n_i}}, \end{aligned}$$

concluding the proof. □

REMARK 3.3. *In the definition (26) of $Q_i$, we used the pseudo-inverse because after filtering by $I_i$, the columns of $W_i$ might become linearly dependent. We emphasize that $Q_i$ is used for theoretical purposes only and never in actual computation.*

Based on the previous two lemmas, we can easily construct a tentative prolongator satisfying the weak approximation property (4) with an a priori given constant $\bar{C}_{\text{req}}$:

ALGORITHM 2.

- *For each subdomain stiffness matrix $A_i$, find all eigenvectors $\hat{w}_j^{(i)}$ corresponding to its eigenvalues $\lambda_j^{(i)} \leq \frac{1}{C_{\mathrm{req}}}$, and denote the number of such eigenvectors by $m_i$.*
- *For each subdomain, create the $n_i \times m_i$ matrix $\hat{W}_i = (\hat{w}_1^{(i)}, \ldots, \hat{w}_{m_i}^{(i)})$, and set $W_i = I_i \hat{W}_i$, where $I_i$ is defined by* (20).
- *Construct the prolongator* (21) *so that $N_1 W_1$ forms the first $m_1$ columns of $\hat{P}$, $N_2 W_2$ forms columns $m_1 + 1$ through $m_1 + m_2$ of $\hat{P}$, etc.*

**4. Prototype Algorithm.** In this section, we give a simple example of a possible implementation taking advantage of the coarse-space enriching principle discussed in the previous section. Our algorithm is described in Figure 3. The input of the algorithm is the right-hand side $b$ of problem (1), the aggregate local stiffness matrices $\{A_i\}$, tolerance parameter $\tau \in (0, 100)[\%]$ and $r_{req}$, the rate of convergence to be guaranteed. For simplicity, we assume that initial tentative prolongator $\hat{P}$ whose columns consist of the zero energy modes is supplied. This is not generally required if the algorithm were modified to first find the zero energy modes of the aggregate problems, but many finite element packages provide them. The global stiffness matrix may either be supplied or assembled from $A_i$. For simplicity, we assume that $A$ is also input.

We emphasize that Figure 3 gives only a prototype implementation. Suitable choice of parameter $r_{req}$ is essential, as it will determine not only the convergence rate, but also the expense of the method. The enrichment of the coarse space is intended to complement the prolongator smoothing, for which good convergence properties were proved and observed [7]. The practical aim is, for problems with local difficulties, to achieve the favorable convergence properties guaranteed by the smoothed aggregation procedure for problems lacking such difficulties. In a typical situation, with a limited number of such singularities this can be done at a justifyable cost.

REMARK 4.1. *For good performance, we suggest using the algorithm for the diagonally scaled system $D^{-1/2} A D^{1/2} y = D^{-1/2} b$, $D = \mathrm{diag}(A)$ instead of $Ax = b$. This scaling suppresses the influence of possible jumps in the equation coefficients, but also guarantees that application of Geršgorin theorem yields a sharper estimate for the spectral radius of the problem matrix.*

REFERENCES


[1] J. H. BRAMBLE, J. E. PASCIAK, J. WANG, AND J. XU, *Convergence estimates for multigrid algorithms without regularity assumptions*, Math. Comp., 57 (1991), pp. 23–45.

[2] A. BRANDT, *Algebraic multigrid theory: The symmetric case*, Appl. Math. Comput., 19 (1986), pp. 23–56.

[3] M. BREZINA AND P. VANĚK, *One black-box iterative solver.* Submitted to Computing., 1997.

[4] J. MANDEL, *Adaptive iterative solvers in finite elements*, in Solving Large-scale Problems in Mechanics, M. Papandrakis, ed., John Wiley & Sons Ltd., 1993, pp. 65–86.

[5] ——, *Iterative methods for p-version finite elements: Preconditioning thin solids*, Comput. Methods Appl. Mech. Engrg., 133 (1996), pp. 247–257.

