Stability of the residual free bubble method for bilinear finite elements on rectangular grids

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Abstract

We consider the nature of the stabilizing term arising in the residual free bubble approach for piecewise bilinear functions on rectangular grids. We show, that on the subspace of piecewise linear functions the stabilizing term is identical to that in the streamline diffusion approach. However, on the space of piecewise bilinear functions there is a case in which the stabilizing term is weaker compared to the term used in the streamline diffusion method. In the particular case when the direction of the convection is directed parallel to a diagonal of the quadrilateral, control is lost over the mixed derivatives in the convection-dominated limit.

1 Introduction

The standard Galerkin method with piecewise linear elements enriched with cubic bubbles and eliminating the bubbles results in the streamline-diffusion (or SUPG) method [4]. In a number of papers this approach has been extended to the linearized Navier-Stokes equations in [7] to a more general framework in [1]. The concept of residual free bubbles has been developed to recover the correct asymptotic behaviour of the streamline-diffusion parameter in both the convection-dominated and the diffusion dominated case [13, 3, 2]. It turns out that the stabilizing term arising in the residual free bubble approach is identical to that in the streamline diffusion method in the piecewise linear case but differs from it for polynomials of degree $k > 1$.

Much less is known in the case of quadrilateral elements. Recently, in [12] some type of equivalence between standard Galerkin methods with bubbles and the SUPG method for the Stokes and Navier-Stokes equations has been established by using the approach of reduced discretization [11]. The local spaces of bubble functions which have been used in [12] are rather general. As a first step to analyse the relationship between the stability of the residual free bubble approach and the streamline diffusion method in case of quadrilaterals, the *convection-dominated limit* has been studied for a scalar convection-diffusion equation in [5]. It has been shown that – in general – the arising stabilizing term vanishes if and only if the corresponding streamline diffusion term vanishes. However, it has been also demonstrated that in the exceptional case that the streamline direction is parallel to a diagonal of the quadrilateral the stabilizing term can vanish without vanishing the corresponding streamline diffusion term. In this sense, the stabilizing term resulting from the residual free bubble approach admits less stability compared to the usual streamline diffusion method at least in the limit of vanishing viscosity. Further investigations to explain the nature of the stabilizing term in the residual free bubble approach has been done in [8]. Although the assumption of the vanishing viscosity has been removed the results are still restricted to the triangular case.

Here, we consider the special case of a decomposition of Ω into rectangles where the sides are parallel to the coordinate axes. For this situation we derive the stabilizing term of the residual free bubble approach for the whole range of the diffusion dominated and convection dominated regime in Section 2. First, we show that the stabilizing term of the residual free bubble method vanishes if and only if the stabilizing term commonly used in the SUPG method is zero. Next, we show in Section 3 that the stabilizing term of the streamline diffusion method when taking the standard choice of the streamline diffusion parameters is at least so stable as it is the stabilizing term of the residual free bubble approach. Moreover, the equivalence of both terms will be established on the subspace of piecewise linear elements. Finally, in Section 4 we show that in case of an alignment of the diagonals of the rectangles with the streamline direction, the stabilizing term of the residual free bubble approach is weaker than the corresponding term of the streamline diffusion approach.

2 Residual free bubble approach and stabilized methods

Let us consider the convection-diffusion problem

$$-\varepsilon\Delta u + a \cdot \nabla u = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{1}$$

in a two-dimensional domain Ω. We are primarily interested in the convection-dominated case which is characterized by $0 < \varepsilon \ll 1$. In order to get a first insight into the behaviour of the stabilizing term produced by the residual free bubble approach we assume:

(H1) Ω can be decomposed into rectangles K such that the sides are parallel to the coordinate axes,

(H2) a, f are constant functions.

We start with the Galerkin approximation of (1) in the following space

$$V_h := V_L + V_B = V_L + \bigoplus_K \mathcal{B}(K), \tag{2}$$

where V_L denotes the finite element space of continuous, piecewise bilinear elements and $\mathcal{B}(K)$ is a finite dimensional subspace of $H_0^1(K)$ which will be specified later. Note that due to our assumptions the spaces V_L and V_B are orthogonal with respect to the inner product $(\nabla ., \nabla .)$. Indeed, for all $v_L \in V_L$ and $v_B \in V_B$ it holds

$$\begin{aligned} (\nabla v_L, \nabla v_B) &= \sum_K (\nabla v_L, \nabla v_B)_K, \\ (\nabla v_L, \nabla v_B) &= \sum_K \left\{ \int_{\partial K} \frac{\partial v_L}{\partial n_K} v_B \, d\gamma - \int_K v_B \, \Delta v_L \, dx \right\} = 0, \end{aligned} \tag{3}$$

since $v_B = 0$ on ∂K and $\Delta v_L = 0$ on each K. The Galerkin approach reads as follows:

