The Construction of Consistent Possibility and Necessity Measures

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Abstract: Given a general measure μ (finite or infinite), we develop possibility and necessity measures as upper and lower estimators of μ. We provide a method for constructing such fuzzy measures and show that the measure can be approximated with arbitrary closeness using fuzzy measures constructed this way. Using the extension principle, these consistent possibility and necessity measures are used to produce possibility and necessity measures on the range space of a measurable function which are consistent with the measure on the range space induced by the measurable function. This induced measure can be approximated with arbitrary closeness by extending consistent possibility and necessity measures constructed on the domain space.

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1 Introduction

Possibility and necessity measures have been studied as an offshoot of fuzzy set theory ([21]). In the usual development, the measures are defined without reference to a (standard) measure or are presented as providing an upper and lower bound on an unknown probability measure ([4]). In [3] de Cooman generalizes the concept of possibility measures which take values in the general lattice. In this paper we examine a special case of this approach. We examine special possibility and necessity measures which take values on $[0, \mu(X)]$, were $(X, \mathcal{L}, \mu)$ is a measure space. In particular, starting with the measure μ we say that a possibility measure *pos* and a necessity measure *nec* are consistent with μ if they bound the measure μ from above and below respectively. It will be shown that consistent possibility and necessity measures can be constructed from nested families of measurable sets. Moreover, the consistent possibility and necessity measures on the range space of a measurable function can be constructed from consistent possibility and necessity measures on the domain using the extension principle. Possibility and necessity measures constructed this way completely determine the measure on the range space. This process provides a method for estimating the measure of sets in the range space which is our primary interest. It provides a computationally convenient method for handling problems in optimization. This paper is a generalization of the ideas presented in ([11]) where we restricted the discussion to probability measures.

This article is divided into four sections. The first section provides a list of basic definitions used throughout the article including the definition of a consistent possibility or necessity measure. The second section develops some basic theorems concerning consistent possibility and necessity measures. These include (1) a method for constructing consistent fuzzy measures, (2) a proof that the set of consistent fuzzy measures completely determine the measure (3) a proof that the extension of consistent fuzzy measures is consistent and (4) a proof that the set of such extended fuzzy measures completely determines the induced measure. The third section provides an application in which consistent possibility measures are used to estimate the expected value of a function of random variables. The fourth section concludes the paper.

2 Definitions

From the usual notation call $(X, \mathcal{L}, \mu)$ a measure space if X is a set, $\mathcal{L}$ is a σ-field of subsets of X and μ is a measure defined on $\mathcal{L}$. We will call μ a finite measure if $\mu(X) < \infty$. Throughout this paper we will use $\mathcal{P}(X)$ to denote the power set of set X, R_∞ to denote the extended real numbers and A^c to denote the complement of set A.

Definition 1 *Given measure space $(X, \mathcal{L}, \mu)$, a set function pos : $\mathcal{P}(X) \to R_\infty$ is called a* ***possibility measure consistent with the measure*** μ *and the set function nec : $\mathcal{P}(X) \to R_\infty$ is called a* ***necessity measure consistent with the measure*** μ *if*

(1) $\forall E \in \mathcal{L}$, $pos(E) \geq \mu(E) \geq nec\,(E)$
(2) $\forall \{A_\alpha\}_{\alpha\in\Lambda} \subseteq \mathcal{P}(X)$, $pos\,(\cup_{\alpha\in\Lambda}A_\alpha) = \sup\{pos\,(A_\alpha) \mid \alpha \in \Lambda\}$ *and* $nec\,(\cap_{\alpha\in\Lambda}A_\alpha) = \inf\{nec\,(A_\alpha) \mid \alpha \in \Lambda\}$
(3) $pos\,(X) = \mu\,(X) = nec\,(X)$ *and* $pos\,(\emptyset) = 0 = nec\,(\emptyset)$.

There exist a large number of possibility and necessity measures consistent with any measure. For example, consider the unit interval with Lebesgue measure. Define $pos_1\,(A) = 1$ if $A \neq \emptyset$ and $pos_1\,(\emptyset) = 0$. This satisfies (1)-(3) and is a consistent possibility measure (the least informative). But so is $pos_2\,(A) = \sup\{x \mid x \in A\}$ if $A \neq \emptyset$ and $pos_2\,(\emptyset) = 0$ since for all measurable A, $\mu\,(A) \leq \mu\,[0, \sup\{x \mid x \in A\}] = \sup\{x \mid x \in A\} = pos_2\,(A)$.

Noting that $A = \cup_{x\in A}\{x\}$ implies $pos\,(A) = \sup\{pos\,(\{x\}) \mid x \in A\}$ and that $A = \cap_{x\in A^c}\{x\}^c$ implies $nec\,(A) = \inf\{nec\,(\{x\}^c) \mid x \in A^c\}$ one can define distribution functions which completely determine *pos* or *nec* (see definition 2). We use the term distribution function only to imply that they are extended real valued functions that completely characterize the possibility or necessity measure.

Definition 2 *Let* pos, nec *be possibility and necessity measures, the functions* $p, n : X \to R_\infty$ *defined by* $p\,(x) = pos\,(\{x\})$ *and* $n\,(x) = nec\,(\{x\}^c)$ *are called* ***possibility and necessity distribution functions*** *respectively.*

Given either *pos* or p one can determine the other and similarly for *nec* and n since $pos\,(A) = \sup\{p\,(x) \mid x \in A\}$ and $nec\,(A) = \inf\{n\,(x) \mid x \in A^c\}$. In addition, if μ is a finite measure (for example, a probability measure) then necessity is the dual of possibility (see Theorem 1 below). However, since we are allowing more general measure spaces whose measure may be infinite, the duality of necessity fails; that is, $n\,(x) = \mu\,(X) - p\,(x)$ does not hold when $\mu\,(X) = \infty$.

The level sets of possibility distributions are used quite frequently. When $\mu\,(X) = 1$, the level sets are called $\alpha-$cuts (see e.g. [14]). Since we have removed the restriction that possibility levels fall in the unit interval, we will call the level sets $s-$cuts. There is a dual definition for necessity distributions.

Definition 3 *Let* p *and* n *be possibility and necessity distribution functions respectively. Define the* ***s-cuts*** *of* p *and* n *to be the sets*

$$p^s = \{x \mid p\,(x) \geq s\} \text{ and } n^s = \{x \mid n\,(x) \leq s\}$$

and the ***strong*** *s-****cuts*** *to be the sets*

$$p^{s+} = \{x \mid p(x) > s\} \text{ and } n^{s+} = \{x \mid n(x) < s\}.$$

The following definition will be used in the method for constructing consistent possibility and necessity measures to follow.

