Analytic Centers and Repelling Inequalities

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Abstract

The new concepts of repelling inequalities, repelling paths, and prime analytic centers are introduced. A repelling path is a generalization of the analytic central path for linear programming, and we show that this path has a unique limit. Furthermore, this limit is the prime analytic center if the set of repelling inequalities contains only those constraints that "shape" the polytope. Because we allow lower dimensional polytopes, the proof techniques are non-standard and follow from data perturbation analysis. This analysis overcomes the difficulty that analytic centers of lower dimensional polytopes are not necessarily continuous with respect to the polytope's data representation. A second concept introduced here is that of the "prime analytic center," in which we establish its uniqueness in the absence of redundant inequalities. Again, this is well known for full dimensional polytopes, but it is not immediate for lower dimensional polytopes because there are many different data representations of the same polytope, each without any redundant inequalities. These two concepts combine when we introduce ways in which repelling inequalities can interact.

Keywords: polyhedral theory, linear programming, computational economics, interior point methods, analytic center, sensitivity analysis, central path, strict complementarity

1 Introduction

In 1979, Khachiyan [18] proved that the class of linear programs has polynomial time complexity. Although some attention was devoted to understanding the behavior of this algorithm, its implementation was disappointing. In 1984, the situation changed with Karmarkar's algorithm [17], where the claim was not only a theoretical complexity argument, but also an implementation that solved large linear programs more than an order of magnitude faster than a commercial quality simplex algorithm. Karmarkar's algorithm is called an "interior point algorithm" because it generates elements away from the boundary of the polyhedron. This launched a flood of research activity into the theory and implementation of interior point algorithms.

We now know that these algorithms are polynomial because they generate points near an infinitely smooth curve, called the *central path*. This path is contained within the interior of the feasible polyhedron and terminates at a unique, strictly complementary, optimal solution [13, 22, 30]. Each element of the central path is an *analytic center*, a concept that Huard [15] used in his method of centers, and which was more recently introduced to the mathematical programming community by Sonnevend [24, 25, 26].

The central path has several parameterizations, which are described in detail by Gonzaga [8]. One of these is developed from a purely primal perspective in which the central path is generated by an objective cut, which repels the center towards the solution. This was the method of Renegar [21], and because this is one of the concepts from which we build, a brief description is given in the next section.

The pathway associated with Renegar's algorithm is the central path, and we tend to think of it as unique, just as we think of *the* analytic center of a polytope. However, the fact is that these centers depend upon the representation of the polytope. This data reliance was understood by Sonnevend [27] when he wrote 'the analytic center depends not only on the polytope's representation, but also on those data elements which perhaps do not 'shape' the polytope. This is the price we have to pay for the smooth dependence on the data.' While this reliance has been recognized by other authors (e.g., Ye [31], Schrijver [23]), it has not been addressed in depth. To understand this dependence, we are concerned with a *minimal* representation of a polytope as defined by Telgen [28, 29]. If there are no implicit equalities, Telgen's minimal representation is a *prime* representation [5] that is, a representation that is redundancy free. In general, there can be more than one prime representation.

There are two new concepts introduced in this paper. First, we present the concept of a *repelling inequality*. Our aim is to have a fundamental understanding of the paths of analytic centers generated by a set of repelling inequalities. Second, there is a *prime analytic center* that is unique for a class of representations, which we call semi-prime, that capture the geometry of the polytope. This class includes all prime representations.

The rest of this paper is organized as follows. The next section presents some of the basic terms and concepts. Additional material about Renegar's method is included to motivate the idea of repelling inequalities. We note that the definitions of the technical terms used throughout this paper, including those taken for granted in the linear programming literature, can be found in the *Mathematical Programming Glossary* [11].

Section 3 contains two important results. First, we show that the analytic center of lower dimensional polytopes is continuous over a specified set of data, and we provide an example demonstrating that these analytic centers need not be continuous outside this set. Second, we introduce the concepts of repelling inequalities and repelling paths. Here, the main result is the establishment of a limit. When there is only one repelling inequality, the repelling path is the central path of a linear program and the limiting properties can be established by standard proof techniques [12, 19, 22, 30, 31]. However, when there is more than a single repelling inequality, these proof techniques fail to apply, so we provide new proof techniques based on data perturbations. Further, these proof techniques overcome the difficulty that analytic centers of lower dimensional polytopes are not necessarily continuous with respect to the polytope's data representation.

Section 4 is concerned with the transient behavior of repelling paths. Repelling paths are shown to be continuous not only over their parameterization variable, but also over problem data. We end the section by showing that a repelling path is either a single point, or a simple curve.

We then turn our attention to prime representations, bringing the second major topic into focus with the fact that the analytic center is the same for all semi-prime representations. This is immediate when the polytope is full dimensional. However, the result for lower dimensional polytopes is more complicated because the existence of equalities allows many semi-prime representations.

In §6 we introduce the relative effects of sets of repelling inequalities, categorizing them according to whether they *oppose* or *ally* with each other. Finally, we present avenues for future research that build on the foundations introduced here.

2 Terms and Concepts

For any vector, v, its *support set* is $\sigma(v) \stackrel{\text{def}}{=} \{j : v_j \neq 0\}$. If I is any subset of indices, $\overline{I}$ denotes its complement (where context determines the original index set). In particular, $\overline{\sigma}(v) = \{j : v_j = 0\}$.

A set subscript on a vector is used to indicate the sub-vector whose components correspond to the elements in the set. For example, $v_{\sigma(v)}$ is the sub-vector of v with non-zero coordinates. When we partition $v = (v_{\sigma(v)}, v_{\overline{\sigma}(v)})$, it is understood that this is really a permutation of v. This notation extends to matrices. Let $U \in \mathbb{R}^{k \times n}$, and let I be a set of row indices. Then, U_I is the sub-matrix with only those rows indexed by I. In the special case of a single row, we simply let the index denote the corresponding sub-matrix; U_i is the i-th row of U.

Sequences are indicated with set notation, and a sequence converges if, and only if, this set of elements has a unique limit. (Please note that we use the term *limit* in its strict sense, not just a cluster point, which could be one of many.) Sequences are always indexed by superscripts. For example, $\{x^k \in X\} \to \hat{x}$ is a sequence of vectors contained in the set X that converges to $\hat{x}$.

A polyhedron is denoted by $\mathcal{P}$, and a *representation* of $\mathcal{P}$ is $P(U, u) = \{x : Ux \leq u\}$. We say that $\mathcal{P}$ is a *polytope* if it is bounded, and because the analytic center for unbounded polyhedra is not uniquely defined (see [20] for extensions), we assume that all polyhedra are polytopes. Given the data, (U, u), the system $Ux \le u$ is divided into the implied equalities $Ax = a$, and the inequalities $Bx \le b$. So, $(A^T \mid B^T)^T$ and $(a^T \mid b^T)^T$ are partitions of U and u, respectively. We call (A, a, B, b) the *separated data* for (U, u), and we take b to be an m-vector. For notational clarity, the dependence of (A, a, B, b) on (U, u) is not explicit, but is understood throughout. Furthermore, we assume that $Ax = a$ contains all the implied equalities, which means that there exists $x \in \mathcal{P}$ such that $Bx < b$. (An economical way to separate the implied equalities from the inequalities is described by Freund, Roundy and Todd [7]; for details see [9, 10, 22]. A probabilistic approach by Boneh is found in [4].) We denote the *strict interior* by $P^0(U, u) = \{x : Ax = a,\ Bx < b\}$, and $\mathrm{ri}(\mathcal{P})$ denotes the *relative interior* of $\mathcal{P}$. If $P(U, u)$ is bounded, $\mathrm{rank}(U) = n$ or equivalently $\mathcal{N}(U) = \{0\}$, where $\mathcal{N}$ denotes the null space.

A constraint is *redundant* if its removal provides another representation of the same polytope. When there are no redundancies, the representation is *prime* [5]. We say that a representation is *semi-prime* if there are no redundant inequalities. Hence, every prime representation is semi-prime regardless of whether or not there are redundant equations.

Given a representation, $P(U, u)$, its *analytic center* is the unique solution to the following problem (sometimes called *logarithmic barrier*):

$$\{x^*(U, u)\} = \operatorname{argmax} \left\{ \sum_{i=1}^{m} \ln(b_i - B_i x) : Ax = a, Bx < b \right\}.$$

Figure 1 illustrates the effect that redundancies have on the analytic center. In the case shown, the analytic center moves upward and away from the two redundant inequalities. This is due to the presence of the associated logarithmic terms in the objective function.

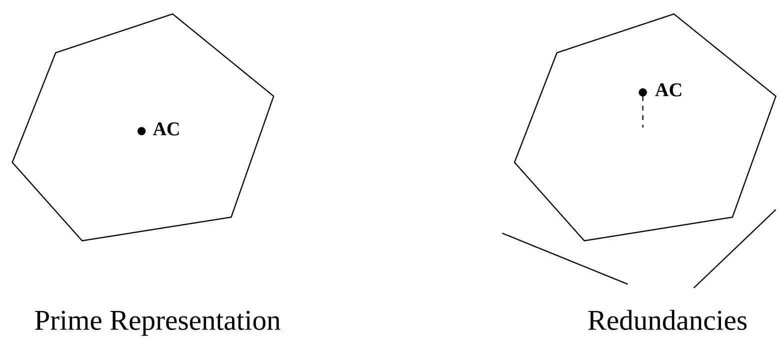

Figure 1: Effect of Redundant Inequalities on Analytic Center

As suggested in the introduction, some subsequent results are motivated by Renegar's method. Consider

$$\max \left\{ \sum_{i=1}^{m} \ln(b_i - B_i x) + \hat{\mu} \ln(K - cx) : Ax = a, Bx < b, cx < K \right\},$$

where the objective cut, $cx \leq K$, is a redundant inequality. Because the objective function is strictly concave for any $\hat{\mu} \geq 0$, this mathematical program has a unique solution, denoted $x(\hat{\mu})$. The analytic center of the feasible region is $x(0)$. Furthermore, the *central path* for this linear program is $\{x(\hat{\mu}) : \hat{\mu} \geq 0\}$, and $\lim_{\hat{\mu}\to\infty} x(\hat{\mu})$ exists and is called the analytic center solution. Notice that as $\hat{\mu}$ increases, say from 1 to 2, that this corresponds to adding the objective cut, $cx \leq K$, twice instead of once. This extra redundant constraint repels the center farther away from the original objective cut.