Find $u_h \in V_h$ such that for all $v_h \in V_h$

$$\varepsilon(\nabla u_h, \nabla v_h) + (a \cdot \nabla u_h, v_h) = (f, v_h). \tag{4}$$

Choosing test functions v_L and b_K from the spaces V_L and $\mathcal{B}(K)$, respectively, we get for $u_h = u_L + u_B$, with $u_L \in V_L$ and $u_B \in V_B$,

$$\varepsilon(\nabla u_L, \nabla v_L) + (a \cdot \nabla u_L, v_L) + (a \cdot \nabla u_B, v_L) = (f, v_L), \tag{5}$$

$$\varepsilon(\nabla u_B, \nabla b_K) + (a \cdot \nabla u_L, b_K) + (a \cdot \nabla u_B, b_K) = (f, b_K), \tag{6}$$

where we used the orthogonality (3) and $b_K \in \mathcal{B}(K) \subset H_0^1(K)$. Now, we reformulate (6) to make it apparent how the linear part drives the bubble part:

$$\varepsilon(\nabla u_B^K, \nabla b_K)_K + (a \cdot \nabla u_B^K, b_K)_K = (f - a \cdot \nabla u_L, b_K)_K, \tag{7}$$

where $u_B^K = u_B\Big|_K$. Let us fix a point $x_K \in K$, for definiteness we choose the barycentre of K. Then, since $u_L \in V_L$ is bilinear on K the representation

$$(a \cdot \nabla u_L)(x) = (a \cdot \nabla u_L)(x_K) + \frac{\partial^2 u_L}{\partial x_1 \partial x_2} a^T \cdot (x - x_K) \tag{8}$$

holds true where $a^T = (a_2, a_1)$. Now we define the bubble space to be the two-dimensional space

$$\mathcal{B}(K) := \mathrm{span}(b_0^K, b_1^K), \tag{9}$$

where b_0^K and b_1^K are the (weak) solutions of the local problems

$$\begin{aligned} -\varepsilon\Delta b_0 + a\cdot\nabla b_0 &= 1 && \text{in } K, \quad b_0 = 0 \quad \text{on } \partial K, && (10)\\ -\varepsilon\Delta b_1 + a\cdot\nabla b_1 &= a^T\cdot(x - x_K) && \text{in } K, \quad b_1 = 0 \quad \text{on } \partial K. && (11)\end{aligned}$$

Note that the solutions b_0^K and b_1^K of (10) and (11), respectively, are uniquely defined. Using these bubble functions we can express the solution of the local problems (7) in the following form

$$u_B^K = \alpha_0^K b_0^K + \alpha_1^K b_1^K, \tag{12}$$

where

$$\alpha_0^K = f - a\cdot\nabla u_L(x_K), \qquad \alpha_1^K = -\frac{\partial^2 u_L}{\partial x_1\partial x_2}\,.$$

Thus, we are able to eliminate

$$u_B = \sum_K (\alpha_0^K\, b_0^K + \alpha_1^K\, b_1^K)$$

in (5) and obtain

$$\varepsilon(\nabla u_L, \nabla v_L) + (a\cdot\nabla u_L, v_L) + S_{RFB}(u_L, v_L) = (f, v_L) + F_{RFB}(v_L) \tag{13}$$

with

$$\begin{aligned} S_{RFB}(u_L, v_L) &:= \sum_K \left(a\cdot\nabla u_L(x_K)b_0^K + \frac{\partial^2 u_L}{\partial x_1\partial x_2} b_1^K, a\cdot\nabla v_L\right)_K, && (14)\\ F_{RFB}(v_L) &:= \sum_K (f b_0^K, a\cdot\nabla v_L)_K. && (15)\end{aligned}$$

Let us define a projection $P: V_L \to V_B$ by

$$P(v_L)|_K := (a\cdot\nabla v_L)(x_K)b_0^K + \frac{\partial^2 v_L}{\partial x_1\partial x_2} b_1^K. \tag{16}$$