Definition 4 *Let $(X, \mathcal{L}, \mu)$ be a measure space. A collection of measurable sets, $PN = \{E_r \mid r \in S \subseteq R_\infty\}$, will be called a* ***possibility nest*** *if it satisfies the following properties:*
1) $r < s \Rightarrow E_r \subset E_s$ (i.e. PN is nested)
2) $X, \emptyset \in PN$
3) $\forall t\ \exists E_r \in PN$ such that $\mu(E_r) = \mu\left(\cup_{\mu(E_s)<t} E_s\right)$ and $E_u \in PN$ such that $\mu(E_u) = \mu\left(\cap_{\mu(E_s)>t} E_s\right)$.

Note that property 3) insures that there are no "gaps" in PN in the sense that the set $T = \{\mu(E_r) \mid r \in S\}$ must be closed This follows since for any limit point t of T we can construct a sequence of points $\{t_n\} \subseteq T$ converging to t either from above or below. Then property 3) insures there is an $E_r \in PN$ such that $\mu(E_r) = t$.

Example 1 *Consider $X = [0, 1]$ with the Lebesgue measure and*
$PN = \left\{\emptyset, X, \left\{\left[.25 + \frac{1}{n}, .75 - \frac{1}{n}\right] \mid n = 4, \infty\right\}\right\}$. Then
$\mu\left(\cup_{\mu(E_s)<.5} E_s\right) = \mu\left(\cup_{n=4}^{\infty}\left[.25 + \frac{1}{n}, .75 - \frac{1}{n}\right]\right) = .5$ but $\nexists E_r \in PN$ such that $\mu(E_r) = .5$..PN fails property 3 and is not a possibility nest.

Example 2 *For $X = [0, 1]$ with Lebegue measure the collection of sets $PN = \{X, (1 - \alpha, 1] \mid \alpha \in [0, 1]\}$ satisfies properties 1,2 and 3 and is a possibility nest.*

3 Development

In some cases, the mathematics of possibility and necessity measures are somewhat simpler than that of probability measures. For example, such measures can be extended by performing interval arithmetic on α-cuts. Our ultimate goal is to try and take advantage of these properties in problems such as the use of consistent possibility and necessity measures in the problem formulation to approximate measures of interest to the decision maker, for example in chance constrained problems or problems seeking to optimize an expected value. Although we primarily have probability measures in mind, our results in this paper are presented in the more general framework of measure theory. Our objective in this paper is illustrate a general method for constructing such measures and to establish some simple properties.

We begin by demonstrating that the usual conversion of possibility to necessity measures preserves consistency on a finite measure space.

Theorem 1 *Given a possibility measure pos consistent with a finite measure μ, then the set function nec,*

$$nec(A) = \mu(X) - pos(A^c),$$

is a necessity measure consistent with μ. If nec is a consistent necessity measure then the set function,

$$pos(A) = \mu(X) - nec(A^c),$$

is a possibility measure consistent with μ.

Proof. See e.g. [14] for a proof that *nec* and *pos* defined above are necessity and possibility measures when $\mu(X) = 1$. The proof of consistency and for the slight generalization to the case $\mu(X) < \infty$ follows:
For the first equation we have
(1) $nec(E) = \mu(X) - pos(E^c) \leq \mu(X) - \mu(E^c) = \mu(E)$
(2) $nec\left(\bigcap_{k\in\Lambda} A_k\right) = \mu(X) - pos\left(X - \bigcap_{k\in\Lambda} A_k\right) =$
$\mu(X) - pos\left(\bigcup_{k\in\Lambda}(X - A_k)\right) = \mu(X) - \sup_{k\in\Lambda} pos(X - A_k) =$
$\inf_{k\in\Lambda}(\mu(X) - pos(X - A_k)) = \inf_{k\in\Lambda} nec(A_k)$ and
(3) $nec(X) = \mu(X) - pos(\emptyset) = \mu(X)$ and $nec(\emptyset) = \mu(X) - pos(X) = 0$.
The proof for the second equation is completely analogous.□

In the next theorem we establish a general method for construction of consistent possibility and necessity measures. We show that these measures can be constructed from nested families of measurable sets (provided the family is a possibility nest). In essence consistent possibility and necessity measures are special outer and inner measures of a set respectively. Although other methods might be devised for constructing consistent fuzzy measures, this is the only method studied in this paper.

Theorem 2 *Let $(X, \mathcal{L}, \mu)$ be a measure space and $PN = \{E_r \mid r \in S \subseteq R_\infty\}$ be a possibility nest. Then the set functions $pos, nec : P(X) \to R_\infty$ defined by*

$$pos(A) = \inf \{\mu(E_r) \mid A \subseteq E_r,\ E_r \in PN\}$$

and

$$nec(A) = \sup \{\mu(E_r) \mid E_r \subseteq A,\ E_r \in PN\}$$

are possibility and necessity measures consistent with μ.