For different values of $\hat{\mu}$, the optimal solution, $x(\hat{\mu})$, is a *weighted analytic center*. In general, for the positive vector $\omega \in \mathbf{R}^{m+1}$, called a weighting vector, the weighted analytic center is the unique solution to

$$\max \left\{ \sum_{i=1}^{m} \omega_i \ln(b_i - B_i x) + \omega_{m+1} \ln(K - cx) : Ax = a, Bx < b, cx < K \right\}.$$

So, $x(\hat{\mu})$ is a special case with $\omega_i = 1$, $i = 1, 2, \ldots, m$, and $\omega_{m+1} = \hat{\mu}$. We would like to point out that a positive weighting vector could be incorporated throughout the paper, but we do not include such a vector because it offers no new insights.

For notational convenience, the vector of Lagrange multipliers is assumed to be a row vector, so transpose notation is not required. The matrix A^+ denotes the Moore-Penrose generalized inverse of the matrix A (see [6]), and capital letters indicate the diagonal matrix of the corresponding vector (e.g., $S = \text{diag}\{s_1, s_2, \ldots, s_n\}$). Additional notation is introduced as needed.

3 Repelling Inequalities and Limiting Properties of Repelling Paths

Let I be a non-empty index set of inequalities, and consider

$$\{x(\mu, U, u, I)\} = \text{argmax} \left\{ \mu \sum_{i \in I} \ln(b_i - B_i x) + \sum_{i=1}^{m} \ln(b_i - B_i x) : Ax = a, Bx < b \right\}.$$

The inequalities indicated by I are called *repelling* because as μ increases, there is incentive from the objective function to make $\sum_{i \in I} \ln(b_i - B_i x)$ as large as possible. The repelling path, $\{x(\mu, U, u, I) : \mu \geq 0\}$, begins at the analytic center of $P(U, u)$, which is $x(0, U, u, I) = x^*(U, u)$, and traces a curve as $\mu \to \infty$.

To help fix ideas, consider the unit square in $\mathbb{R}^2$, given by the prime representation:

$$(1)\ -x_1 \leq 0, \quad (2)\ -x_2 \leq 0, \quad (3)\ x_1 \leq 1, \quad (4)\ x_2 \leq 1.$$

As shown in figure 2, the analytic center is $x^*(U, u) = (\frac{1}{2}, \frac{1}{2})$. The repelling paths, with various choices of I, begin there and are shown with dashed arrows. We explain some of these.

For $I = \{1\}$, the repelling path is being repelled from the boundary where $x_1 = 0$. We have

$$\begin{aligned}\{x(\mu, U, u, I)\} &= \operatorname{argmax}\left\{(1+\mu)\ln x_1 + \ln x_2 + \ln(1-x_1) + \ln(1-x_2) : 0 < x_1, x_2 < 1\right\} \\ &= \left\{\left(\frac{1+\mu}{2+\mu}, \frac{1}{2}\right)\right\} \to \left(1, \tfrac{1}{2}\right), \text{ as } \mu \to \infty.\end{aligned}$$

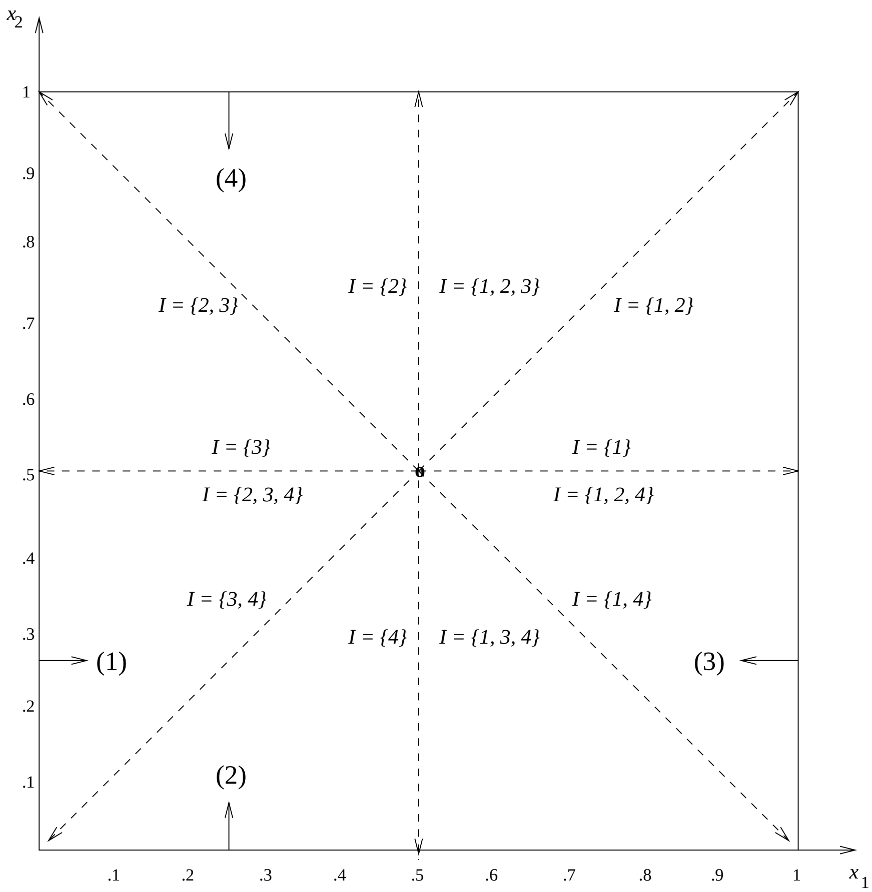

Figure 2: Example Trajectories of Repelling Inequalities

Similarly, when $I = \{4\}$, the path is repelled from the boundary where $x_2 = 1$, and we have

$$\begin{aligned}\{x(\mu, U, u, I)\} &= \operatorname{argmax}\left\{(1+\mu)\ln(1-x_2) + \ln x_1 + \ln x_2 + \ln(1-x_1) : 0 < x_1, x_2 < 1\right\} \\ &= \left\{\left(\frac{1}{2}, \frac{1}{2+\mu}\right)\right\} \to \left(\tfrac{1}{2}, 0\right), \text{ as } \mu \to \infty.\end{aligned}$$

Now, suppose $|I| = 2$. If the repelling constraints correspond to bounds on different variables, the repelling path converges to one of the corners of the square: $x(\mu, U, u, I) \to (\hat{x}_1, \hat{x}_2) \in \{0,1\}^2$, as $\mu \to \infty$. For example, $I = \{1,2\}$ corresponds to $x_1, x_2 \geq 0$, and the limit is $(1,1)$. On the other hand, if the repelling inequalities are the bounds on the same variable, such as $I = \{1,3\}$, the entire repelling path is the analytic center:

$$\begin{aligned}\{x(\mu, U, u, I)\} &= \operatorname{argmax}\{(1+\mu)(\ln x_1 + \ln(1-x_1)) \; + \; \ln x_2 + \ln(1-x_2) : 0 < x_1, x_2 < 1\} \\ &= \left\{\left(\tfrac{1}{2}, \tfrac{1}{2}\right)\right\}, \text{ for all } \mu \geq 0.\end{aligned}$$

Similarly, $x(\mu, U, u, I)$ is invariant with respect to μ if $I = \{1,2,3,4\}$. It is also possible that the repelling paths for different sets of repelling inequalities are equal, which is the case for $I = \{2\}$ and $I' = \{1,2,3\}$.

Notice in the example, that no repelling inequality holds with equality in the limit. Lemma 3.1 shows that this is true in general. In words, the slack values associated with repelling inequalities are uniformly bounded away from zero.

Lemma 3.1 *There exists $\varepsilon > 0$, such that whenever $\{x(\mu^k, U, u, I)\} \to \hat{x}$ for $\mu^k \to \infty$, $\hat{s}_I = b_I - B_I \hat{x} \geq \varepsilon$.*

Proof: Let $x^0 \in P^0(U, u)$ and $s^0 = b - Bx^0 > 0$. Suppose $\mu^k \to \infty$, and let $s^k = b - Bx(\mu^k, U, u, I)$. Since $\mathcal{P}$ is bounded, the sequence $\{(x(\mu^k, U, u, I), s^k)\}$ has a cluster point, $(\hat{x}, \hat{s})$. The optimality of $x(\mu^k, U, u, I)$ implies

$$\mu^k \sum_{i \in I} \ln(s_i^0) + \sum_{i=1}^{m} \ln(s_i^0) \leq \mu^k \sum_{i \in I} \ln(s_i^k) + \sum_{i=1}^{m} \ln(s_i^k).$$

Dividing by μ^k gives,

$$\sum_{i \in I} \ln(s_i^0) + \frac{1}{\mu^k} \sum_{i=1}^{m} \ln(s_i^0) \leq \sum_{i \in I} \ln(s_i^k) + \frac{1}{\mu^k} \sum_{i=1}^{m} \ln(s_i^k).$$

Since the left side converges to $\sum_{i \in I} \ln(s_i^0)$, the right side is bounded below. Suppose $\{s_i^k \in \mathbb{R}_{++}\} \to 0$ for some $i \in I$. Then, the fact that $\mathcal{P}$ is bounded implies

$$\sum_{i \in I} \ln(s_i^k) \to -\infty,$$

which in turn implies

$$\frac{1}{\mu^k} \sum_{i=1}^{m} \ln(s_i^k) \to \infty.$$

This contradicts the boundedness of $\mathcal{P}$, so $\hat{s}_I$ is bounded away from 0. ■

In the special situation where $I = \{i\}$, the slack value, s_i, is not only bounded away from zero, but is also maximized as $\mu \to \infty$. This is the foundation of Renegar's method. With $|I| > 1$, there is no such guarantee of componentwise maximization (see the previous example when $I = \{1, 3\}$).