Using again the representation (8), now for an arbitrary element $v_L \in V_L$, we have

$$\begin{aligned} (a\cdot\nabla v_L)(x) &= (a\cdot\nabla v_L)(x_K)\cdot 1 + \frac{\partial^2 v_L}{\partial x_1\partial x_2} a^T\cdot(x - x_K),\\ &= (a\cdot\nabla v_L)(x_K)\; L b_0^K + \frac{\partial^2 v_L}{\partial x_1\partial x_2}\; L b_1^K,\\ (a\cdot\nabla v_L)(x) &= L(P(v_L)),\end{aligned}$$

consequently the stabilizing term can be written in the form

$$S_{RFB}(u_L, v_L) = \sum_K (P(u_L), L(P(v_L)))_K = \sum_K \varepsilon(\nabla P(u_L), \nabla P(v_L))_K. \quad (17)$$

From the last representation we immediately conclude

$$S_{RFB}(v_L, v_L) \geq 0 \quad \forall v_L \in V_L. \quad (18)$$

Let us compare (17) with the corresponding term of the streamline-diffusion method

$$S_{SD}(u_L, v_L) := \sum_K \tau_K (a \cdot \nabla u_L, a \cdot \nabla v_L)_K, \quad (19)$$

where τ_K are positive, user chosen parameters.

LEMMA 1 *There are positive constants τ_K^1 and τ_K^2 (depending on ε and the local meshsize $h_K = \text{diam}(K)$) such that*

$$\sum_K \tau_K^1 \|a \cdot \nabla v_L\|_{0,K}^2 \leq S_{RFB}(v_L, v_L) \leq \sum_K \tau_K^2 \|a \cdot \nabla v_L\|_{0,K}^2 \quad \forall v_L \in V_L. \quad (20)$$

Proof. Let us denote by $\mathcal{N}$ the subspace of $Q_1(K)$ on which $a \cdot \nabla v_L|_K$ vanishes. Since the dimension of the factorspace $Q_1(K)/\mathcal{N}$ is finite it is sufficient to show that

$$v_L|_K \longmapsto \|\nabla P(v_L)\|_{0,K} \qquad \text{and} \qquad v_L|_K \longmapsto \|a \cdot \nabla v_L\|_{0,K}$$

are norms on $Q_1(K)/\mathcal{N}$. Then, the statement follows by the fact that all norms on a finite dimensional space are equivalent. Most of the properties of a norm are obvious. We show that from $\|a \cdot \nabla v_L\|_{0,K} = 0$ and $\|\nabla P(v_L)\|_{0,K} = 0$, respectively, $v_L = 0$ in $Q_1(K)/\mathcal{N}$ follows. The first statement is a direct consequence of the definition of $\mathcal{N}$. Now let $\|\nabla P(v_L)\|_{0,K}$ be equal to zero. Then, $P(v_L)$ is constant on K and due to $P(v_L)|_{\partial K} = 0$ this constant has to be zero. Taking into consideration that b_0^K and b_1^K are linearly independent we obtain from (16)

$$(a \cdot \nabla v_L)(x_K) = 0 \qquad \text{and} \qquad \frac{\partial^2 v_L}{\partial x_1 \partial x_2} = 0,$$

thus, the representation (8) implies that $a \cdot \nabla v_L$ vanishes on K, i.e., $v_L \in \mathcal{N}$. The equivalence of norms yields the existence of positive constants τ_K^1 and τ_K^2 with

$$\tau_K^1 \|a \cdot \nabla v_L\|_{0,K}^2 \leq \varepsilon \|\nabla P(v_L)\|_{0,K}^2 \leq \tau_K^2 \|a \cdot \nabla v_L\|_{0,K}^2 \quad \forall v_L \in Q_1(K)/\mathcal{N}.$$

Summation over all elements K gives the statement of the Lemma. ♣

Remark 1 *Lemma 1 does not imply that the stabilizing effect of the residual free bubble approach is identically to that of the SUPG method. In order to compare both approaches in more detail we have to study the asymptotic behaviour of τ_K^1 and τ_K^2, respectively, and compare it with the common choice of the parameters τ_K in the SUPG method. This will be done in the next section.*

3 Relationship to the streamline diffusion norm

In this section we study the relationship of the norm

$$|||v|||_{RFB} := \left(\varepsilon|v|_1^2 + \sum_K \varepsilon\|\nabla P(v)\|_{0,K}^2\right)^{1/2}. \tag{21}$$

connected with the residual free bubble approach and the streamline diffusion norm given by

$$|||v|||_{SD} := \left(\varepsilon|v|_1^2 + \sum_K \tau_K\|a\cdot\nabla v\|_{0,K}^2\right)^{1/2}. \tag{22}$$