Proof. Since $X, \emptyset \in PN$, pos and nec are well defined.
(1) Given $E_1, E_2 \in \mathcal{L}$, since $E_1 \subseteq E_2 \Rightarrow \mu(E_1) \leq \mu(E_2)$. By definition of pos and nec, $pos(E) \geq \mu(E) \geq nec(E)$ for all measurable E.
(2) Proof for pos: Given $\{A_\alpha\}_{\alpha \in \Lambda}$, if $\beta \in \Lambda$ then $A_\beta \subseteq \cup_{\alpha \in \Lambda} A_\alpha$ implies $\{E_r \mid \cup_{\alpha \in \Lambda} A_\alpha \subseteq E_r\} \subseteq \{E_s \mid A_\beta \subseteq E_s\}$ which implies $pos(A_\beta) \leq pos(\cup_{\alpha \in \Lambda} A_\alpha)$ and thus $\sup_{\alpha \in \Lambda} pos(A_\alpha) \leq pos(\cup_{\alpha \in \Lambda} A_\alpha)$.
Let $t = \sup \{\mu(E_s) \mid \cup_{\alpha \in \Lambda} A_\alpha \nsubseteq E_s\}$ where t could be ∞. By assumption on PN $\exists E_r \in PN$ such that $\mu(E_r) = \mu\left(\cup_{\mu(E_s)<t} E_s\right) = \sup\{\mu(E_s) \mid \mu(E_s) < t\}$ (the latter identity holds since PN is nested). By definition of t and because PN is nested, $\mu(E_s) < t$ implies $\cup_{\alpha \in \Lambda} A_\alpha \nsubseteq E_s$ so for some $\beta \in \Lambda$ $A_\beta \nsubseteq E_s$. This means that $pos(A_\beta) \geq \mu(E_s)$ so $\sup_{\alpha \in \Lambda} pos(A_\alpha) \geq \mu(E_r)$.
Case 1 - Assume $\cup_{\alpha \in \Lambda} A_\alpha \subseteq E_r$. Then $pos(\cup_{\alpha \in \Lambda} A_\alpha) \leq \mu(E_r) \leq \sup_{\alpha \in \Lambda} pos(A_\alpha)$.
Case 2 - Assume $\cup_{\alpha \in \Lambda} A_\alpha \nsubseteq E_r$. Then for some $\beta \in \Lambda$ $A_\beta \nsubseteq E_r$. But $\forall E_s \in PN$ with $\mu(E_r) < \mu(E_s)$ we have $A_\beta \subseteq \cup_{\alpha \in \Lambda} A_\alpha \subseteq E_s$ and $\forall E_s \in PN$ with $\mu(E_s) \leq \mu(E_r)$ $A_\beta \nsubseteq E_s$ and $\cup_{\alpha \in \Lambda} A_\alpha \nsubseteq E_s$. This means that $pos(A_\beta) = pos(\cup_{\alpha \in \Lambda} A_\alpha)$ giving $\sup_{\alpha \in \Lambda} pos(A_\alpha) \geq pos(\cup_{\alpha \in \Lambda} A_\alpha)$.
(3) Since $X \in PN$, $pos(X) = \inf\{\mu(E_\tau) \mid X \subseteq E_\tau\} = \inf\{\mu(X)\} = \mu(X)$ and $nec(X) = \sup\{\mu(E_r) \mid E_r \subseteq X\} = \mu(X)$. Similarly, $pos(\emptyset) = nec(\emptyset) = 0$ since $\emptyset \in PN$.
The proof for nec is completely analogous to that for pos.□

Example 3 *To see why Property 3 of Definition 4 is needed consider the sets PN of Example 1 which failed the definition of a possibility nest. Then $pos((.25, .75)) = 1$ but $\sup\left\{pos\left(\left[.25 + \frac{1}{n}, .75 - \frac{1}{n}\right]\right) \mid n = 4, \infty\right\} = .5$ even though $(.25, .75) = \cup_{n=4}^{\infty} \left[.25 + \frac{1}{n}, .75 - \frac{1}{n}\right]$ so pos constructed from this PN is not a possibility measure.*

The next theorem establishes the relationships between the E_r's and the s-cuts of p and n.

Theorem 3 *Let p and n be possibility and necessity distribution functions for possibility and necessity measures constructed from PN with properties 1-3 as in Theorem 2. Then* $\forall s$,

$$p^{s+} \subseteq \left(\cup_{\mu(E_r)\leq s} E_r\right)^c \subseteq \left(\cup_{\mu(E_r)<s} E_r\right)^c = p^s$$

and

$$n^{s+} \subseteq \cap_{\mu(E_r)\geq s} E_r \subseteq \cap_{\mu(E_r)>s} E_r = n^s.$$

Proof. We first show that $\left(\cup_{\mu(E_r)<s} E_r\right)^c = p^s$.
1.a. Suppose $p^s = \emptyset$ then $\forall x \in X$ $p(x) < s$. Thus for each x $\exists E_r \in PN$ such that $x \in E_r$ and $\mu(E_r) < s$ so $X = \cup_{\mu(E_r)<s} E_r$ and $\left(\cup_{\mu(E_r)<s} E_r\right)^c = \emptyset$.
b. Suppose $\left(\cup_{\mu(E_r)<s} E_r\right)^c = \emptyset$. Then $\cup_{\mu(E_r)<s} E_r = X$ so for each $x \in X$ $\exists E_r \in PN$ so that $x \in E_r$ and $\mu(E_r) < s$. So for this x $p(x) < s$. Thus $p^s = \emptyset$.
2.a. Suppose $p^s \neq \emptyset$ and let $x \in p^s \Rightarrow p(x) \geq s$. Thus $\inf\{\mu(E_r) \mid x \in E_r\} \geq s$. So if $\mu(E_r) < s$ then necessarily $x \notin E_r$ which means $x \in \left(\cup_{\mu(E_r)<s} E_r\right)^c$ $\Rightarrow p^s \subseteq \left(\cup_{\mu(E_r)<s} E_r\right)^c$.
2.b. Suppose $x \in \left(\cup_{\mu(E_r)<s} E_r\right)^c \neq \emptyset$. Then $\forall E_r$ such that $x \in E_r$ $\mu(E_r) \geq s$. Since $X \in PN$ $x \in X$ and therefore $\exists$ at least one E_r (namely X) for which $x \in E_r$ and $\mu(E_r) \geq s$. Therefore $p(x) = \inf\{\mu(E_r) \mid x \in E_r\} \geq s$ and $\left(\cup_{\mu(E_r)<s} E_r\right)^c \subseteq p^s$.
Next we show that $p^{s+} \subseteq \left(\cup_{\mu(E_r)\leq s} E_r\right)^c$.
Suppose $x \in p^{s+} \Rightarrow p(x) > s \Rightarrow \inf\{\mu(E_r) \mid x \in E_r\} > s$. Now for $\mu(E_r) \leq s$ $x \notin E_r$ implies $x \in \left(\cup_{\mu(E_r)\leq s} E_r\right)^c$ and $p^{s+} \subseteq \left(\cup_{\mu(E_r)\leq s} E_r\right)^c$.
The proof for the relationships involving necessities is similar.□

One implication of this result is that the s-cuts for possibility and necessity distribution functions constructed in this way are measurable. This follows from the nested property of PN.

As stated earlier, possibility and necessity measures have a nice sup,inf calculus. For example, the possibility measure of an arbitrary union of sets is the supremum over the possibility measure of each individual set. Usually this is simpler than determining the standard measure of the arbitrary union unless the sets are disjoint. We will use these fuzzy measures in applications as approximations for a measure. But first, we need to know if there are a sufficient number of possibility and necessity measures that approximate any given measure.

Theorem 4 *Let* $(X, \mathcal{L}, \mu)$ *be a measure space and*

$$P = \{pos \mid pos \text{ is a possibility measure consistent with } \mu\}$$

and let

$$N = \{nec \mid nec \text{ is a neccesity measure consistent with } \mu\}.$$

Then $\forall E \in \mathcal{L},\ \mu(E) = \min\{pos(E) \mid pos \in P\} = \max\{nec(E) \mid nec \in N\}$.