This section's forthcoming main result says that there is a unique cluster point – i.e., that every repelling path has a limit as $\mu \to \infty$. Neither the classical approach of McLinden [19], the recent results by Güler and Ye [12], nor the target following analysis of Jansen, Roos, Terlaky, and Vial [16, 22] extend to our problem formulation. This is because some of the non-repelling inequalities can have positive slack in the limit, as was seen in the previous example with $I = \{1, 3\}$.

Our development begins with the following lemma, which simply shows that a collection of polytopes is bounded, provided the representations converge.

Lemma 3.2 *Let* $\left\{\left(U^k, u^k\right)\right\} \to (U, u)$ *and* $P(U, u)$ *be bounded and nonempty. Then, there exist a natural number* K *such that,*

$$\mathcal{Q} = \bigcup_{k \geq K} P(U^k, u^k)$$

is bounded.

Proof: We first show that there exists K such that $k \geq K$ implies $P(U^k, u^k)$ is bounded. Suppose not. Then, there exists subsequence $\{k^j\}$ such that $P(U^{k^j}, u^{k^j})$ is unbounded for each j. Subsequently, there exists x^j such that $\|x^j\| = 1$ and $U^{k^j} x^j \leq 0$. Since $\{x^j\}$ is a bounded sequence, we assume for clarity of notation that $\{x^j\} \to x$. We now have that $\{U^{k^j} x^j\} \to Ux \leq 0$, which is a contradiction to the assumption that $P(U, u)$ is bounded. Hence, there exists K such that $k \geq K$ implies $P(U^k, u^k)$ is bounded.

Let $\mathcal{Q}$ be defined with the K in the preceding paragraph and suppose $\mathcal{Q}$ is unbounded. Then there exists $\{x^j \in \mathcal{Q}\}$ such that $\|x^j\| \to \infty$. Set $J^1 = 1$, and let $x^{J^1} \in P(U^{k^1}, u^{k^1})$. Since $P(U^{k^1}, u^{k^1})$ is bounded, there exists $J^2 > J^1$ such that $j \geq J^2$ implies $x^j \notin P(U^{k^1}, u^{k^1})$. Let $x^{J^2} \in P(U^{k^2}, u^{k^2})$. Again, since $P(U^{k^2}, u^{k^2})$ is bounded, there exists $J^3 > J^2$ such that $j \geq J^3$ implies $x^j \notin P(U^{k^2}, u^{k^2})$. Continuing, we form the subsequence $\{x^{J^i} \in P(U^{k^i}, u^{k^i})\}$, which has the following two properties,

$$\|x^{J^i}\| \to \infty, \text{ and } U^{k^i} \frac{x^{J^i}}{\|x^{J^i}\|} \leq \frac{u^{J^i}}{\|x^{J^i}\|}.$$

However, $\left\{\frac{x^{J^i}}{\|x^{J^i}\|}\right\}$ is bounded, and any cluster point, x, has the property that $Ux \leq 0$, which contradict the assumption that $P(U, u)$ is bounded. ■

The next objective is to show that the analytic center is a continuous function over a particular set of data. Unlike the analytic center of a full dimensional polytope, which is analytic in its data, the analytic center of a lower dimensional polytope can have discontinuities with respect to data changes. The following example demonstrates the problem. Consider

$$\left\{(U^k,u^k)\right\} = \left\{ \left[\begin{array}{cc} 1+\frac{1}{k} & 1 \\ -1 & -1 \\ -1 & 0 \\ \hline 0 & -1 \end{array} \right], \left(\begin{array}{c} 1 \\ -1 \\ 0 \\ \hline 0 \end{array} \right) \right\} \to \left\{ \left[\begin{array}{cc} 1 & 1 \\ -1 & -1 \\ \hline -1 & 0 \\ 0 & -1 \end{array} \right], \left(\begin{array}{c} 1 \\ -1 \\ \hline 0 \\ 0 \end{array} \right) \right\} = \{(U,u)\},$$

where the partition indicates the separated data (A^k, a^k, B^k, b^k) and (A, a, B, b). It is easy to check that $x^*(U^k, u^k) = (0, 1)$, for all k (in fact this is the only element in $P(U^k, u^k)$), but $x^*(U, u) = \left(\frac{1}{2}, \frac{1}{2}\right)$.

Before proceeding, we present some notation that is particularly useful. Let $\{(U^k, u^k)\} \to (U, u)$. Furthermore, let (A, a, B, b) and (A^k, a^k, B^k, b^k) be the separated data for (U, u) and (U^k, u^k), respectively. Define $(\delta A^k, \delta a^k, \delta B^k, \delta b^k)$ such that

$$U^k = \left[\begin{array}{c} A + \delta A^k \\ B + \delta B^k \end{array} \right] \text{ and } u^k = \left(\begin{array}{c} a + \delta a^k \\ b + \delta b^k \end{array} \right). \tag{1}$$

This notation represents (U^k, u^k) with the partition of the separated data for (U, u). Because this notation does not provide information about the separated data for (U^k, u^k), it is ***not*** necessarily the case that (A^k, a^k, B^k, b^k) is same as $(A + \delta A^k, a + \delta a^k, B + \delta B^k, b + \delta b^k)$. To illustrate, in the previous example we have

$$A = \left[\begin{array}{cc} 1 & 1 \\ -1 & -1 \end{array} \right], \; A + \delta A^k = \left[\begin{array}{cc} 1+\frac{1}{k} & 1 \\ -1 & -1 \end{array} \right], \text{ and } A^k = \left[\begin{array}{cc} 1+\frac{1}{k} & 1 \\ -1 & -1 \\ -1 & 0 \end{array} \right].$$

The next lemma is found in [6] and provides us with a collection of data perturbations that ensure the continuity of the analytic center. The Lemma essentially states that a sequence of Moore-Penrose generalized inverses converges to the Moore-Penrose inverse of the limiting matrix if, and only if, rank is preserved.

Lemma 3.3 *If* $\{(U^k, u^k)\} \to (U, u)$, $\left\{(A + \delta A^k)^+\right\} \to A^+$ *if, and only if,* $\operatorname{rank}(A + \delta A^k) = \operatorname{rank}(A)$ *for all sufficiently large* k.

In light of Lemma 3.3, we say that $\{(U^k, u^k)\} \to (U, u)$ is *equality rank preserving* if $\operatorname{rank}(A + \delta A^k) = \operatorname{rank}(A)$, for all sufficiently large k. The sequence in the last example was not equality rank preserving because

$$2 = \operatorname{rank}(A + \delta A^k) = \operatorname{rank}\left(\left[\begin{array}{cc} 1+\frac{1}{k} & 1 \\ -1 & -1 \end{array} \right]\right) \neq \operatorname{rank}\left(\left[\begin{array}{cc} 1 & 1 \\ -1 & -1 \end{array} \right]\right) = \operatorname{rank}(A) = 1.$$

To characterize these equality rank preserving sequences, we define the following collection,

$$\mathcal{E}(U,u) = \left\{ \left\{ (U^k, u^k) \right\} \to (U,u) : (U^k,u^k) \text{ is equality rank preserving } (U,u) \right\}.$$

The following Lemma shows that equality rank preserving sequences retain the inequalities of the unperturbed system.

Lemma 3.4 *Let* $\{(U^k,u^k)\} \in \mathcal{E}(U,u)$. *Then, there exists* $\{x^k \in P(U^k,u^k)\} \to x \in P(U,u)$ *such that* $(B+\delta B^k)x^k < b + \delta b^k$ *and* $Bx < b$.

Proof: Let $\left\{ \tilde{x}^k \in P(U^k,u^k) \right\} \to \tilde{x} \in P(U,u)$. Then,

$$\begin{aligned} \alpha^k \equiv (A+\delta A^k)\tilde{x}^k &\le a + \delta a^k \\ (B+\delta B^k)\tilde{x}^k &\le b + \delta b^k. \end{aligned}$$

Since $\left\{ (A+\delta A^k)\tilde{x}^k \right\} \to A\tilde{x} = a$, $\{\alpha^k\} \to a$. Let $x^0 \in P^0(U,u)$. Then, $x^0 = A^+ a + q^0$ for some $q^0 \in \mathcal{N}(A)$. Set

$$x^k = (A+\delta A^k)^+ \alpha^k + q^0 - (A+\delta A^k)^+(A+\delta A^k)q^0.$$

Then,

$$\begin{aligned} (A+\delta A^k)x^k &= (A+\delta A^k)(A+\delta A^k)^+\alpha^k + (A+\delta A^k)q^0 - (A+\delta A^k)(A+\delta A^k)^+(A+\delta A^k)q^0 \\ &= \alpha^k + (A+\delta A^k)q^0 - (A+\delta A^k)q^0 \\ &= \alpha^k \\ &\le a + \delta a^k. \end{aligned}$$

So, all that is left is to show that $(b+\delta b^k) - (B+\delta B^k)x^k > 0$, for large k. First,

$$\begin{aligned} &(b+\delta b^k) - (B+\delta B^k)x^k \\ &= (b+\delta b^k) - \left((B+\delta B^k)(A+\delta A^k)^+\alpha^k + (B+\delta B^k)q^0 - (B+\delta B^k)(A+\delta A^k)^+(A+\delta A^k)q^0 \right) \\ &= \left(b - (B+\delta B^k)\left((A+\delta A^k)^+\alpha^k + q^0 \right) + \delta b^k \right) + (B+\delta B^k)(A+\delta A^k)^+(A+\delta A^k)q^0. \end{aligned}$$

Since $\{(U^k,u^k)\} \in \mathcal{E}(U,u)$,

$$\left\{ b - (B+\delta B^k)\left((A+\delta A^k)^+\alpha^k + q^0 \right) + \delta b^k \right\} \to b - B(A^+a + q^0) = b - Bx^0 > 0,$$

and because $q^0 \in \mathcal{N}(A)$,

$$\left\{ (B+\delta B^k)(A+\delta A^k)^+(A+\delta A^k)q^0 \right\} \to 0.$$

Hence, $(b+\delta b^k) - (B+\delta B^k)x^k > 0$ for sufficiently large k, and the proof is complete. ■

The next Lemma proves that the analytic center is a continuous function over $\mathcal{E}(U,u)$. Similar continuity results are found in [13, 20, 24], but unlike our result, those results do not allow either matrix perturbations, or the dimension of the polytope to change.

