Constructing Consistent Fuzzy Surfaces From Fuzzy Data

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Abstract

Given fuzzy data describing a 3-dimensional entity such as terrain, we develop methods to construct surfaces that are consistent with the uncertainty in the data and surface model itself. In particular, surfaces generated from higher α-cut values of the fuzzy data are contained within the surfaces generated by lower α-cut values of the fuzzy data. Moreover, the smoothness and continuity conditions of the surface generating method is maintained by each level surface. We demonstrate the ideas by developing two and three dimensional surfaces from fuzzy data for cubic splines and *digital terrain models* (DTM's) generated from triangulation.

Key words: Fuzzy numbers, Fuzzy interpolation, fuzzy surface, fuzzy DTM

1 Introduction

It often occurs that interpolation data are not sets of real numbers but are ranges of values whose distribution within the range may not be probabilistic but known possibilistically or qualitatively. For example, in mapping the ocean floor, the depths are obtained via sonar measurements. The vessel that is sending and receiving the sound is subject to movement, the sonar models are themselves approximations, and the instruments sending and receiving have finite precision. Moreover, the ocean itself is not a homogeneous medium. Perhaps, taking account of all of these factors a probabilistic model could be developed, but it would be difficult. Nevertheless, accounting of the inherent uncertainty associated with the data is desired. We are especially interested in DTM's that account for its inherent uncertainty. Thus, the focus of our examples are made with DTM's in mind.

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We make a distinction between error and uncertainty as follows. Error assumes that a true value exists. Uncertainty denotes incomplete knowledge that is characterized by whether or not one can say that a proposition is exclusively true or false. A statement is uncertain when its (exclusive) truth or falseness cannot be ascertained. Uncertainty can be modeled using fuzzy set theory and is used here.

We develop methods that allow for the construction of fuzzy surfaces from fuzzy data sets that are consistent and maintain the underlying smoothness of the interpolation method being used to generate the surface. We are assuming that the latitude and longitude can be measured to desired accuracy; that is, the coordinate (x, y) is accurate to as many digits as needed. All of the uncertainty is in the altitude, z, component and this uncertainty is a fuzzy number. This means that its membership function is an upper-semi continuous concave function whose core is non-empty (often a singleton). Thus, all alpha-levels are (closed and bounded) intervals.

Let $D = \{(x, y, z) \mid a_1 \leq x \leq b_1, a_2 \leq y \leq b_2, a_3 \leq z \leq b_3,\ a_i, b_i \in \mathbb{R}$ and $(x, y, z) \in \mathbb{R}^3\}$ be the domain (region) of interest. By **consistent fuzzy surfaces** generated from a fuzzy point data sets $\{(x_i, y_i, \tilde{z}_i), i = 0, 1, ..., N\} \subseteq D$ we mean real-valued surfaces that satisfy two properties:

(1) Surfaces with larger α-cut values are contained within the surfaces with lower α-cut values. That is, let $\alpha_1 \leq \alpha_2$. This gives us four data sets (unless $\alpha_2 = 1$ in which case we get three surfaces since the core is a singleton by the definition of a fuzzy number):

$$\begin{aligned}
&\{(x_i, y_i, z^-_{i\alpha 1})\},\\
&\{(x_i, y_i, z^+_{i\alpha 1})\},\\
&\{(x_i, y_i, z^-_{i\alpha 2})\}, \text{ and}\\
&\{(x_i, y_i, z^+_{i\alpha 2})\}.
\end{aligned}$$

Let $S^-_{\alpha 1}(x, y, z)$ and $S^+_{\alpha 1}(x, y, z)$ be the surfaces generated by $\{(x_i, y_i, z^-_{i\alpha 1})\}$ and $\{(x_i, y_i, z^+_{i\alpha 1})\}$ respectively and $S^-_{\alpha 2}(x, y, z)$ and $S^+_{\alpha 2}(x, y, z)$ be the surfaces generated by $\{(x_i, y_i, z^-_{i\alpha 2})\}$ and $\{(x_i, y_i, z^+_{i\alpha 2})\}$. Then

$$S^-_{\alpha 1}(x, y, z) \leq S^-_{\alpha 2}(x, y, z) \leq S^+_{\alpha 2}(x, y, z) \leq S^+_{\alpha 1}(x, y, z).$$

(2) All generated surfaces $S^{-/+}_{\alpha}(x, y, z)$ possess the underlying smoothness and continuity conditions associated with the interpolation method being use.