Proof. Let $E \in \mathcal{L}$. Consider $PN = \{\emptyset, E, X\}$. Then (1) PN is nested (2) $X, \emptyset \in PN$ and (3) for any $t \geq 0$ $\cup_{\mu(E_s)<t} E_s \in PN$ and $\cap_{\mu(E_s)>t} E_s \in PN$ (as will always be the case when there are a finite number of nested sets in PN). Thus PN satisfies the requirements of theorem 2 and determines a possibility measure *pos* and necessity measure *nec* consistent with measure μ with the property that $pos(E) = \inf\{\mu(E_r) \mid A \subseteq E_r,\ E_r \in PN\} = \mu(E)$ and $nec(E) = \sup\{\mu(E_r) \mid E_r \subseteq A,\ E_r \in PN\} = \mu(E)$. This combined with property (1) of definition 1 proves the theorem.□

Our primary interest is in estimating the measures induced on the range space of a measurable function. For example measurable functions of random variables. The next result establishes the fact that the extension of consistent possibility and necessity measures is itself consistent.

Theorem 5 *Let* $(X, \mathcal{L}, \mu)$ *be a measure space and* $(Y, \mathcal{O})$ *a measurable space. Let* $f : X \rightarrow Y$ *be an* $\mathcal{L}$*-measurable function and let* ν *be the measure on* Y *defined by* $\nu(E) = \mu(f^{-1}(E))\ \forall E \in \mathcal{O}$. *Let* p_X *and* n_X *be possibility and necessity distribution functions for possibility and necessity measures* pos_X *and* nec_X *consistent with* μ. *Then the functions* $p_Y, n_Y : Y \rightarrow R_\infty$ *defined by* $p_Y(y) = \sup\{p_X(x) \mid f(x) = y\}$ *(where we define* $\sup \emptyset = 0$*) and* $n_Y(y) = \inf\{n_X(x) \mid f(x) = y\}$ *(where we define* $\inf \emptyset = \mu(X)$*) are possibility and necessity distribution functions for a possibility measure* pos_Y *and necessity measure* nec_Y *consistent with* ν.

Proof. (1) Let $E \in \mathcal{O}$. Then $pos_X(f^{-1}(E)) \geq \mu(f^{-1}(E)) = \nu(E)$. But $pos_Y(E) = \sup\{p_Y(y) \mid y \in E\} = \sup\{\sup\{p_X(x) \mid f(x) = y\} \mid y \in E\} = \sup\{p_X(x) \mid x \in f^{-1}(E)\} = pos_X(f^{-1}(E))$. Therefore $pos_Y(E) \geq \nu(E)$.
Similarly, $\nu(E) = \mu(f^{-1}(E)) \geq nec_X(f^{-1}(E))$. But
$nec_Y(E) = \inf\{n_Y(y) \mid y \in E^c\} = \inf\{\inf\{n_X(x) \mid f(x) = y\} \mid y \in E^c\} = \inf\{n_X(x) \mid x \in f^{-1}(E^c)\} = \inf\{n_X(x) \mid x \in f^{-1}(E)^c\} = nec_X(f^{-1}(E))$

(2) $pos_Y(\cup_{\gamma\in\Lambda}A_\gamma) = pos_X(f^{-1}(\cup_{\gamma\in\Lambda}A_\gamma)) = pos_X(\cup_{\gamma\in\Lambda}f^{-1}(A_\gamma)) =$
$\sup\{pos_X(f^{-1}(A_\gamma)) \mid \gamma\in\Lambda\} = \sup\{pos_Y(A_\gamma) \mid \gamma\in\Lambda\}$.
Similarly, $nec_Y(\cap_{\gamma\in\Lambda}A_\gamma) = nec_X(f^{-1}(\cap_{\gamma\in\Lambda}A_\gamma)) =$
$nec_X(\cap_{\gamma\in\Lambda}f^{-1}(A_\gamma)) = \inf\{nec_X(f^{-1}(A_\gamma)) \mid \gamma\in\Lambda\} =$
$\inf\{nec_Y(A_\gamma) \mid \gamma\in\Lambda\}$
(3) $pos_Y(Y) = pos_X(f^{-1}(Y)) = pos_X(X) = \mu(X) = \mu(f^{-1}(Y)) = \nu(Y)$ and $pos_Y(\emptyset) = pos_X(f^{-1}(\emptyset)) = pos_X(\emptyset) = 0$.
Similarly, $nec_Y(Y) = nec_X(f^{-1}(Y)) = nec_X(X) = \mu(X) = \mu(f^{-1}(Y)) = \nu(Y)$ and $nec_Y(\emptyset) = nec_X(f^{-1}(\emptyset)) = nec_X(\emptyset) = 0$.□

What is happening here is that by determining $pos_Y(A)$ we're actually determining the infimum over the $\mu(E_r)$'s such that $f^{-1}(A) \subseteq E_r$. Similarly $nec_Y(A)$ gives the supremum over the E_r's such that $E_r \subseteq f^{-1}(A)$. The next theorem, in combination with Theorem 3, formalizes this concept.

Theorem 6 *Given the same hypothesis as Theorem 5, if pos_X and nec_X are constructed using $PN = \{E_s \mid s \in S\}$ as in Theorem 2, and if $pos_Y(A) = r$ then $f^{-1}(A) \subseteq n_X^r$ and if $nec_Y(A) = r$ then $p_X^r \subseteq f^{-1}(A)$.*