Since both norms are of the same structure it is sufficient to study the equivalence of the corresponding semi-norms defined on each element K, i.e.,

$$|v|_{RFB} := \varepsilon^{1/2}\|\nabla P(v)\|_{0,K} \qquad \text{and} \qquad |v|_{SD} := \tau_K^{1/2}\|a\cdot\nabla v\|_{0,K}. \tag{23}$$

Let K be a rectangle of the decomposition of Ω with sides of length l_1 and l_2. As above, the barycentre of K will be denoted by x_K. We assume the standard shape regularity condition that there is a positive constant c_1 such that

$$c_1\, h_K \le \min(l_1,l_2)\ \le \max(l_1,l_2) \le h_K, \tag{24}$$

where h_K denotes the diameter of K. For each $v\in Q_1(K)$ we have the representation

$$(a\cdot\nabla v)(x) = (a\cdot\nabla v)(x_K) + a^T\cdot(x-x_K)\frac{\partial^2 v}{\partial x_1\partial x_2}. \tag{25}$$

Let us define the constants

$$\alpha \;=\; |K|^{1/2}\,(a\cdot\nabla v)(x_K), \tag{26}$$

$$\beta \;=\; \sqrt{\frac{|K|}{12}}|a|h_K\frac{\partial^2 v}{\partial x_1\partial x_2}. \tag{27}$$

Then, a direct calculation shows

$$\|a\cdot\nabla v\|_{0,K}^2 = \alpha^2 + \frac{a_1^2 l_2^2 + a_2^2 l_1^2}{|a|^2 h_K^2}\,\beta^2, \tag{28}$$

which implies

$$c_1(\alpha^2+\beta^2) \le \alpha^2+\frac{\min(l_1^2,l_2^2)}{h_K^2}\beta^2 \le \|a\cdot\nabla v\|_{0,K}^2 \le \alpha^2+\beta^2 \quad \forall v\in Q_1(K). \tag{29}$$

Note, that for any $v \in P_1(K) \subset Q_1(K)$ we have $\beta = 0$ and from (28) we get

$$\|a \cdot \nabla v\|_{0,K}^2 = \alpha^2 \qquad \forall v \in P_1(K). \tag{30}$$

Now, let us return to our stabilized semi-norm $|v|_{RFB}$.
Using the definitions of α and β given above we have

$$|v|_{RFB}^2 = (\alpha, \beta) \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \tag{31}$$

where the entries of the Matrix $A = (a_{ij})$ are given by

$$a_{11} = \frac{\varepsilon |b_0^K|_{1,K}^2}{|K|}, \tag{32}$$

$$a_{12} = a_{21} = \frac{\sqrt{12}\varepsilon(\nabla b_0^K, \nabla b_1^K)}{|a|\ h_K\ |K|}, \tag{33}$$

$$a_{22} = \frac{12\varepsilon\ |b_1^K|_{1,K}^2}{|a|^2 h_K^2 |K|}. \tag{34}$$

It turns out that because of (29) the expression $\|a \cdot \nabla v\|_{0,K}$ is uniformly equivalent to the l_2-norm of (α, β). With the eigenvalues λ_{min}, λ_{max} of the matrix $A = (a_{ij})$ we have

$$\lambda_{min} \|a \cdot \nabla v\|_{0,K}^2 \le |v|_{RFB}^2 \le \frac{\lambda_{max}}{c_1} \|a \cdot \nabla v\|_{0,K}^2 \quad \forall v \in Q_1(K). \tag{35}$$

In order to estimate the eigenvalues we need a more detailed information on the properties of the bubble functions b_0^K and b_1^K. To this end we first give an equivalent formulation of the entries of A.

LEMMA 2 *The matrix A defined in (32)-(34) has the following entries*

$$a_{11} = \frac{1}{|K|} \int_K b_0^K(x)\ dx, \tag{36}$$

$$a_{12} = a_{21} = \frac{\sqrt{3}}{|K|} \int_K \frac{b_1^K(x) + a^T \cdot (x - x_K) b_0^K(x)}{|a|\ h_K}\ dx, \tag{37}$$

$$a_{22} = \frac{12}{|K|} \int_K \frac{a^T \cdot (x - x_K)}{|a|^2\ h_K^2} b_1^K(x)\ dx. \tag{38}$$

Proof. Let us recall the definition of the bubble functions b_0^K and b_1^K.