Remark 1 *Property (1) might be considered as the definition of what is meant by "fuzzy polynomial." However, since we are generating surfaces (in-*

*terpolating polynomials) that in the past have not guaranteed consistency, we explicitly impose these properties. In particularly, property (1) says that the α-cuts are nested. This is an important property to maintain since if a pair of surfaces are likely to the degree α, then any surface it encloses has a **higher** (or equal) likelihood of occurring and not a lower likelihood. This is consistent with the definition of membership functions which requires it to be non-decreasing to the left of the core and non-increasing to the right of the core. Moreover, property (2) maintains that the generated fuzzy surfaces are consistent with the underlying properties of the generator. That is, if splines are used, then the generated fuzzy surfaces inherit the properties associated with the spline methods.*

It is noted that [4] fails with respect to (2) for all splines of degree greater than one. Moreover, their upper and lower functions associated with Lagrange interpolation are not Lagrange polynomials. While fuzzy B-splines constructed according to [1] are contained within the convex hull of their control points, there is no guarantee that it satisfies property (1). Our approach is one that generates consistent fuzzy surfaces.

2 Notation

We use the following notation:

I_N is the index set $\{0, 1, ..., N\}$,

$[z]_\alpha = [z_\alpha^-, z_\alpha^+]$, is an α-cut of fuzzy number $\tilde{z}$,

$(z^-/z^1/z^+)$ is the triangular fuzzy number with support $[z^-, z^+]$ and modal value z^1,

$\mathcal{F}(\mathbb{R})$ is the space of real fuzzy numbers (upper semi-continuous concave membership functions with non-empty core).

3 The model

The models that we consider are derived from inputs that are finite sets of data describing a geographic entity $\{(\mathbf{x}_i, \tilde{z}_i), i \in I_N\}$, where $\mathbf{x}_i \in \mathbb{R}$ or $\mathbf{x}_i \in \mathbb{R}^2$. The output will be an interpolation polynomial that approximates $\tilde{f}$. Positions $\mathbf{x}_i$ and attributes $\tilde{\mathbf{z}}_i$ are related by the function $\tilde{f}$ with

$$\tilde{z}_i = \tilde{f}(\mathbf{x}_i), \text{ for all } i \text{ in } I_N.$$

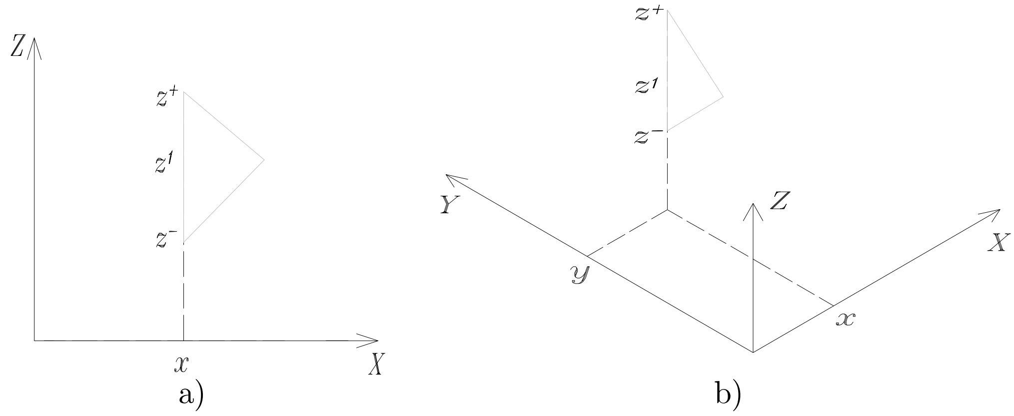

Fig. 1. Points with fuzzy $\tilde{z}$ triangular coordinate, $\tilde{z} = (z^-/z^1/z^+)$: a) two dimensional fuzzy point $(x, \tilde{z})$; b) three dimensional fuzzy point $(x, y, \tilde{z})$.