Proof. We begin by showing that $f^{-1}(A) \subseteq \cap_{\mu(E_s)>r}E_s$. Then apply Theorem 3. Note that the containment is true if $f^{-1}(A) = \emptyset$. Assume $f^{-1}(A) \neq \emptyset$ and that $x \in f^{-1}(A)$, i.e. $f(x) = y \in A$ and assume $pos_Y(A) = r$. Then $p_Y(y) \leq r$. But $p_Y(y) = \sup\{p_X(x) \mid f(x) = y\}$. Thus $p_X(x) \leq r$. But this means $pos_X(\{x\}) = \inf\{\mu(E_s) \mid x \in E_s\} \leq r$. Since the E_s are nested, this means that $x \in E_s$ whenever $\mu(E_s) > r$ so $x \in \cap_{\mu(E_s)>r}E_s$.
We now show that $\left(\cup_{\mu(E_s)<r}E_s\right)^c \subseteq f^{-1}(A)$ and again apply Theorem 3. Assume $f^{-1}(A) = \emptyset$. Then by definition $nec_Y(A) = \mu(X)$. Since $X \in PN$ $\cup_{\mu(E_s)<\mu(X)}E_s = X$ so $\left(\cup_{\mu(E_s)<r}E_s\right)^c = \emptyset$ and the original statement holds.
Now assume $f^{-1}(A) \neq \emptyset$ and $nec_Y(A) = r$. Then $\forall y \in A$, $n_Y(y) \geq r$. But $n_Y(y) = \inf\{n_X(x) \mid f(x) = y\}$ which implies that $\forall x \in f^{-1}(A)$ $n_X(x) = nec_X(\{x\}^c) \geq r$. Thus $\sup\{\mu(E_s) \mid E_s \subseteq \{x\}^c\} = \sup\{\mu(E_s) \mid x \notin E_s\} \geq r$. Suppose $x \notin \left(\cup_{\mu(E_s)<r}E_s\right)^c$, i.e.$\forall E_t$ such that $x \in E_t$ then $\mu(E_t) < r$. Then $x \in E_q$ for all q such that $\mu(E_q) > \mu(E_t)$ since PN is nested. But this along with the fact that $\emptyset \in PN$ implies $n(x) \leq \mu(E_s) < r$ which is a contradiction. Thus $x \in \left(\cup_{\mu(E_s)<r}E_s\right)^c$ and so $f^{-1}(A) \cap \left(\cup_{\mu(E_s)<r}E_s\right) = \emptyset$.□

The idea is that we may be able to construct *pos* and *nec* quite readily and use these to estimate the induced measure on the range space, ν. In the next theorem we show that we can construct all the possibility and necessity distributions on Y consistent with ν we need (i.e. sufficient to determine ν)

from possibility and necessity distributions on X consistent with μ using the extension principle.

Theorem 7 *Let $(X,\mathcal{L},\mu)$ be a measure space and $(Y,\mathcal{O})$ a measurable space. Let $f : X \to Y$ be an $\mathcal{L}$-measurable function and let ν be the measure on Y defined by $\nu(E) = \mu(f^{-1}(E))$ $\forall E \in \mathcal{O}$. Let pos_Y and nec_Y be possibility and necessity measures respectively that are consistent with ν with possibility distribution function p_Y and necessity distribution function n_Y for which the strong s-cuts, p_Y^{s+} and n_Y^{s+} are measurable. Then the functions $p_X, n_X : X \to R_\infty$ defined by $p_X(x) = p_Y(f(x))$ and $n_X(x) = n_Y(f(x))$ are possibility and necessity distribution functions for a possibility measure, pos_X and a necessity measure nec_X, consistent with μ where p_Y and n_Y are the possibility and necessity distribution functions produced from p_X and n_X by the extension principle.*

Proof. Proof for *pos*.
First, note that from our definition $pos_X(A) = \sup\{p_X(x) \mid x \in A\} = \sup\{p_Y(f(x)) \mid x \in A\} = \sup\{p_Y(y) \mid y \in f(A)\} = pos_Y(f(A))$.
(1) Let $E \in \mathcal{L}$. Then $f(E) \subseteq \left(p_Y^{pos_X(E)+}\right)^c$ since otherwise $\exists x \in E$ for which $f(x) \in p_Y^{pos_X(E)+}$. But this would mean that $p_Y(f(x)) = p_X(x) > pos_X(E) = pos_Y(f(E))$ which is a contradiction. By assumption $p_Y^{pos_X(E)+}$, and hence $\left(p_Y^{pos_X(E)+}\right)^c$ is measurable and then so is $f^{-1}\left(\left(p_Y^{pos_X(E)+}\right)^c\right)$. Since $E \subseteq f^{-1}(f(E)) \subseteq f^{-1}\left(\left(p_Y^{pos_X(E)+}\right)^c\right)$ we have
$\mu(E) \le \mu\left(f^{-1}\left(\left(p_Y^{pos_X(E)+}\right)^c\right)\right) = \upsilon\left(\left(p_Y^{pos_X(E)+}\right)^c\right) \le pos_Y\left(\left(p_Y^{pos_X(E)+}\right)^c\right)$.
We claim that $pos_Y\left(\left(p_Y^{pos_X(E)+}\right)^c\right) = pos_X(E)$. First assume that $E \ne \emptyset$. Then clearly $pos_Y\left(\left(p_Y^{pos_X(E)+}\right)^c\right) \ge pos_X(E)$ since for each $x \in E$ $p_X(x) = p_Y(f(x)) \le pos_X(E)$ so $f(x) \in \left(p_Y^{pos_X(E)+}\right)^c$. On the other hand, by definition, $y \in \left(p_Y^{pos_X(E)+}\right)^c$ implies that $p_Y(y) \le pos_X(E)$ so that $pos_Y\left(\left(p_Y^{pos_X(E)+}\right)^c\right) \le pos_X(E)$. Now assume that $E = \emptyset$. Then $pos_X(E) = 0$ and $\left(p_Y^{pos_X(E)+}\right)^c = \{y \mid p_Y(y) = 0\}$ (which may be empty) giving $pos_Y\left(\left(p_Y^{pos_X(E)+}\right)^c\right) = 0$.
(2) $pos_X(\cup_{\gamma\in\Lambda} A_\gamma) = pos_Y(f(\cup_{\gamma\in\Lambda} A_\gamma)) = pos_Y(\cup_{\gamma\in\Lambda} f(A_\gamma)) =$

$\sup\{pos_Y(f(A_\gamma)) \mid \gamma \in \Lambda\} = \sup\{pos_X(A_\gamma) \mid \gamma \in \Lambda\}$.
(3) $pos_X(X) = pos_X(f^{-1}(Y)) = pos_Y(Y) = \nu(Y) = \mu(f^{-1}(Y)) = \mu(X)$ and $pos_X(\emptyset) = pos_X(f^{-1}(\emptyset)) = pos_Y(\emptyset) = 0$.
Let p be the possibility distribution on Y constructed from p_X via the extension principle. Then $p(y) = \sup\{p_X(x) \mid f(x) = y\} = p_Y(y)$ by definition of p_X.
The proof for *nec* is very similar.□

Next we consider product measures over the product of a finite number of measure spaces.