Find b_0^K, $b_1^K \in H_0^1(K)$ such that for all $v \in H_0^1(K)$

$$\begin{aligned} \varepsilon(\nabla b_0^K, \nabla v)_K + (a \cdot \nabla b_0^K, v)_K &= (1, v)_K, \\ \varepsilon(\nabla b_1^K, \nabla v)_K + (a \cdot \nabla b_1^K, v)_K &= (a^T \cdot (x - x_s), v)_K. \end{aligned}$$

Then, by choosing $v = b_0^K$ and $v = b_1^K$ we obtain by integrating by parts the convection term

$$\varepsilon |b_0^K|_{1,K}^2 = \int_K b_0^K(x)\, dx, \qquad \varepsilon |b_1^K|_{1,K}^2 = \int_K a^T \cdot (x - x_s) b_1^K(x)\, dx.$$

Further, setting in the first equation $v = b_1^K$ and in the second $v = b_0^K$ and adding both expressions we get

$$\varepsilon(\nabla b_0^K, \nabla b_1^K)_K = \frac{1}{2} \int_K \Big(b_1^K(x) + a^T \cdot (x - x_s) b_0^K(x) \Big)\, dx,$$

by taking into consideration that

$$(a \cdot \nabla b_0^K, b_1^K)_K + (a \cdot \nabla b_1^K, b_0^K)_K = (a \cdot \nabla(b_0^K\, b_1^K), 1) = 0.$$

Using these expressions in the definition of the matrix entries a_{ij}, $i,j = 1,2$, we have the statement of the Lemma. ♣

In the following we apply the maximum principle for weak solutions. For its formulation we introduce the bilinear form $\mathcal{L} : H^1(K) \times H^1(K) \to \mathbf{R}$ by

$$\mathcal{L}(u,v) := \varepsilon(\nabla u, \nabla v)_K + (a \cdot \nabla u, v)_K. \tag{39}$$

Then, we have

LEMMA 3 *Let $w \in H^1(K)$ satisfy*

$$\mathcal{L}(w,v) \le 0 \quad \textit{for all } v \in H_0^1(K) \textit{ with } v \ge 0. \tag{40}$$

Then,

$$\sup_K w \le \sup_{\partial K} w^+,$$

where $w^+ = \max\{w, 0\} \in H^1(K)$.

Proof. See [9] Chapter 8, Theorem 8.1. ♣

Following the lines of [2] where the case of triangular mesh cells has been considered we estimate the bubble functions b_0^K and b_1^K. We have the following result

LEMMA 4 *The bubble functions b_0^K and b_1^K defined in (10), (11) satisfy the following estimates*

$$0 \le b_0^K(x) \le |a|^{-1} h_K, \qquad |b_1^K(x)| \le \frac{|a| h_K}{2} b_0^K(x). \tag{41}$$

Proof. We define (not necessarily unique) the "upwind-most" point x_a in $\overline{K}$ by the inequality

$$a \cdot (x - x_a) \geq 0 \qquad \forall x \in K.$$

Applying Lemma 3 to $w(x) = -b_0^K(x)$ and $w(x) = b_0^K(x) - |a|^{-2} a \cdot (x - x_0)$ we obtain the bounds

$$0 \leq b_0^K(x) \leq |a|^{-2} a \cdot (x - x_0) \leq |a|^{-1} h_K.$$

Next we apply Lemma 3 to $w(x) = |a| h_K b_0^K(x)/2 \pm b_1^K(x)$ resulting in the second estimate of (41) due to

$$|a^T \cdot (x - x_K)| \leq \frac{|a| h_K}{2}$$

and $w(x) = 0$ for $x \in \partial K$. ♣

Let us define

$$\tilde{h}_K := a_{11} = |K|^{-1} \int_K b_0^K(x)\, dx. \tag{42}$$

Now we are in a position to formulate our first main result.