Only the univariate, $\tilde{z} = \tilde{f}(x)$, and bivariate, $\tilde{z} = \tilde{f}(x, y)$, cases will be considered , with one dimensional fuzzy attribute $\tilde{z} \in \mathcal{F}(\mathbb{R})$. Construction of consistent $\tilde{f}$ is our aim. It is assumed that the reader is familiar with the basic definitions, properties, algebraic operations, metrics of fuzzy intervals (see [6], [2]).

A *fuzzy point* $\tilde{P}$ will be defined by and ordered set $(x, \tilde{z})$ or $(x, y, \tilde{z})$. In figure 1 we can see a graphic representation of a fuzzy point where the fuzzy coordinate $\tilde{z}$ is a triangular fuzzy number

$$\tilde{z} = (z^-/z^1/z^+), \text{ with } z^- \leq z^1 \leq z^+ \in \mathbb{R}.$$

3.1 Construction of consistent fuzzy surfaces: The univariate case

Kaleva[4] and Lowen[5] have developed fuzzy interpolation of univariate functions $\tilde{z} = \tilde{f}(x)$ applying the *extension principle* to the real-valued Lagrange and spline interpolation polynomials. Thus, given a partition $\Delta : a = x_0 < x_1 < ... < x_N = b$, where every x_i has an associated fuzzy number $\tilde{z}_i \in \mathcal{F}(\mathbb{R})$, they find a polynomial $\tilde{p} : \mathbb{R} \to \mathcal{F}(\mathbb{R})$, such that $\tilde{p}(x_i) = \tilde{z}_i$ for all i in I_N. Kaleva used the concept of fuzzy interpolation polynomial $\widetilde{p}(x)$ given by Lowen. Representing $\widetilde{p}(x)$ using α-cuts, we can write

$$[p(x)]_\alpha \equiv [p^-_\alpha(x), p^+_\alpha(x)] = \{z \in \mathbb{R} : z = p_{d\alpha}(x), d_{i\alpha} \in [z_i]_\alpha, \alpha \in I\},$$

where $p_{d\alpha}(x)$ is an interpolation polynomial such that $p_{d\alpha}(x_i) = d_{i\alpha}$ for the same partition Δ.

3.1.1 Fuzzy Lagrange interpolation

Using Lagrange polynomials base

$$L_i(x) \equiv \prod_{\substack{j=0 \\ j \neq i}}^{N} \frac{(x - x_j)}{(x_i - x_j)}, i \in I_N,$$

the fuzzy interpolation polynomial given by Lowen[5] is written in the form

$$[p(x)]_\alpha = \sum_{i=0}^{N} [z_i]_\alpha L_i(x), \alpha \in I.$$

The upper limit $p^+_\alpha(x)$ of the interval $[p(x)]_\alpha$ is the solution of the optimization problem

$$\max_{z^-_{i\alpha} \leq d_{i\alpha} \leq z^+_{i\alpha}} p_{d\alpha}(x)$$

where $[z_i]_\alpha$ is denoted by $[z^-_{i\alpha}, z^+_{i\alpha}]$.