Theorem 3.1 *The analytic center, $x^*(U,u)$, is continuous over $\mathcal{E}(U,u)$.*

Proof: Let $(U^k,u^k) \in \mathcal{E}(U,u)$, and let (A,a,B,b) be the separated data for (U,u). Furthermore, let $(\delta A^k, \delta a^k, \delta B^k, \delta b^k)$ be as in (1). From Lemma 3.4, there exists $\{\tilde{x}^k \in P(U^k,u^k)\} \to \tilde{x} \in \mathcal{P}(U,u)$ such that $\tilde{s}^k = b + \delta b^k - (B + \delta B^k)\tilde{x}^k > 0$ and $\tilde{s} = b - B\tilde{x} > 0$, for sufficiently large k. This means that for large k, the separated data for $P(U^k,u^k)$ is

$$\begin{aligned}(A+\delta A^k)_{\overline{J}^k} x &= (a+\delta a^k)_{\overline{J}^k}\\ (A+\delta A^k)_{J^k} x &\leq (a+\delta a^k)_{J^k}\\ (B+\delta B^k)x &\leq (b+\delta b^k),\end{aligned}$$

where $(J^k|\overline{J}^k)$ partitions the implied equalities, $Ax = a$, of $P(U,u)$, and the inequalities indicated by J^k are not implied equalities in the perturbed system $P(U^k,u^k)$.

From Lemma 3.2, we have that the sequence of analytic centers, $\{x^*(U^k,u^k)\}$ is bounded. Let $\hat{x}$ be a cluster point of $\{x^*(U^k,u^k)\}$. Define $x^k = x^*(U^k,u^k)$ and the associated slack variables:

$$\begin{aligned}s^k_{J^k} = (a+\delta a^k)_{J^k} - (A+\delta A^k)_{J^k}x^k &\to a_{J^k} - A_{J^k}\hat{x} = 0\\ s^k = (b+\delta b^k) - (B+\delta B^k)x^k &\to b - B\hat{x} = \hat{s}.\end{aligned}$$

For sufficiently large k, (x^k,s^k) is the analytic center of the polytope described by

$$\begin{aligned}(A+\delta A^k)_{\overline{J}^k} x &= (a+\delta a^k)_{\overline{J}^k}\\ (A+\delta A^k)_{J^k} x &= (a+\delta a^k)_{J^k} - s^k_{J^k}\\ (B+\delta B^k)x + s &= (b+\delta b^k)\\ s &\geq 0.\end{aligned}$$

The Lagrange conditions describing (x^k,s^k) are the existence of α^k, β^k, and γ^k such that

$$\begin{aligned}(A+\delta A^k)_{\overline{J}^k} x^k &= (a+\delta a^k)^k_{\overline{J}^k}\\ (A+\delta A^k)_{J^k} x^k &= (a+\delta a^k)^k_{J^k} - s^k_{J^k}\\ (B+\delta B^k)x^k + s^k &= (b+\delta b^k)\\ \alpha^k(A+\delta A^k)_{\overline{J}^k} + \beta^k(A+\delta A^k)_{J^k} + \gamma^k(B+\delta B^k) &= 0\\ \gamma^k S^k &= e^T\\ s^k &> 0.\end{aligned}$$

Since

$$\sum_{i=1}^m \ln(\tilde{s}^k_i) \leq \sum_{i=1}^m \ln(s^k_i) \Rightarrow \sum_{i=1}^m \ln(\tilde{s}_i) \leq \sum_{i=1}^m \ln(\hat{s}_i),$$

an argument analogous to that of the proof of Lemma 3.1 shows that $\hat{s} > 0$. Hence, $\{\gamma^k = e^T(S^k)^{-1}\}$ converges, say to $\hat{\gamma}$. Set

$$\left[\alpha^k, \beta^k\right] = -\gamma^k(B+\delta B^k)(A+\delta A^k)^+.$$

Since $\left\{(U^k, u^k)\right\} \in \mathcal{E}(U,u)$, Lemma 3.3 implies $\left\{\left[\alpha^k, \beta^k\right]\right\} \to -\hat{\gamma}BA^+ \equiv \left[\hat{\alpha}, \hat{\beta}\right]$. Hence, $(\hat{\alpha}, \hat{\beta}, \hat{\gamma})$ satisfy the the optimality conditions for $\hat{x}$ to be the analytic center of $P(U,u)$. Since this analytic center is unique, we have the desired result:

$$\lim_{k\to\infty} x^*(U^k, u^k) = x^*(U,u).$$

■

The following corollary is immediate.

Corollary 3.1.1 *$x^*(U,u)$ is a continuous function of u, keeping U fixed.*

Our efforts are now directed towards describing a region of the original polytope that we later show contains the limit of the repelling path. This region is defined as an optimal set and is in general not contained in the boundary of the polytope. Specifically, define

$$P^I(U,u) = \operatorname{argmax}\left\{\sum_{i\in I} \ln(b_i - B_i x) : x \in P(U,u), B_I x < b_I\right\}.$$

To illustrate, consider the example in Figure 2, where

$$\begin{aligned} P^{\{1,2\}}(U,u) &= \operatorname{argmax}\{\ln x_1 + \ln x_2 : 0 < x_1, x_2 \le 1\} \\ &= \{(1,1)\}; \\ P^{\{1,3\}}(U,u) &= \operatorname{argmax}\{\ln x_1 + \ln(1-x_1) : 0 < x_1 < 1, 0 \le x_2 \le 1\} \\ &= \{(\tfrac{1}{2}, \zeta) : 0 \le \zeta \le 1\}. \end{aligned}$$

The former optimal set is on the boundary of $P(U,u)$, but the latter is not. The following lemma establishes that the slacks associated with the repelling constraints are invariant over $P^I(U,u)$.

Lemma 3.5 *$b_I - B_I x$ is constant in $P^I(U,u)$, say $s_I^*(U,u)$. Conversely, if $x \in P(U,u)$ and $b_I - B_I x = s_I^*(U,u)$, then $x \in P^I(U,u)$.*

Proof: Rewrite the defining problem using slacks as variables:

$$\max\left\{\sum_{i\in I} \ln(s_i) : Ax = a, Bx + s = b, s_I > 0, s_{\overline{I}} \ge 0\right\}.$$

Since the objective function is strictly concave in s_I, its optimal value, $s_I^*(U,u)$, is unique. It follows that $s_I = b_I - B_I x = s_I^*(U,u)$ for all $x \in P^I(U,u)$. The converse follows from the definition of optimality. ■

A consequence of Lemma 3.5 is that,

$$P^I(U,u) = \{x \in P(U,u) : s_I^*(U,u) = b_I - B_I x\}.$$

The next lemma completes the work required to show that $\{x(\mu, U, u, I)\}$ has a unique limit as $\mu \to \infty$. The strategy of proof is divided into two parts, with the first part showing that the cluster points of $\left\{x(\mu^k, U, u, I) : \mu^k \to \infty\right\}$ are contained in $P^I(U,u)$. The second part proves the existence of a unique limit by showing that the only cluster point is the analytic center of $P^I(U,u)$.

Lemma 3.6 *If $\{x(\mu^k, U, u, I)\} \to \hat{x}$, for $\mu^k \to \infty$, then $\hat{x} \in P^I(U,u)$. Further, partition $\overline{I}$ into $\overline{J}$ and J, where $\overline{J}$ indicates the implied equalities in $\overline{I}$ upon fixing $B_I x = b_I - s_I^*(U,u)$. Then,*

$$\{\hat{x}\} = \text{argmax}\left\{\sum_{i \in J} \ln(b_i - B_i x) : x \in P^I(U,u), B_J x > b_J\right\}.$$

Proof: Let $x^k = x(\mu^k, U, u, I) \to \hat{x}$, $s^k = b - Bx^k \to b - B\hat{x} = \hat{s}$, and $s_I^* = s_I^*(U,u)$. We first prove $\hat{x} \in P^I(U,u)$. Let $x^0 \in P^0(U,u)$ and $s^0 = b - Bx^0 > 0$. The optimality of x^k implies for any $\beta \in (0,1)$,

$$\mu^k \sum_{i \in I} \ln(s_i^k) + \sum_{i=1}^m \ln(s_i^k) \geq \mu^k \sum_{i \in I} \ln(s_i^* + \beta(s_i^0 - s_i^*)) + \sum_{i=1}^m \ln(s_i^* + \beta(s_i^0 - s_i^*)).$$