Theorem 8 *Let $\left(X_{i=1}^N X_i, \mathcal{L}, \mu\right)$ be a product measure space. Let PN, having the same properties as in Theorem 2, be of the form* $PN = \left\{X_{i=1}^N E_r^i \mid r \in S \text{ and } E_r^i \subseteq X_i\right\}$ *so that* $\mu\left(X_{i=1}^N E_r^i\right) = \prod_{i=1}^N \mu_i\left(E_r^i\right)$ *where μ_i is the measure on X_i. For each i define $\pi_i, \tau_i : X_i \to R_\infty$ by* $\pi_i(x_i^*) = \inf\left\{\mu\left(X_{i=1}^N E_r^i\right) \mid r \in S \text{ and } x_i^* \in E_r^i\right\}$ *and* $\tau_i(x_i^*) = \sup\left\{\mu\left(X_{i=1}^N E_r^i\right) \mid r \in S \text{ and } x_i^* \notin E_r^i\right\}$. *If $f : X_{i=1}^N X_i \to Y$ is measurable then* $p_Y(y) = \sup\{\max\{\pi_i(x_i) \mid i = 1 \text{ to } N\} \mid f(x_1, ..., x_N) = y\}$ *is a possibility distribution function consistent with the measure ν on Y where $\nu(E) = \mu(f^{-1}(E))$ and* $n_Y(y) = \inf\{\min\{\tau_i(x_i) \mid i = 1 \text{ to } N\} \mid f(x_1, ..., x_N) = y\}$ *is a necessity distribution function consistent with ν.*

Proof. From Theorem 5 it is sufficient to show that the function $p : X_{i=1}^N X_i \to Y$ defined by $p(x_1, ..., x_N) = \max\{\pi_i(x_i) \mid i = 1 \text{ to } N\}$ is the possibility distribution function associated with PN and that the function $n(x_1, ..., x_N) = \min\{\tau_i(x_i) \mid i = 1 \text{ to } N\}$ is the necessity distribution function associated with PN. From Theorem 2
$p(x) = \inf\left\{\mu\left(X_{i=1}^N E_r^i\right) \mid x \in X_{i=1}^N E_r^i\right\} =$
$\max\left\{\inf\left\{\mu\left(X_{i=1}^N E_r^i\right) \mid x_i \in E_r^i\right\} \mid i = 1 \text{ to } N\right\} = \max\{\pi_i(x_i) \mid i = 1 \text{ to } N\}$.
The identity
$\inf\left\{\mu\left(X_{i=1}^N E_r^i\right) \mid x \in X_{i=1}^N E_r^i\right\} =$
$\max\left\{\inf\left\{\mu\left(X_{i=1}^N E_r^i\right) \mid x_i \in E_r^i\right\} \mid i = 1 \text{ to } N\right\}$ follows from the nested property of PN. Next, $n(x) = \sup\left\{\mu\left(X_{i=1}^N E_r^i\right) \mid x \notin X_{i=1}^N E_r^i\right\} =$
$\min\left\{\sup\left\{\mu\left(X_{i=1}^N E_r^i\right) \mid x_i \notin E_r^i\right\} \mid i = 1 \text{ to } N\right\} = \min\{\tau_i(x_i) \mid i = 1 \text{ to } N\}$.
The identity
$\sup\left\{\mu\left(X_{i=1}^N E_r^i\right) \mid x \notin X_{i=1}^N E_r^i\right\} =$
$\min\left\{\sup\left\{\mu\left(X_{i=1}^N E_r^i\right) \mid x_i \notin E_r^i\right\} \mid i = 1 \text{ to } N\right\}$ also follows from the nested property of PN. □

Example 4 *The purpose of this example is to illustrate the construction of consistent possibility and necessity measures using Theorem 2 and the extension of these measures using Theorems 5 and 8. Let* $f : [0,1] \times [0,1] \to [0,1]$ *by* $f(x,y) = x \cdot y$ *and let* $[0,1] \times [0,1]$ *have the usual Lebesgue measure. Now consider the family of sets* $\{[(1-\alpha),1] \times [(1-\alpha),1] \mid \alpha \in [0,1]\}$ *each with measure* α^2*. These define a possibility distribution function* $p_X : [0,1] \times [0,1] \to [0,1]$ *where* $p_X(x,y) = \inf\{\alpha^2 \mid (x,y) \in [(1-\alpha),1] \times [(1-\alpha),1]\}$ *which occurs if* $1-\alpha = \min\{x,y\}$*, thus* $p_X(x,y) = (1-\min\{x,y\})^2$*. Then* $p_Y(z) = \sup\left\{(1-\min\{x,y\})^2 \mid x \cdot y = z\right\}$*. To maximize* $(1-\min\{x,y\})^2$ *we need to minimize* $\min\{x,y\}$ *which occurs at* $(1,z)$ *and* $(z,1)$*. Thus* $p_Y(z) = (1-z)^2$*.*
Alternatively, using Theorem 8,
$\pi_1(x) = \inf\{\alpha^2 \mid (x,y) \in [(1-\alpha),1] \times [(1-\alpha),1]\} = (1-x)^2$*. Similarly* $\pi_2(y) = (1-y)^2$*.*
Then $p_Y(z) = \sup\left\{\max\left\{(1-x)^2, (1-y)^2\right\} \mid x \cdot y = z\right\} = (1-z)^2$*.*
This family of sets also define a necessity distribution function $n_X : [0,1] \times [0,1] \to [0,1]$ *where*
$n_X(x,y) = \sup\{\alpha^2 \mid [(1-\alpha),1] \times [(1-\alpha),1] \subseteq [0,1] - (x,y)\}$ *which occurs if* $1-\alpha = \min\{x,y\}$*, thus* $n_X(x,y) = (1-\min\{x,y\})^2$*. Then*
$n_Y(z) = \inf\left\{(1-\min\{x,y\})^2 \mid x \cdot y = z\right\}$*. To minimize* $(1-\min\{x,y\})^2$ *we need to maximize* $\min\{x,y\}$ *which occurs at* $(\sqrt{z},\sqrt{z})$*. Thus* $n_Y(z) = (1-\sqrt{z})^2$*.*
Alternatively $\tau_1(x) = \sup\{\alpha^2 \mid x \notin [(1-\alpha),1]\} = (1-x)^2$ *and similarly* $\tau_2(y) = (1-y)^2$*. Then* $n_Y(z) = \inf\left\{\min\left\{(1-x)^2, (1-y)^2\right\} \mid x \cdot y = z\right\} = (1-\sqrt{z})^2$*.*
Consider $[a,b] \subseteq [0,1]$*.*
The Lebesgue measure of $[a,b]$ *is*

$$\nu([a,b]) = \int_b^1 \left(\frac{b}{x} - \frac{a}{x}\right) dx + \int_a^b \left(1 - \frac{a}{x}\right) dx = b - a + a \ln a - b \ln b$$

while

$$pos_Y([a,b]) = \sup\{p_Y(z) \mid z \in [a,b]\} = (1-a)^2$$

and

$$nec_Y([a,b]) = \inf\{n_Y(z) \mid z \in [0,1] - [a,b]\} = \begin{cases} 0 & \text{if } b \neq 1 \\ (1-\sqrt{a})^2 & \text{otherwise} \end{cases}.$$