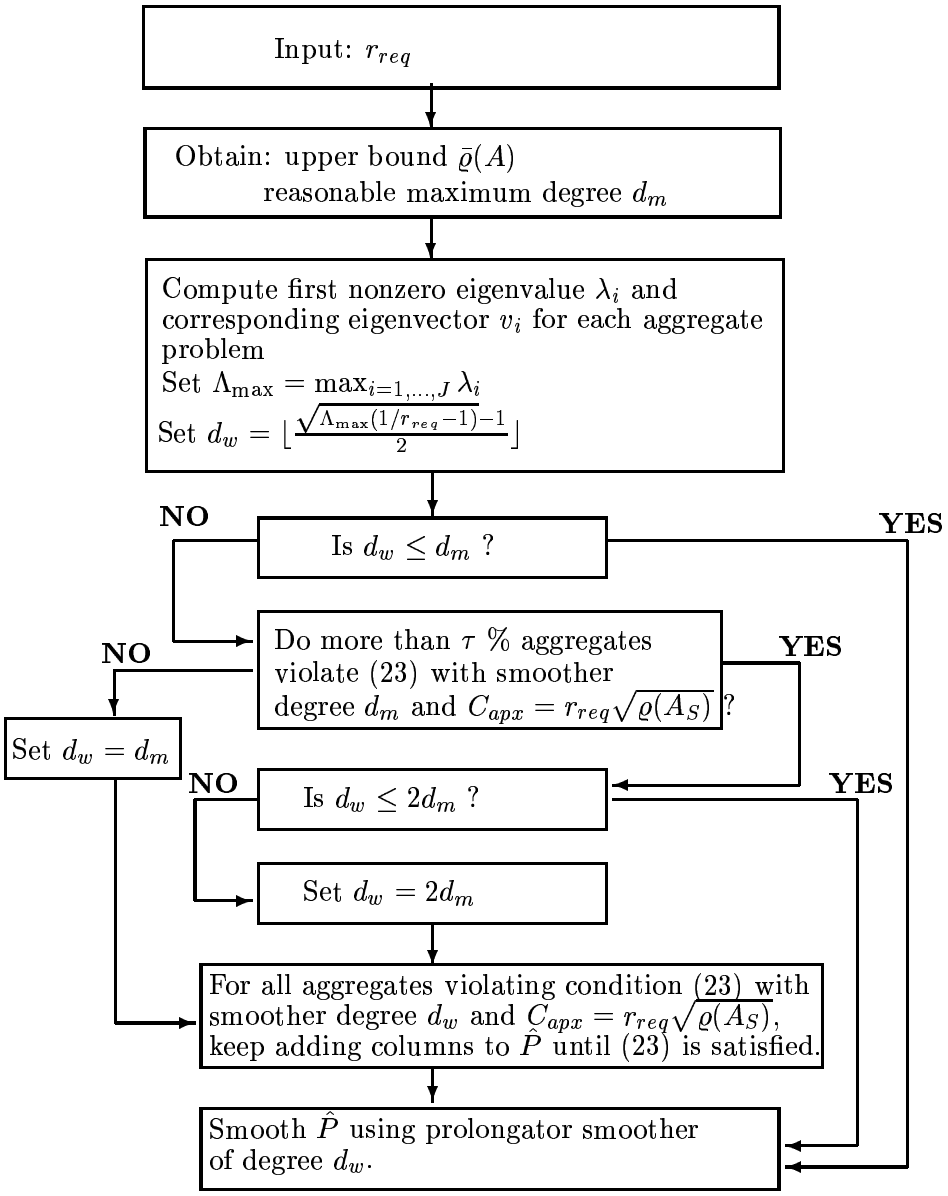


FIG. 3. *Simple algorithm for construction of prolongator $P$.*

[6] P. VANĚK, *Acceleration of convergence of a two-level algorithm by smoothing transfer operator*, Applications of Mathematics, 37 (1992), pp. 265–274.

[7] P. VANĚK, M. BREZINA, AND R. TEZAUR, *Two-grid method for linear elasticity on unstructured meshes*. To appear in SIAM J. Sci. Comp., 1998.

[8] P. VANĚK, J. MANDEL, AND M. BREZINA, *Algebraic multigrid on unstructured meshes*, UCD/CCM Report 34, Center for Computational Mathematics, University of Colorado at Denver, December 1994.

[9] P. VANĚK, J. MANDEL, AND M. BREZINA, *Algebraic multigrid by smoothed aggregation for second and fourth order elliptic problems*, Computing, 56 (1996), pp. 179–196.

[10] P. VANĚK, R. TEZAUR, M. BREZINA, AND J. KŘÍŽKOVÁ, *Two–level method with coarse space size independent convergence*, in Domain Decomposition Methods in Sciences and Engineering, 8th International Symposium on Domain Decomposition, Beijing, P.R. China, 1996, R. Glowinski, J. Périaux, Z. Shi, and O. Widlund, eds., John Wiley & Sons Ltd., New York, N.Y., 1997, pp. 233–240.

[11] J. WANG, *Convergence analysis without regularity assumptions for multigrid algorithms based on SOR smoothing*, SIAM J. Numer. Anal., 29 (1992), pp. 987–1001.

[12] ——, *Convergence analysis of Schwartz algorithm and multilevel decomposition iterative methods II: Non-selfadjoint and positive indefinite elliptic problems*, SIAM J. Numer. Anal., 30 (1993), pp. 953–970.