Equivalently,

$$\begin{aligned}
&\sum_{i \in I} \ln(s_i^k) \\
&\quad \geq \sum_{i \in I} \ln(s_i^* + \beta(s_i^0 - s_i^*)) + \frac{1}{\mu^k} \sum_{i=1}^m \ln(s_i^* + \beta(s_i^0 - s_i^*)) - \frac{1}{\mu^k} \sum_{i=1}^m \ln(s_i^k) \\
&\quad = \sum_{i \in I} \ln(s_i^* + \beta(s_i^0 - s_i^*)) + \frac{1}{\mu^k} \sum_{i=1}^m \ln(s_i^* + \beta(s_i^0 - s_i^*)) - \frac{1}{\mu^k} \sum_{i \in \sigma(\hat{s})} \ln(s_i^k) - \frac{1}{\mu^k} \sum_{i \in \bar{\sigma}(\hat{s})} \ln(s_i^k).
\end{aligned}$$

As $k \to \infty$,

$$\begin{aligned}
\frac{1}{\mu^k} \sum_{i=1}^m \ln(s_i^* + \beta(s_i^0 - s_i^*)) &\to 0 \text{ and} \\
\frac{1}{\mu^k} \sum_{i \in \sigma(\hat{s})} \ln(s_i^k) &\to 0.
\end{aligned}$$

Furthermore, as $k \to \infty$, $\{s_{\bar{\sigma}(\hat{s})}\} \to 0$, which implies $\frac{1}{\mu^k} \sum_{i \in \bar{\sigma}(\hat{s})} \ln(s_i^k) < 0$ for sufficiently large k. Hence,

$$\sum_{i \in I} \ln \hat{s}_i \geq \sum_{i \in I} \ln(s_i^* + \beta(s_i^0 - s_i^*)),$$

for any $\beta \in (0,1)$. However, upon allowing $\beta \to 0$ (recall $s_I^* > 0$) we have,

$$\sum_{i \in I} \ln \hat{s}_i \geq \sum_{i \in I} \ln s_i^*,$$

and the optimality of s_I^* implies $\hat{x} \in P^I(U,u)$.

Let $s(\mu) = b - Bx(\mu, U, u, I)$. So far we have,

$$\begin{aligned} \lim_{\mu \to \infty} s_I(\mu) &= s_I^* \text{ and} \\ \lim_{\mu \to \infty} s_{\overline{J}}(\mu) &= 0. \end{aligned}$$

Since $(x(\mu, U, u, I), s_J(\mu))$ is the analytic center of the polytope described by

$$\begin{aligned} Ax &= a \\ B_I x &= b_I - s_I(\mu) \\ B_{\overline{J}} x &= b_{\overline{J}} - s_{\overline{J}}(\mu) \\ B_J x + s_J &= b_J \\ s_J &\geq 0, \end{aligned}$$

Corollary 3.1.1 not only implies the existence of

$$\lim_{\mu \to \infty} \left(x(\mu, U, u, I), s_J(\mu) \right),$$

but also shows that this limit is the analytic center of

$$\left\{ x : x \in P^I(U,u), B_J x < b_J \right\}.$$

■

Lemma 3.6 has an interesting interpretation. First, the limit maximizes the logarithmic barrier function over the repelling inequalities, defining P^I. Second, the limit is the analytic center of P^I – i.e., it maximizes the logarithmic barrier function over the non-repelling inequalities to decide the specific limit. For example, consider $I = \{4\}$ in figure 2 – i.e., the repelling inequality is $x_2 \leq 1$. The limit first maximizes $\ln(s_4)$, but the entire line segment, $\{[0, \zeta] : \zeta \in [0,1]\}$, does this with $s_4^* = 1$. The second criterion is to find the analytic center of the set of feasible elements satisfying $s_4 = s_4^*$. This second criterion decides that the limit is the unique point, $\left(0, \frac{1}{2}\right)$.

We conclude with the main result of this section, which shows that a repelling path has unique limit as μ goes to zero or infinity. Furthermore, the limit as μ goes to zero is the analytic center of $P(U,u)$, and the limit as μ goes to infinity is the analytic center of $P^I(U,u)$.

Theorem 3.2 *Let $P(U,u)$ be a representation of a polytope, and $\hat{x}(U,u,I)$ be the analytic center of $P^I(U,u)$. Then,*

$$\begin{aligned}\lim_{\mu\to\infty} x(\mu,U,u,I) &= \hat{x}(U,u,I) \quad \textit{and} \\ \lim_{\mu\to 0^+} x(\mu,U,u,I) &= x^*(U,u).\end{aligned}$$

Proof: The first equality follows directly from Lemma 3.6. Let $\{\mu^k \in \mathbb{R}_{++}\} \to 0$ and set $s(\mu^k) = b - Bx(\mu^k,U,u,I)$. Furthermore, let $(\bar{x},\bar{s})$ be a cluster point of $\{x(\mu^k,U,u,I), s(\mu^k)\}$. The necessary and sufficient Lagrange conditions for $(x(\mu^k,U,u,I), s(\mu^k))$ are the existence of $\alpha(\mu^k)$ and $\beta(\mu^k)$ such that

$$\begin{aligned}Ax(\mu^k,U,u,I) &= a \\ Bx(\mu^k,U,u,I) &< b \\ \alpha(\mu^k)A + \beta(\mu^k)B &= 0 \\ \beta_I(\mu^k)S_I(\mu^k) &= (\mu^k+1)e^T \\ \beta_{\overline{I}}(\mu^k)S_{\overline{I}}(\mu^k) &= e^T.\end{aligned}$$

Because $\mu^k \sum_{i\in I} \ln(\tilde{s}_I) + \sum_{i=1}^{m} \ln(\tilde{s}_i) \le \mu^k \sum_{i\in I} \ln(s_i(\mu^k)) + \sum_{i=1}^{m} \ln(s_i(\mu^k))$ for any $\tilde{s} \in P^0(U,u)$, we have that $\bar{s} > 0$ by the same argument found in the proof of Lemma 3.1. So, $\{\beta(\mu^k)\}$ and $\{-\beta(\mu)BA^+\}$ converge, say to $\bar{\beta}$ and $\bar{\alpha}$, respectively. We now have

$$\begin{aligned}A\bar{x} &= a \\ B\bar{x} &< b \\ \bar{\alpha}A + \bar{\beta}B &= 0 \\ \bar{\beta}_I\bar{S}_I &= e^T \\ \bar{\beta}_{\overline{I}}\bar{S}_{\overline{I}} &= e^T,\end{aligned}$$

which are the necessary and sufficient Lagrange conditions describing $x^*(U,u)$. Hence, any cluster point of $\{x(\mu^k,U,u,I)\}$ is $x^*(U,u)$, and the second equality holds. ■

Henceforth, we denote the analytic center of $P^I(U,u)$ by $\hat{x}(U,u,I)$, and we define $\hat{s}(U,u,I) = b - B\hat{x}(U,u,I)$. When there is no confusion, we simplify this notation to $\hat{x}$ and $\hat{s}$, respectively.

4 Transient Behavior of Repelling Paths

We first show that $x(\mu,U,u,I)$ is continuous over $\mathbb{R}_{++} \times \mathcal{E}(U,u)$, which has the immediate corollary that $x(\mu,U,u,I)$ is a continuous function of μ, keeping (U,u,I) fixed.

Theorem 4.1 *For any I, $x(\mu,U,u,I)$ is continuous over $\mathbb{R}_{++} \times \mathcal{E}(U,u)$.*

Proof: Let $\left\{(\mu^k, U^k, u^k) \in \mathbb{R}_{++} \times \mathcal{E}(U, u)\right\} \to (\mu, U, u)$, where $\mu > 0$. Let $(\delta A^k, \delta a^k, \delta B^k, \delta b^k)$ be as in (1) and let $(J^k|\overline{J}^k)$ partition of the implied equalities $Ax = a$ as in Theorem 3.1. We have by definition that $x(\mu^k, U^k, u^k, I)$ maximizes

$$\mu^k \sum_I \ln(s_i) + \sum_{i=1}^m \ln(s_i) + \sum_{i \in J^k} \ln(s_i)$$

subject to

$$\begin{aligned} (A + \delta A^k)_{\overline{J}^k} x^k &= (a + \delta a^k)^k_{\overline{J}^k}, \\ (A + \delta A^k)_{J^k} x + s_{J^k} &= (a + \delta a^k)^k_{J^k}, \\ (B + \delta B^k) x^k + s^k &= (b + \delta b^k), \\ s &> 0, \\ s_{J^k} &> 0. \end{aligned}$$

Let

$$\begin{aligned} x^k &= x(\mu^k, U^k, u^k, I), \\ s^k_{J^k} &= s_{J^k}(\mu^k, U^k, u^k, I) = (a + \delta a^k)_{J^k} - (A + \delta A^k)_{J^k} x(\mu^k, U^k, u^k, I), \text{ and} \\ s^k &= s(\mu^k, U^k, u^k, I) = (b + \delta b^k) - (B + \delta B^k) x(\mu^k, U^k, u^k, I). \end{aligned}$$

Then, for sufficiently large k, (x^k, s^k) is the analytic center of the polytope described by

$$\begin{aligned} (A + \delta A^k)_{\overline{J}^k} x &= (a + \delta a^k)^k_{\overline{J}^k}, \\ (A + \delta A^k)_{J^k} x &= (a + \delta a^k)^k_{J^k} - s^k_{J^k}, \\ (B + \delta B^k) x + s &= (b + \delta b^k) \\ s &\geq 0. \end{aligned}$$

Hence, the Lagrange conditions for (x^k, s^k) are

$$\begin{aligned} (A + \delta A^k)_{\overline{J}^k} x^k &= (a + \delta a^k)^k_{\overline{J}^k}, \\ (A + \delta A^k)_{J^k} x^k &= (a + \delta a^k)^k_{J^k} - s^k_{J^k}, \\ (B + \delta B^k) x^k + s^k &= (b + \delta b^k), \\ \alpha^k (A + \delta A^k)_{\overline{J}^k} + \beta^k (A + \delta A^k)_{J^k} + \gamma^k (B + \delta B^k) &= 0, \\ \gamma^k S^k &= (\mu + 1) e^T \\ s^k &> 0. \end{aligned}$$

So, an argument like that found in the proof of Theorem 3.1 shows that $\left\{s(\mu^k, U^k, u^k, I)\right\} \to s(\mu, U, u, I)$. ■

Corollary 4.1.1 *The repelling path, $\{x(\mu, U, u, I) : \mu \geq 0\}$ is a continuous function with respect to μ.*

Proof: When $\mu > 0$, the result follows directly from Theorem 4.1. The case when $\mu = 0$ is a direct consequence of Theorems 3.1 and 3.2. ■

The similarity of the proofs for Theorems 3.1 and 4.1 is a strength of the technique of proof. The standard method of proof used to show that the central path is analytic with respect to (μ, b, c), where b and c are the right-hand side and cost coefficient vectors of a standard form linear program, uses the implicit function theorem [13]. However, the use of the implicit function theorem does not always provide an equivalent analytic property for the analytic center solution, which is generally discontinuous with respect to data perturbations (see [14] for details). While the sensitivity analysis type proofs presented for Theorems 3.1 and 4.1 establish only continuity, the same method of proof works for both the repelling path and its limits. The differential properties of a repelling path and its limits are currently unexplored.