For example $\nu([.75,1]) = .0342$ *and* $pos_Y([.75,1]) = (1-.75)^2 = .0625$ *while* $nec_Y([.75,1]) = \left(1-\sqrt{.75}\right)^2 = .0179$*, demonstrating consistency.*

4 An Application

In this section we illustrate the use of consistent possibility and necessity measures as a means of approximating an expected value. Let $Z = f(X_1, ..., X_N)$ where the X_i's are continuous independent random variables. Assume that the support of each X_i is defined on a closed interval of real numbers,supp$(X_i) = [b_i, c_i]$ and that f is monotone increasing in each X_i. Note that since we are working with probability measures, i.e. a finite measure taking values on the unit interval, we use α-cuts and necessity measures are duals of possibility measures.

In [11] we show that

$$^R\mu_{X_i}(x) = 1-(1-F_{X_i}(x))^N \text{ and } {}^L\mu_{X_i}(x) = 1-F_{X_i}(x)^N$$

are possibility distributions for each X_i that, when extended to Z, produce possibility distributions with corresponding possibility measures that are consistent with the probability measure on Z. Here F_{X_i} denotes the cumulative distribution function corresponding to the random variable X_i. These are the possibility measures constructed from the families of sets

$$^RPN = \left\{E_r = \left(X_{i=1}^N\left(c_i - r(c_i-b_i), c_i\right]\right)^c \mid r \in [0,1]\right\}$$

and

$$^LPN = \left\{E_r = \left(X_{i=1}^N\left[b_i, b_i + r(c_i-b_i)\right)\right)^c \mid r \in [0,1]\right\}$$

and utilize the independence assumption. The superscript R and L are for right and left, from the fact that the α-cuts of these possibility distributions are fixed at the right and left endpoints respectively of the supports of each X_i. The α-cuts of the possibility distributions extended to Z are constructed as follows.

We first calculate the α-cuts for each distribution function $^R\mu_{X_i}(x)$ which we will denote $^R(X_i)_\alpha$. Let $\alpha = 1-(1-F_{X_i}(x))^N$ then $x = F_{X_i}^{-1}\left(1-(1-\alpha)^{\frac{1}{N}}\right)$ which is increasing in α giving

$$^R(X_i)_\alpha = \left[F_{X_i}^{-1}\left(1-(1-\alpha)^{\frac{1}{N}}\right), c_i\right].$$

Then, since f is monotone increasing in each X_i, the α-cut for the extended consistent possibility distribution on Z, denoted ${}^R\mu_Z$ is

$$^R Z_\alpha = \left[f\left(F_{X_1}^{-1}\left(1-(1-\alpha)^{\frac{1}{N}}\right), ..., F_{X_N}^{-1}\left(1-(1-\alpha)^{\frac{1}{N}}\right)\right), f\left(c_1, ..., c_N\right)\right].$$

Next we calculate the α-cuts for each ${}^L\mu_{X_i}(x)$ denoted ${}^L(X_i)_\alpha$. Let $\alpha = 1 - F_{X_i}(x)^N$ then $x = F_{X_i}^{-1}\left((1-\alpha)^{\frac{1}{N}}\right)$ which is decreasing in α giving

$$^L (X_i)_\alpha = \left[b_i, F_{X_i}^{-1}\left(1-(1-\alpha)^{\frac{1}{N}}\right)\right].$$

Then the α-cut for the extended consistent possibility distribution on Z, denoted ${}^L\mu_Z$ is

$$^L Z_\alpha = \left[f\left(b_1, ..., b_N\right), f\left(F_{X_1}^{-1}\left(1-(1-\alpha)^{\frac{1}{N}}\right), ..., F_{X_N}^{-1}\left(1-(1-\alpha)^{\frac{1}{N}}\right)\right)\right].$$

Let ${}^R pos$ and ${}^L pos$ be the consistent possibility measures associated with ${}^R\mu_Z$ and ${}^L\mu_Z$ respectively. Likewise, let ${}^R nec$ be the necessity measure derived from ${}^R pos$ (see Theorem 1). Then these measures provide a bound on the cumulative distribution function of Z since

$$^R nec\left(\left[f\left(b_1, ..., b_N\right), z\right]\right) \leq F_Z(z) \leq^L pos\left(\left[f\left(b_1, ..., b_N\right), z\right]\right)$$

Moreover, since f is continuous in each X_i we have ${}^R nec\left(\left(z, , f\left(c_1, ..., c_N\right)\right]\right) =^R \mu_Z(z)$ and ${}^L pos\left(\left(x, , f\left(c_1, ..., c_N\right)\right]\right) =^L \mu_Z(z)$, i.e. the endpoints of these intervals. Taken together we get the main result (proven in [11]) that

$$E(Z) \in \left[\int_0^1 \left({}^R Z_\alpha\right) d\alpha, \int_0^1 \left({}^L Z_\alpha\right) d\alpha\right]$$

and a midpoint approximation is

$$EE(Z) = \frac{1}{2}\left(\int_0^1 \left({}^R Z_\alpha\right) d\alpha + \int_0^1 \left({}^L Z_\alpha\right) d\alpha\right).$$

This can be a rather wide approximation. We can improve the approximation by partitioning the domain space and conditioning on the partition. If $\{A_j \mid j = 1, M\}$ is a partition of X then
$E(Z) = \sum_{j=1}^{M} E\left(Z \mid X \in A_j\right) P\left(X \in A_i\right)$ (were $P(E)$ equals the probability

of the event E) which can be approximated by $\sum_{j=1}^{M} EE\left(Z \mid X \in A_j\right) P\left(X \in A_j\right)$. We note that this partitioning approach is closely related to that of R.E. Moore in [19].

As an example let X,Y be independent identically distributed $u\left[0,1\right]$ random variables and let $Z = X^3Y$. We will use the above left and right possibility distributions to estimate $E\left(Z\right)$. The actual value is $E\left(X^3\right) E\left(Y\right) = \left(\frac{1}{4}\right)\left(\frac{1}{2}\right) = .1250$.