THEOREM 1 *For any v belonging to the subspace $P_1(K)$ of $Q_1(K)$ we have*

$$\tilde{h}_K \|a \cdot \nabla v\|_{0,K}^2 = |v|_{RFB}^2 \quad \forall v \in P_1(K), \tag{43}$$

where there is a positive constant c_2 such that $\tilde{h}_K$ satisfies

$$\frac{c_2\, h_K}{|a|} \min\left(1, \frac{|a|\, h_K}{\varepsilon}\right) \leq \tilde{h}_K \leq \frac{h_K}{|a|}. \tag{44}$$

Moreover, on the whole space $Q_1(K)$ we have

$$|v|_{RFB}^2 \leq \frac{4}{c_1}\, \tilde{h}_K \|a \cdot \nabla v\|_{0,K}^2, \tag{45}$$

with the shape regularity constant c_1 from (24).

Proof. We first consider $v \in P_1(K)$. In this case (27) implies $\beta = 0$. Thus, from (31), (30) and (42) we get

$$|v|_{RFB}^2 = a_{11}\, \alpha^2 = \tilde{h}_K \|a \cdot \nabla v\|_{0,K}^2.$$

Now consider the general case $v \in Q_1(K)$. We start with (35) and estimate λ_{max}. Note that the eigenvalues of A are given by

$$\lambda_{max} = \frac{a_{11} + a_{22}}{2} + \sqrt{\left(\frac{a_{11} + a_{22}}{2}\right)^2 + a_{12}^2 - a_{11} a_{22}}, \tag{46}$$

$$\lambda_{min} = \frac{a_{11} + a_{22}}{2} - \sqrt{\left(\frac{a_{11} + a_{22}}{2}\right)^2 + a_{12}^2 - a_{11} a_{22}}. \tag{47}$$

Since $a_{12}^2 \le a_{11}a_{22}$ it holds

$$\lambda_{max} \le a_{11} + a_{22}.$$

Using Lemma 2 and Lemma 4 we get

$$\begin{aligned} a_{22} &= \frac{12}{|K|}\int_K \frac{a^T\cdot(x-x_K)}{|a|^2\,h_K^2} b_1^K(x)\,dx, \\ &\le \frac{12}{|K|}\frac{|a|h_K}{2|a|^2\,h_K^2}\int_K \frac{|a|h_K}{2} b_0^K(x)\,dx, \\ &\le \frac{3}{|K|}\int_K b_0^K(x)\,dx, \end{aligned}$$

and consequently

$$\lambda_{max} \le a_{11} + a_{22} \le 4\tilde{h}_K.$$

Together with (35) this proves (45). It remains to show the estimates (43) for $a_{11} = \tilde{h}_K$. The upper bound follows directly from Lemma 4. In order to get a lower bound for a_{11} we try to bound b_0^K from below by a quadratic bubble function. For this we transform K with the side length l_1, l_2 onto the reference quadrilateral $[-1,+1]\times[-1,+1]$ by means of

$$x_i = x_{K,i} + \frac{l_i}{2}\xi_i, \quad i = 1,2$$

and define

$$\Phi(x_1,x_2) := (1-\xi_1^2)(1-\xi_2^2).$$

Then, a direct calculation gives

$$\begin{aligned} (L\Phi)(\xi_1,\xi_2) &= \frac{8\varepsilon}{l_1^2}(1-\xi_2^2) + \frac{8\varepsilon}{l_2^2}(1-\xi_1^2) \\ &\quad -\frac{4a_1}{l_1}\xi_1(1-\xi_2^2) - \frac{4a_2}{l_2}\xi_2(1-\xi_1^2), \\ (L\Phi)(\xi_1,\xi_2) &\le \frac{16\,|a|}{c_1\,h_K}\max\left(1,\frac{\varepsilon}{|a|\,h_K}\right), \end{aligned} \qquad (48)$$

where we used (24). Setting

$$\gamma_K := \max\left(1,\frac{\varepsilon}{|a|\,h_K}\right),$$

we have for the function

$$w(x) = \frac{c_1\,h_K}{16\,|a|\,\gamma_K}\Phi(x) - b_0^K(x),$$

that $w = 0$ on ∂K and

$$\mathcal{L}(w,v) = (L\Phi - 1, v) \le 0 \quad \forall v \in H_0^1(K),\ v \ge 0.$$