The second result of this section shows that a repelling path either degenerates to a single point or is a simple curve – i.e. the repelling path does not cross itself.

Theorem 4.2 *Either $x(\mu^1, U, u, I) = x(\mu^2, U, u, I)$ for all μ^1 and μ^2, or $x(\mu^1, U, u, I) \neq x(\mu^2, U, u, I)$ for any $\mu^1 \neq \mu^2$.*

Proof: Suppose that $\mu^1 \neq \mu^2$ and $x(\mu^1, U, u, I) = x(\mu^2, U, u, I)$. Setting $s^1 = b - Bx(\mu^1, U, u, I)$ and $s^2 = b - Bx(\mu^2, U, u, I)$, the Lagrange conditions are

$$\begin{aligned} Ax(\mu^1, U, u, I) &= a & \qquad Ax(\mu^2, U, u, I) &= a \\ Bx(\mu^1, U, u, I) &< b & \qquad Bx(\mu^2, U, u, I) &< b \\ \alpha^1 A + \beta^1 B &= 0 & \qquad \alpha^2 A + \beta^2 B &= 0 \\ \beta^1_I S^1_I &= (\mu^1 + 1)e^T & \qquad \beta^2_I S^2_I &= (\mu^2 + 1)e^T \\ \beta^1_{\bar{I}} S^1_{\bar{I}} &= e^T & \qquad \beta^2_{\bar{I}} S^2_{\bar{I}} &= e^T. \end{aligned}$$

Let $\mu > 0$ and θ be such that $\mu = (1-\theta)\mu^1 + \theta\mu^2$. Since $x(\mu^1, U, u, I) = x(\mu^2, U, u, I)$ implies $s^1 = s^2$, the corresponding linear combinations of the Lagrange multipliers, $(1-\theta)\alpha^1 + \theta\alpha^2$ and $(1-\theta)\beta^1 + \theta\beta^2$, satisfy the Lagrange conditions for μ with the common primal values. ■

5 The Prime Analytic Center

Recall that a representation is prime if it does not contain any redundancy and is semi-prime if there are no redundant inequalities. When the dimension of $\mathcal{P}$ is full, prime and semi-prime are equivalent, and representations differ only by scale and the ordering of the inequalities (see Schrijver [23]). Row scaling does not affect the analytic center because

$$\max\left\{\sum_i \ln(r_i(b_i - B_i x)) : Ax = a, r_i B_i x < r_i b_i \text{ for } i = 1, \dots m\right\}$$
$$= \sum_i \ln r_i + \max\left\{\sum_i \ln(b_i - B_i x) : Ax = a, B_i x < b_i \text{ for } i = 1, \dots m\right\},$$

where $r_i > 0$ is the scale of the i-th inequality. This leads directly to the following:

Theorem 5.1 *The analytic center of a full-dimensional polytope is the same for each prime representation.*

The situation is more complicated when $\dim(\mathcal{P}) < n$. To illustrate, consider four prime representations of the diagonal of a unit square:

$$\begin{aligned}
P_1 &= \{x \in \mathbb{R}^2 : x_1 - x_2 = 0,\ -x_1 \le 0,\ x_1 \le 1\};\\
P_2 &= \{x \in \mathbb{R}^2 : x_1 - x_2 = 0,\ -x_1 \le 0,\ x_2 \le 1\};\\
P_3 &= \{x \in \mathbb{R}^2 : x_1 - x_2 = 0,\ -x_2 \le 0,\ x_2 \le 1\};\\
P_4 &= \{x \in \mathbb{R}^2 : x_1 - x_2 = 0,\ -x_2 \le 0,\ x_1 \le 1\}.
\end{aligned}$$

Putting these into matrix terms, we have $[A^k | a^k] = [1\ -1\,|\,0]$ for $k = 1, 2, 3, 4$, but the inequalities differ:

$$\begin{aligned}
[B^1\,|\,b^1] &= \left[\begin{array}{rr|r} -1 & 0 & 0\\ 1 & 0 & 1 \end{array}\right];\\
[B^2\,|\,b^2] &= \left[\begin{array}{rr|r} -1 & 0 & 0\\ 0 & 1 & 1 \end{array}\right];\\
[B^3\,|\,b^3] &= \left[\begin{array}{rr|r} 0 & -1 & 0\\ 0 & 1 & 1 \end{array}\right];\\
[B^4\,|\,b^4] &= \left[\begin{array}{rr|r} 0 & -1 & 0\\ 1 & 0 & 1 \end{array}\right].
\end{aligned}$$

In general, with one set of variables dependent on the others (viz., $x_1 = x_2$), we can have different inequalities represent the same set. Despite this, we shall prove that the analytic center of a polytope is the same for all semi-prime representations – i.e., Theorem 5.1 does extend to polytopes of lower dimension. The following result is used to prove the extension of Theorem 5.1 and appears to be new.

Lemma 5.1 *Suppose $P(U, u)$ and $P(U', u')$ are two representations of $\mathcal{P}$ with corresponding separated data (A, a, B, b) and (A', a', B', b'). Then,*

1. $\mathcal{N}(A) = \mathcal{N}(A')$.

2. $A^+A = A'^+A'$.

3. $A^+a = A'^+a'$.

Proof: Define $\mathcal{A} = \{x : Ax = a\}$ and $\mathcal{A}' = \{x : A'x = a'\}$. We begin by showing $\mathcal{A} = \mathcal{A}'$. Let $x^0 \in P^0(U,u)$ and consider $x \in \mathcal{A}$. Then, there exists $\alpha \in (0,1)$ such that $Av = a$ and $Bv < b$ for $v = x^0 + \alpha(x - x^0)$. This implies $v \in \mathcal{P}$, and $A'v = a'$. Since $A'x^0 = a'$, it follows that $A'x = a'$, which shows $x \in \mathcal{A}'$. Hence, $\mathcal{A} \subseteq \mathcal{A}'$. Similarly, $\mathcal{A}' \subseteq \mathcal{A}$, so $\mathcal{A} = \mathcal{A}'$. We now have the first proposition:

$$\mathcal{N}(A) = \mathcal{A} - \{x^0\} = \mathcal{A}' - \{x^0\} = \mathcal{N}(A').$$

The orthogonal projection operators onto $\mathcal{N}(A)$ and $\mathcal{N}(A')$ are $I - A^+A$ and $I - A'^+A'$, respectively. Since $\mathcal{N}(A) = \mathcal{N}(A')$ and the projection operator onto this space is unique, we have $A^+A = A'^+A$. From this we obtain the third simply by substitution: $A^+a = A^+Ax^0 = A'^+A'x^0 = A'^+a'$. ■

To extend Theorem 5.1, we also need the following result of Bayer and Lagarias [3]:

Lemma 5.2 *Consider the affine transformation $Rx + \tilde{x}$, where R is non-singular. Then, if x^* is the analytic center of $\{x : Bx \leq b\}$, $Rx^* + \tilde{x}$ is the analytic center of $\{w : BR^{-1}w \leq b + BR^{-1}\tilde{x}\}$.*

We are now ready to establish the main result of this section.

Theorem 5.2 *The analytic center for any semi-prime representation of a polytope is the same.*

Proof: Let $P(U,u)$ and $P(U',u')$ be two semi-prime representations of $\mathcal{P}$ with corresponding separated data (A,a,B,b) and (A',a',B',b'). From lemma 5.1, $\mathcal{N}(A) = \mathcal{N}(A')$, and we denote this space by $\mathcal{V}$. Also, $A^+a = A'^+a'$, and this vector is denoted by v. Let $q = \dim(\mathcal{V})$. If $q = n$, we have the full dimensional case covered by Theorem 5.1.