In general if X is uniformly distributed on $[b,c]$, the c.d.f. for $x \in [b,c]$ is $F_X\left(x\right) = \frac{x-b}{c-b}$. If there are N independent random variables that are arguments of the function to be analyzed we form right possibility distribution function ${}^R\mu_X\left(x\right) = 1 - \left(1 - \frac{x-b}{c-b}\right)^N$. Setting $\alpha = 1 - \left(1 - \frac{x-b}{c-b}\right)^N$ gives $x = c - \left(c-b\right)\left(1-\alpha\right)^{\frac{1}{N}}$ so the α-cut for ${}^R\mu_X$ is

$$ {}^R X_\alpha = \left[c - \left(c-b\right)\left(1-\alpha\right)^{\frac{1}{N}}, c\right]. $$

We also form the left possibility distribution ${}^L\mu_X\left(x\right) = 1 - \left(\frac{x-b}{c-b}\right)^N$ and setting $\alpha = 1 - \left(\frac{x-b}{c-b}\right)^N$ yields $x = b + \left(c-b\right)\left(1-\alpha\right)^{\frac{1}{N}}$ so

$$ {}^L X_\alpha = \left[b, b + \left(c-b\right)\left(1-\alpha\right)^{\frac{1}{N}}\right]. $$

Now returning to our problem, let $A = [b,c]\text{x}[d,e] \subseteq [0,1]\text{x}[0,1]$ then since Z is monotone increasing in X and Y and $N = 2$ we get the right α-cut for Z to be

$$ {}^R\left(Z \mid A\right)_\alpha = \left[\left(c - \left(c-b\right)\left(1-\alpha\right)^{\frac{1}{2}}\right)^3 \left(e - \left(e-d\right)\left(1-\alpha\right)^{\frac{1}{2}}\right), c^3 e\right] $$

and the left α-cut for Z is

$$ {}^L\left(Z \mid A\right)_\alpha = \left[b^3 d, \left(b + \left(c-b\right)\left(1-\alpha\right)^{\frac{1}{2}}\right)^3 \left(d + \left(e-d\right)\left(1-\alpha\right)^{\frac{1}{2}}\right), b^3 e\right]. $$

Then $E\left(Z \mid A\right)$ is between
$\int_0^1 \left(c - \left(c-b\right)\left(1-\alpha\right)^{\frac{1}{2}}\right)^3 \left(e - \left(e-d\right)\left(1-\alpha\right)^{\frac{1}{2}}\right) d\alpha$ and
$\int_0^1 \left(b + \left(c-b\right)\left(1-\alpha\right)^{\frac{1}{2}}\right)^3 \left(d + \left(e-d\right)\left(1-\alpha\right)^{\frac{1}{2}}\right) d\alpha$ and an estimate of the conditional expected value is the midpoint of this interval.

FIRST APPROXIMATION: Let $A = [0,1]\text{x}[0,1]$. Applying the above formula we find that the expected value is between $\int_0^1 \left(1-(1-\alpha)^{\frac{1}{2}}\right)^4 d\alpha = .0667$ and $\int_0^1 (1-\alpha)^{\frac{1}{2}} d\alpha = .6667$ and a midpoint approximation is $\frac{1}{2}(.0667 + .6667) = .3667$. As pointed out, this is a rather wide approximation.

SECOND APPROXIMATION: To tighten the estimate we partition $[0,1]^2$ into four squares of area (and probability) $\frac{1}{4}$ and define possibility distributions on each piece.

On $A_1 = \left[0, \frac{1}{2}\right]\text{x}\left[0, \frac{1}{2}\right]$,

$E(Z \mid A_1)$ is between $\int_0^1 \left(\frac{1}{2} - \frac{1}{2}(1-\alpha)^{\frac{1}{2}}\right)^4 d\alpha = .0041667$ and

$\int_0^1 \left(\frac{1}{2}\right)^4 (1-\alpha)^2 d\alpha = .020833$

On $A_2 = \left[0, \frac{1}{2}\right]\text{x}\left[\frac{1}{2}, 1\right]$

$E(Z \mid A_2)$ is between $\int_0^1 \left(\frac{1}{2} - \frac{1}{2}(1-\alpha)^{\frac{1}{2}}\right)^3 \left(1 - \frac{1}{2}(1-\alpha)^{\frac{1}{2}}\right) d\alpha = .010417$ and

$\int_0^1 \left(\frac{1}{2}(1-\alpha)^{\frac{1}{2}}\right)^3 \left(\frac{1}{2} + \frac{1}{2}(1-\alpha)^{\frac{1}{2}}\right) d\alpha = .045833$

On $A_3 = \left[\frac{1}{2}, 1\right]\text{x}\left[0, \frac{1}{2}\right]$

$E(Z \mid A_3)$ is between $\int_0^1 \left(1 - \frac{1}{2}(1-\alpha)^{\frac{1}{2}}\right)^3 \left(\frac{1}{2} - \frac{1}{2}(1-\alpha)^{\frac{1}{2}}\right) d\alpha = .075$ and

$\int_0^1 \left(\frac{1}{2} + \frac{1}{2}(1-\alpha)^{\frac{1}{2}}\right)^3 \left(\frac{1}{2}(1-\alpha)^{\frac{1}{2}}\right) d\alpha = .23125.$

On $A_4 = \left[\frac{1}{2}, 1\right]\text{x}\left[\frac{1}{2}, 1\right]$

$E(Z \mid A_4)$ is between $\int_0^1 \left(1 - \frac{1}{2}(1-\alpha)^{\frac{1}{2}}\right)^4 d\alpha = .2375$ and

$\int_0^1 \left(\frac{1}{2} + \frac{1}{2}(1-\alpha)^{\frac{1}{2}}\right)^4 d\alpha = .5375.$

Then $E(Z)$ is between $\frac{1}{4}(.0041667 + .010417 + .075 + .2375) = .081771$ and $\frac{1}{4}(.020833 + .045833 + .23125 + .5375) = .20885$

The midpoint estimate is $\frac{1}{2}(.081771 + .20885) = .14531$. This is a significant improvement.

5 Conclusion

In summary, we have shown how consistent possibility and necessity measures can be constructed from nested families of measurable sets and that these fuzzy measures provide upper and lower bounds on the given measure. Using the distribution functions for these fuzzy measures, the extension principle produces distribution functions for possibility and necessity measures that bound the measure of sets in the range space (when the measure is the one induced from the domain space). We showed that the measure on the range space can be completely determined by extending consistent possibility and necessity distributions on the domain. An area of further research is to show that these distributions are useful in approximation theory and in optimization problems.

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