Applying the maximum principle (Lemma 3) we obtain $w(x) \leq 0$ which means

$$b_0^K(x) \geq \frac{c_1\, h_K}{16\, |a|\, \gamma_K} \Phi(x).$$

Thus,

$$\begin{aligned} \tilde{h}_K &= \frac{1}{|K|} \int_K b_0^K(x)\, dx, \\ &\geq \frac{c_1\, h_K}{16\, |a|\, \gamma_K\, |K|} \int_K \Phi(x)\, dx, \\ &\geq \frac{c_2\, h_K}{|a|} \min\left(1, \frac{|a|\, h_K}{\varepsilon}\right), \end{aligned}$$

where c_2 is given by

$$c_2 := \frac{c_1}{16\, |K|} \int_K \Phi(x)\, dx.$$

This proves the left hand side of (44). ♣

Remark 2 *Theorem 1 shows that on the subspace of continuous, piecewise linear functions the stabilizing terms in the residual free bubble approach and the streamline diffusion method (with the standard choice of τ_K in the diffusion-dominated and convection-dominated case, respectively), coincide. Moreover, in the general case of continuous, piecewise bilinear functions, stability with respect to the streamline diffusion norm (23) forces stability in the residual free bubble norm (21), due to (45). Note that getting a lower bound for λ_{min} is tricky. In the next section we are dealing with this question in the convection-dominated case when ε tends to zero.*

4 The convection-dominated case

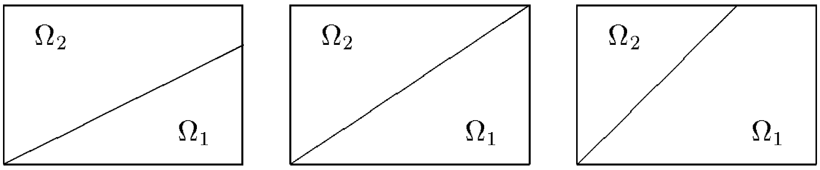
From Lemma 1 we know that there is a positive constant τ_K^1, depending of h_K and ε, such that

$$\sum_K \tau_K^1 \|a \cdot \nabla v\|_{0,K}^2 \leq S_{RFB}(v, v) \quad \forall v \in V_L.$$

If this constant would behave like $\tau_K^1 \sim c_3\, \tilde{h}_K$ then, the stability concepts of both the residual free bubble approach and the streamline diffusion method with the standard choice of the streamline-diffusion parameter τ_K coincide. We will see that in the convection-dominated case, i.e., sufficiently small $\varepsilon > 0$ there are exceptional cases in which the stability concepts do not coincide. Due to Theorem 1 this means that – in general – the stability which is achieved by the residual free bubble approach is weaker than that of the streamline diffusion method.

Let us assume without restricting the generality that $a_1 > 0$ and $a_2 \geq 0$. Further, let $\kappa = (a_2 l_1)/(a_1 l_2)$. Then, we can distinguish the following situations indicated in Figure 1.

Figure 1: Streamline direction with respect to the element K



$0 \leq \kappa < 1$ $\kappa = 1$ $1 < \kappa < \infty$

If κ is positive the element K is splitted into the open subdomains Ω_1 and Ω_2 by the line

$$x_2 = \frac{a_2}{a_1}(x_1 - x_{K,1} + \frac{l_1}{2}) + x_{K,2} - \frac{l_2}{2}.$$

In case of $a_2 = 0$ we get $\kappa = 0$ and Ω_1 becomes the empty set. Now, the idea is to replace the bubble functions b_0^K and b_1^K by their asymptotic limits

$$p_0^K := \lim_{\varepsilon \to 0} b_0^K \qquad \text{and} \qquad p_1^K := \lim_{\varepsilon \to 0} b_1^K \tag{49}$$

where

$$p_0^K = \begin{cases} \frac{1}{a_2}\left(x_2 - x_{K,2} + \frac{l_2}{2}\right) & \text{in } \Omega_1, \\ \frac{1}{a_1}\left(x_1 - x_{K,1} + \frac{l_1}{2}\right) & \text{in } \Omega_2 \end{cases}, \tag{50}$$

$$p_1^K = \begin{cases} \frac{1}{a_2}\left(x_2 - x_{K,2} + \frac{l_2}{2}\right)\left(a_2(x_2 - x_{K,2}) - \frac{a_1 l_2}{2}\right) & \text{in } \Omega_1, \\ \frac{1}{a_1}\left(x_1 - x_{K,1} + \frac{l_1}{2}\right)\left(a_1(x_1 - x_{K,1}) - \frac{a_2 l_1}{2}\right) & \text{in } \Omega_2 \end{cases}. \tag{51}$$