Suppose $q < n$. Then, there exists a non-singular $R \in \mathbb{R}^{n \times n}$ such that for any $w \in \mathcal{V}$,

$$Rw = \begin{pmatrix} u \\ 0 \end{pmatrix},$$

where $u \in \mathbb{R}^q$. So, $Ax = a \Leftrightarrow x = v + R^{-1}\begin{pmatrix} u \\ 0 \end{pmatrix}$, and

$$x \in \mathcal{P} \Leftrightarrow BR^{-1}\begin{pmatrix} u \\ 0 \end{pmatrix} \leq b - Bv \Leftrightarrow [BR^{-1}]_{J_u} u \leq b - Bv,$$

where J_u is the index set associated with u and the set subscript indicates columns of BR^{-1}. Similarly, $x \in \mathcal{P} \Leftrightarrow [B'R^{-1}]_{J_u} u \leq b' - B'v$. Now consider the following full q dimensional polytope:

$$\begin{aligned} \mathcal{P}' &= \{u \in \mathbb{R}^q : [BR^{-1}]_{J_u} u \leq b - Bv\} \\ &= \{u \in \mathbb{R}^q : [B'R^{-1}]_{J_u} u \leq b' - B'v\}. \end{aligned}$$

These inequalities cannot contain a redundancy since that would violate the assumption that there are no redundant inequalities in our original representations of $\mathcal{P}$. Therefore, Theorem 5.1 implies that both representations have the same analytic center:

$$u^*([BR^{-1}]_{J_u}, b - Bv) = u^*([B'R^{-1}]_{J_u}, b' - B'v)) \stackrel{\text{def}}{=} u^*.$$

Lemma 5.2 implies that $R^{-1}\begin{pmatrix} u^* \\ 0 \end{pmatrix}$ is the analytic center of both representations:

$$x^*(U, u) = R^{-1}\begin{pmatrix} u^* \\ 0 \end{pmatrix} = x^*(U', u').$$

■

To illustrate the proof of Theorem 5.2, consider the previous example. We have $v = 0$ and $\mathcal{V} = \{x \in \mathbb{R}^2 : x_1 = x_2\}$. Let R be the rotation matrix such that $Rx = (u, 0)^T$ for any $x \in \mathcal{V}$. The matrix, R, and its inverse are:

$$R = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \text{ and } R^{-1} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

In all four different representations,

$$\left[B^i R^{-1}\right]_{J_u} = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix} \text{ and } b - B^i v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \text{ for } i = 1, 2, 3, 4.$$

Thus, the reduced full dimensional polytope is simply $\{u : 0 \leq u \leq \sqrt{2}\}$, and the prime analytic center of this polytope is $u^* = \frac{1}{\sqrt{2}}$. Mapping this back to $\mathcal{P}$, we have the analytic center:

$$x^* = R^{-1}\begin{pmatrix} u^* \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix},$$

which is the same for all prime representations.

This example shows the importance of transforming $\mathcal{V}$ so that it appears full dimensional. We chose to "zero-out" the last $n - \dim(\mathcal{V})$ variables, but any collection of $n - \dim(\mathcal{V})$

variables whose columns in A are linearly independent could have been eliminated. What matters is that the transformation is invertible.

In light of Theorem 5.2, we refer to *the* prime analytic center, which gives us a point that is independent from any particular semi-prime representation of $\mathcal{P}$. Formally, we introduce the following:

Definition. The *prime analytic center* of $\mathcal{P}$ is the unique analytic center obtained from any semi-prime representation, $P(U, u)$.

Since the prime analytic center is independent of a particular choice of (U, u), we denote it as a function of the polytope, $x(\mathcal{P})$. The following corollary to Theorem 5.2 shows that the prime analytic center is the limit of a repelling path.

Corollary 5.2.1 *Let $P(U, u)$ be a representation of $\mathcal{P}$ with semi-prime representation $P(U_I, u_I)$. Then,*

$$\lim_{\mu \to \infty} x(\mu, U, u, I) = x(\mathcal{P}).$$

Proof: The definition of P^I implies

$$P^I(U, u) = P^I(U_I, u_I).$$

From Theorem 5.2, $P^I(U_I, u_I) = \{x(\mathcal{P})\}$. Theorem 3.2 now implies

$$\lim_{\mu \to \infty} x(\mu, U, u, I) = \hat{x}(U, u, I) \in P^I(U, u) = P^I(U_I, u_I) = \{x(\mathcal{P})\}.$$

■

The corollary says that if the repelling constraints comprise a semi-prime representation, the repelling path terminates at the prime analytic center of $\mathcal{P}$. Interestingly, once the implied equalities are separated from the inequalities, a semi-prime representation can be found by removing the redundant inequalities in any fashion, see Telgen [29]. So, once all the redundant inequalities are identified, the indices of the remaining inequalities may be used to form I.

We now establish a continuity result for the prime analytic center. Unfortunately, even with equality rank preserving sequences, the prime analytic center is not a continuous function over semi-prime representations. This follows because it is possible to have semi-prime representations, $P(U^k, u^k) = \mathcal{P}^k$, where $\{(U^k, u^k)\} \to (U, u)$, but have that $P(U, u)$ is not a prime representation of $\mathcal{P}$. An example is shown in figure 3. The unit square is perturbed by replacing the constraint $x_1 \le 1$ by the two constraints $x_1 + \frac{1}{k}x_2 \le 1 + \frac{1}{k}$ and $x_1 - \frac{1}{k}x_2 \le 1$. The sequence of prime representations converges to a non-prime representation of the unit square, due to the redundancy of these inequalities. In this case the analytic center moves to $(\frac{1}{3}, \frac{1}{2})$, skewed by having a weight of 2 on the upper bound, $x_1 \le 1$.

We want to restrict perturbing $\mathcal{P}$ such that this kind of pathology does not happen. We say $\{\mathcal{P}^k\}$ is a *semi-prime sequence* if there exist semi-prime representations $P(U^k, u^k) = \mathcal{P}^k$

such that $\{(U^k, u^k)\} \to (U, u)$, and $P(U, u)$ is a semi-prime representation of $\mathcal{P}$. A function, $f(\mathcal{P})$ is *semi-prime continuous* if

$$\lim_{\mathcal{P}^k \to \mathcal{P}} f(\mathcal{P}^k) = f(\mathcal{P}) \text{ for all semi-prime sequences, } \{\mathcal{P}^k\} \to \mathcal{P}.$$

Theorem 5.3 *Let (U, u) be a semi-prime representation of $\mathcal{P}$. Then $x(\mathcal{P})$ is semi-prime continuous over $\mathcal{E}(U, u)$.*

Proof: Let $\left\{(U^k, u^k)\right\}$ be a semi-prime sequence in $\mathcal{E}(U, u)$. Using Theorem 3.1, we have

$$\{x(\mathcal{P}^k)\} = \{x^*(U^k, u^k)\} \to x^*(U, u) = x(\mathcal{P}),$$

and the result is proven. ■

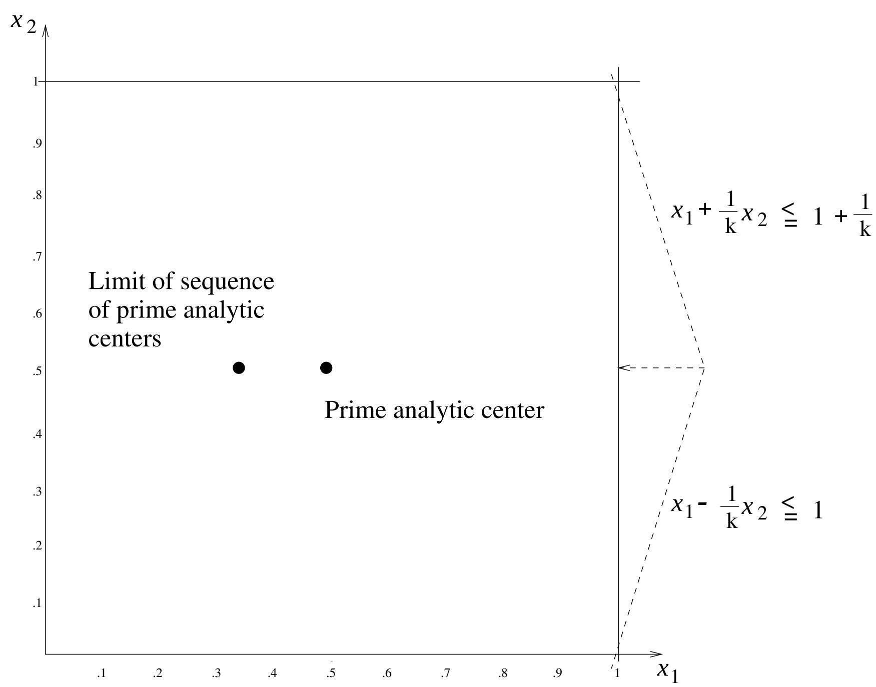

Figure 3: A Sequence of Prime Representations with Non-Prime Limit

In this section we have shown that there is a unique analytic center for all semi-prime representations of a polytope. This prime analytic center is tied more closely to the geometry of a polytope than is a non-prime analytic center because it does not depend on redundant inequalities. Furthermore, the prime analytic center is continuous over semi-prime, equality rank preserving sequences.

6 Relative Effects of Repelling Inequalities

In this section we investigate how repelling inequalities interact. Specifically, collections of repelling inequalities are characterized by whether or not they have common objectives. We begin with an inquiry into collections with conflicting objectives and conclude with a discussion of collections that have common objectives.

As mentioned in Section 3, the repelling path can be the same for different collections of repelling inequalities. For example, in Figure 2 the repelling paths corresponding to $I = \{4\}$ and $I = \{1, 3, 4\}$ are the same. This is because inequalities 1 and 3 *oppose* each other; a concept that is formally defined below.

Definition. The inequalities indexed by J are *opposing inequalities*, relative to the set I, if

$$x(\mu, U, u, I) = x(\mu, U, u, I \cup J), \text{ for all } \mu > 0.$$

The next result shows that the analytic center of a polytope is the prime analytic center only when the inequalities can be partitioned into a prime representation and a corresponding collection of opposing inequalities. Consequently, this result completely defines the set of representations for which the analytic center is the prime analytic center.

Theorem 6.1 *We have $x^*(U, u) = x(\mathcal{P})$ if, and only if, the inequalities, $Bx \le b$, can be partitioned into $B_I x \le b_I$ and $B_{\overline{I}} x \le b_{\overline{I}}$ such that $\{x : Ax = a, B_I x \le b_I\}$ is a semi-prime representation of $\mathcal{P}$, and the inequalities indexed by $\overline{I}$ are opposing inequalities relative to I.*

Proof: We first show that $x^*(U, u) = x(\mathcal{P})$ is a sufficient condition to guarantee the stated partition of the inequalities. From [29], once the implied equalities are separated from $Ux \le u$, a semi-prime representation can be found by removing redundant inequalities in any fashion. Let I be comprised of the indices from the remaining inequalities, so that $\overline{I}$ indexes the redundant inequalities. Since $x(\mu, U, u, I \cup \overline{I})$ solves

$$\max\left\{(\mu + 1)\sum_{i=1}^{m} \ln(b_i - B_i x) : x \in P(U, u)\right\},$$

we have $x(\mu, U, u, I \cup \overline{I}) = x^*(U, u) = x(\mathcal{P})$, for all $\mu > 0$. From Lemma 3.6 and the definition of $x^*(U, u)$ we have

$$\begin{aligned}
\sum_{i \in I} \ln(b_i - B_i x(\mathcal{P})) &\ge \sum_{i \in I} \ln(b_i - B_i x), \text{ for all } x \in P(U, u), \text{ and} \\
\sum_{i=1}^{m} \ln(b_i - B_i x^*(U, u)) &\ge \sum_{i \in I} \ln(b_i - B_i x), \text{ for all } x \in P(U, u).
\end{aligned}$$

Since $x^*(U, u) = x(\mathcal{P})$, these last two inequalities imply that for any $\mu > 0$,

$$\begin{aligned}
&\mu \sum_{i \in I} \ln(b_i - B_i x^*(U, u)) + \sum_{i=1}^{m} \ln(b_i - B_i x^*(U, u)) \\
&\ge \quad \mu \sum_{i \in I} \ln(b_i - B_i x) + \sum_{i=1}^{m} \ln(b_i - B_i x), \text{ for all } x \in P(U, u).
\end{aligned}$$

Since the right-hand side is maximized for $x = x(\mu, U, u, I)$, we have $x(\mu, U, u, I) = x^*(U, u) = x(\mathcal{P})$. This completes the proof of sufficiency because we now have

$$x(\mu, U, u, I) = x^*(U, u) = x(\mathcal{P}) = x(\mu, U, u, I \cup \overline{I}).$$

We now establish necessity. As previously mentioned, since $I \cup \overline{I}$ indexes all the inequalities, $x(\mu, U, u, I \cup \overline{I}) = x^*(U, u)$. Furthermore, using Corollary 5.2.1 and the assumption that the inequalities indexed by $\overline{I}$ are opposing inequalities relative to I,

$$\begin{aligned} x(\mathcal{P}) &= \lim_{\mu \to \infty} x(\mu, U, u, I) \\ &= \lim_{\mu \to \infty} x(\mu, U, u, I \cup \overline{I}) \\ &= \lim_{\mu \to \infty} x^*(U, u) \\ &= x^*(U, u). \end{aligned}$$

■

We now consider collections of repelling inequalities that have common repelling effects. In particular, the inequalities $B_I x \leq b_I$ and $B_J x \leq b_J$ are said to have a *common repelling effect* if $P^I(U, u) = P^J(U, u)$. When this happens, the inequalities indexed by I are said to *ally* with the inequalities indexed by J.

Definition. The inequalities indexed by I *ally pairwise* if, for any pair $i, j \in I$, we have $P^{\{i\}}(U, u) = P^{\{j\}}(U, u)$.

Below, Theorem 6.2 shows that if a collection of inequalities ally pairwise, then subcollections ally collectively.

Theorem 6.2 *If the inequalities indexed by I ally pairwise, then for any subsets I_1 and I_2 of I, $P^{I_1}(U, u) = P^{I_2}(U, u)$.*

Proof: Let I_1 and I_2 be subsets of I. Then, for all $i, j \in I$,

$$P^{\{i\}}(U, u) = \text{argmax}\{b_i - B_i x : x \in P(U, u)\} = \text{argmax}\{b_j - B_j x : x \in P(U, u)\} = P^{\{j\}}(U, u).$$

Using the fact the sum of the individual maxima is less than or equal to the maximum of the sum, we have

$$\text{argmax}\left\{\sum_{i \in I_1} \ln(b_i - B_i x) : x \in P(U, u)\right\} = \text{argmax}\left\{\sum_{i \in I_2} \ln(b_i - B_i x) : x \in P(U, u)\right\},$$

which is equivalent to $P^{I_1}(U, u) = P^{I_2}(U, u)$. ■

In light of Theorem 6.2, we define a collection of inequalities to be *mutual allies* if every subcollection has a common repelling affect.

Definition. The inequalities indexed by I are said to be *mutual allies* if $P^{I_1}(U,u) = P^{I_2}(U,u)$ for any subsets I_1 and I_2 of I.

We can thus restate Theorem 6.2 as follows.

Corollary 6.2.1 *The inequalities indexed by I ally pairwise if, and only if, they are mutual allies.*

The last concept of this section is that of domination. The idea here is that a collection of repelling inequalities can be a super ally to another collection of repelling inequalities and hence dominate the repelling effect.

Definition. The collection I_1 is said to *dominate* the collection I_2 if $P^{I_1}(U,u) \subseteq P^{I_2}(U,u)$.

In figure 2, the set $\{1,2\}$ dominates the set $\{1\}$ because

$$P^{\{1,2\}}(U,u) = \{(1,1)\} \subseteq \{(1,\zeta) : 0 \le \zeta \le 1\} = P^{\{1\}}(U,u).$$

Just because I_2 is a subset of I_1 does not necessarily imply that I_1 dominates I_2. This is seen from the example illustrated in figure 2, where $P^{\{1,2,3,4\}}(U,u) = \{(\frac{1}{2},\frac{1}{2})\}$ and $P^{\{2\}} = \{(\zeta,1) : 0 \le \zeta \le 1\}$. In fact, this example shows that a subset need not ally with the collective repelling effect. However, the last theorem of this section shows that a subset relationship does imply an ally structure provided the individual collections of repelling inequalities are mutual allies.

Theorem 6.3 *Let I_1 and I_2 index two sets of mutual allies. Then, if I_1 dominates I_2, $P^{I_1 \cup I_2}(U,u) = P^{I_1}(U,u)$.*

Proof: Let $\bar{x} \in P^{I_1}(U,u) \subseteq P^{I_2}(U,u)$. Then, for any $i \in I_1 \cup I_2$,

$$b_i - B_i\bar{x} \ge \max\{b_i - B_i x : x \in P(U,u)\}.$$

So,

$$\sum_{i \in I_1 \cup I_2} \ln(b_i - B_i\bar{x}) \ge \max\left\{\sum_{i \in I_1 \cup I_2} \ln(b_i - B_i x) : x \in P(U,u)\right\},$$

which implies $\bar{x} \in P^{I_1 \cup I_2}(U,u)$. Hence, $P^{I_1}(U,u) \subseteq P^{I_1 \cup I_2}(U,u)$.

Let $\tilde{x} \in P^{I_1 \cup I_2}$. Then,

$$\begin{aligned} \sum_{i \in I_1 \cup I_2} \ln(b_i - B_i\bar{x}) &= \sum_{i \in I_1 \cup I_2} \ln(b_i - B_i\tilde{x}) \qquad (2) \\ &= \max\left\{\sum_{i \in I_1 \cup I_2} \ln(b_i - B_i x) : x \in P(U,u)\right\}. \end{aligned}$$

Notice that

$$\sum_{i \in I_1} \ln(b_i - B_i\tilde{x}) > \sum_{i \in I_1} \ln(b_i - B_i\bar{x}).$$

leads to the immediate contradiction that $\bar{x} \notin P^{I_1}(U, u)$. So,

$$\sum_{i \in I_1} \ln(b_i - B_i \tilde{x}) \leq \sum_{i \in I_1} \ln(b_i - B_i \bar{x}). \tag{3}$$

Suppose the inequality in (3) is strict. Then, the equality in (2) implies

$$\sum_{I_2 \setminus I_1} \ln(b_i - B_i \tilde{x}) > \sum_{I_2 \setminus I_1} \ln(b_i - B_i \bar{x}). \tag{4}$$

However, because $\bar{x} \in P^{I_2}(U, u)$ and I_2 is mutually reinforcing, $\bar{x} \in P^{I_2 \setminus I_1}(U, u)$, which is a contradiction to the inequality in (4). Hence,

$$\sum_{i \in I_1} \ln(b_i - B_i \tilde{x}) = \sum_{i \in I_1} \ln(b_i - B_i \bar{x}),$$

and $\tilde{x} \in P^{I_1}(U, u)$. ■

7 Avenues for Further Research

The concept of repelling inequalities and the uniqueness of the analytic center for semi-prime representations comprise the beginning of many possible avenues for fruitful research. Here are some of those.

1. The relative effects of repelling inequalities suggest a new insight into multiple objective linear programming. Different from the approaches of Arbel [2] and Abhyankar, Morin, and Trafalis [1], the results presented here suggest an extension of Renegar's algorithm, considering each objective as repelling. More broadly, this theory can lend insight into the underlying economics of MOLP.

2. There could be some special benefits to having the prime analytic center of the optimality region. For example, the prime analytic center might be a desirable solution from which to conduct parametric analysis.

3. Discovering redundancies during the course of a central path-following algorithm might accelerate convergence by dropping them. If so, an explanation for this might stem from having a path of prime analytic centers.

4. The limit of a repelling path is the analytic center of a polytope whose representation need not be prime. There is, however, always some set of weights for which the point is a weighted prime analytic center (removing redundancies). How do these weights relate to the original system? Is there a connection to weighting multiple objectives if the repelling inequalities are the objective cuts, and the rest of the polytope is given by a prime representation?

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