Note that p_0^K and p_1^K are continuous along the line separating Ω_1 and Ω_2 but do not belong to $H_0^1(K)$. However, in the formulas (36)–(38) b_i^K, $i = 0, 1$, can be replaced by the solutions p_i^K, $i = 0, 1$ of the reduced problems up to boundary layer terms with exponential decay from the boundary into the interior of K and higher order terms in the asymptotic expansion, respectively [10], [6]. Since the L^1-norm of such boundary layer terms vanishes for

$\varepsilon \to 0$, we can assume that

$$\lim_{\varepsilon\to 0} a_{11} = \frac{1}{|K|}\int_K p_0^K(x)\,dx, \tag{52}$$

$$\lim_{\varepsilon\to 0} a_{12} = \lim_{\varepsilon\to 0} a_{21} = \frac{\sqrt{3}}{|K|}\int_K \frac{p_1^K(x) + a^T\cdot(x-x_K)p_0^K(x)}{|a|\;h_K}\,dx, \tag{53}$$

$$\lim_{\varepsilon\to 0} a_{22} = \frac{12}{|K|}\int_K \frac{a^T\cdot(x-x_K)}{|a|^2\;h_K^2} p_1^K(x)\,dx. \tag{54}$$

In the following we calculate the entries

$$m_{ij} := \lim_{\varepsilon\to 0} a_{ij}, \quad i,j = 1,2$$

of the Matrix $M = (m_{ij})$ and study the minimal eigenvalue of M.
First, a direct calculation yields

$$m_{11} = \frac{1}{6}\min\left(\frac{l_1}{a_1}, \frac{l_2}{a_2}\right)(3 - \Lambda(\kappa)), \tag{55}$$

$$m_{22} = \frac{(a_2 l_1 + a_1 l_2)|K|}{2|a|^2 h_K^2}(1-\Lambda(\kappa))^2, \tag{56}$$

$$m_{12} = m_{21} = 0, \tag{57}$$

where $\Lambda(\kappa) \in [0,1]$ is given by

$$\Lambda(\kappa) = \min\left(\kappa, \frac{1}{\kappa}\right).$$

Thus, the eigenvalues of the Matrix M are the diagonal entries. Summarizing we have the following result.

THEOREM 2 *There are positive constants c_3 and c_4 such that for all $\kappa \in [0,\infty)$ and for all $v \in Q_1(K)$ the estimate*

$$c_3(1-\Lambda(\kappa))^2\frac{h_K}{|a|}||a\cdot\nabla v||_{0,K}^2 \le \lim_{\varepsilon\to 0}|v|_{RFB}^2 \le c_4\frac{h_K}{|a|}||a\cdot\nabla v||_{0,K}^2 \tag{58}$$

holds true. The lower bound 0 for $\kappa = 1$ cannot be improved.

Proof. The upper estimate has been already stated in Theorem 1. To show the lower bound we start with (31) and take $\varepsilon \to 0$. This gives

$$\min\,(m_{11}, m_{22})\,||a\cdot\nabla v||_{0,K}^2 \le \lim_{\varepsilon\to 0}|v|_{RFB}^2,$$

where the lower bound is sharp. Since $m_{22} = 0$ for $\kappa = 1$, we have proved that in this case the lower bound is optimal. Further, using (55) and the shape regularity of the mesh (24) we obtain

$$m_{11} \ge \frac{1}{3}\min\left(\frac{l_1}{a_1}, \frac{l_2}{a_2}\right) \ge \frac{c_1 h_K}{3|a|} \ge \frac{c_1 h_K}{3|a|}(1-\Lambda(\kappa))^2.$$

Similarly, we can estimate

$$m_{22} \geq \frac{c_1 h_K}{2|a|} \frac{a_1 + a_2}{\sqrt{a_1^2 + a_2^2}} \frac{|K|}{h_K^2} (1 - \Lambda(\kappa))^2,$$

from which (58) follows by taking into consideration that the inequalities

$$\frac{a_1 + a_2}{\sqrt{a_1^2 + a_2^2}} \geq 1$$

and

$$|K| = l_1 \, l_2 \geq c_1^2 h_K^2$$

hold. ♣

Remark 3 *Theorem 2 shows that, in comparision to the streamline diffusion method with the standard choice of the streamline diffusion parameter* $\tau_K \sim h_K/|a|$*, stability of the residual free bubble approach is lost in the convection dominated limit if the streamlines are aligned with the diagonals of the elements (*$\kappa = 1$*). On the other hand Theorem 1 shows that – also in this exceptional case – on the subspace of continuous, piecewise linear functions the residual free bubble approach is as stable as the streamline diffusion method is.*

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