Since for all $x \in (x_j, x_{j+1})$, the sign of $L_i(x)$,

$$sign(L_i(x)) = \begin{cases} (-1)^{i-j-1} & \text{for } 1 \leq j \leq i-1 \\ (-1)^{j-i} & \text{for } i \leq j \leq N-1 \end{cases},$$

is constant, the optimal value is reached when

$$d^+_{i\alpha} = \begin{cases} z^+_{i\alpha} \text{ if } L_i(x) \geq 0 \\ z^-_{i\alpha} \text{ if } L_i(x) < 0 \end{cases}.$$

The similar can be done for $p^-_\alpha(x)$, getting

$$d^-_{i\alpha} = \begin{cases} z^-_{i\alpha} \text{ if } L_i(x) \geq 0 \\ z^+_{i\alpha} \text{ if } L_i(x) < 0 \end{cases}.$$

Using (1), we have a simple method of obtaining $\tilde{p}(x)$.

$$p_\alpha^-(x) = \sum_{L_i(x)\geq 0} z_{i\alpha}^- L_i(x) + \sum_{L_i(x)<0} z_{i\alpha}^+ L_i(x), \tag{1}$$

$$p_\alpha^+(x) = \sum_{L_i(x)\geq 0} z_{i\alpha}^+ L_i(x) + \sum_{L_i(x)<0} z_{i\alpha}^- L_i(x) \tag{2}$$

Taking the particular case of triangular fuzzy numbers, given the fuzzy data points $\{(x_i, \tilde{z}_i)\}$, with $\tilde{z}_i = (z_i^-/z_i^1/z_i^+)$, we can get $\tilde{p}(x) = (p^-(x)/p^1(x)/p^+(x))$ by using

$$\begin{aligned} p^-(x) &= \sum_{L_i(x)\geq 0} z_i^- L_i(x) + \sum_{L_i(x)<0} z_i^+ L_i(x), \\ p^1(x) &= \sum_{i=0}^{N} z_i L_i(x), \\ p^+(x) &= \sum_{L_i(x)\geq 0} z_i^+ L_i(x) + \sum_{L_i(x)<0} z_i^- L_i(x). \end{aligned} \tag{3}$$

It is important to realize that one difference between real-valued Lagrange polynomial and the generated interpolation fuzzy-valued interpolation function (3) is that the fuzzy interpolating function looses smoothness at the knots at every α-cut. Only when the core is a singleton do the correspondent real-valued smoothness properties hold, like in previous case for $p^1(x)$. Moreover, $p^-(x)$ and $p^+(x)$ are not Lagrange polynomials while $p^1(x)$ is. Therefore, the fuzzy-valued polynomial obtained satisfies property (1) but not property (2), which gives rise to the following proposition:

Proposition 1: Let $[p(x)]_\alpha \equiv [p_\alpha^-(x), p_\alpha^+(x)]$ be an α-cut of fuzzy-valued polynomial given by (1).The real-valued functions $p_\alpha^-(x)$ and $p_\alpha^+(x)$ have discontinuous first derivatives on the nodes of partition Δ.

Proof: Taking the first derivative of $p_\alpha^-(x)$ we have

$$\frac{d}{dx}p_\alpha^-(x) = \sum_{L_i(x)\geq 0} z_{i\alpha}^- \frac{d}{dx}L_i(x) + \sum_{L_i(x)<0} z_{i\alpha}^+ \frac{d}{dx}L_i(x).$$

Evaluating limits for the left polynomial, $p^-(x)$, as $x \to {x_k}^-$ (limit from the left) and comparing this value to that obtained from $x \to {x_k}^+$ (limit from the right) at any node x_k we get

$$\lim x \to {x_k}^- \{\frac{d}{dx}p_\alpha^-(x) = \sum_{L_i(x_k^-)\geq 0} z_{i\alpha}^- \frac{d}{dx}L_i(x) + \sum_{L_i(x_k^-)<0} z_{i\alpha}^+ \frac{d}{dx}L_i(x)\}, \tag{4}$$

$$\lim x \to {x_k}^{+}\{\frac{d}{dx}p_\alpha^-(x) = \sum_{L_i(x_k^+)\geq 0} z_{i\alpha}^-\frac{d}{dx}L_i(x) + \sum_{L_i(x_k^+)<0} z_{i\alpha}^+\frac{d}{dx}L_i(x)\}. \quad (5)$$

Since the sign of $L_i(x)$ changes in succeeding intervals (x_j, x_{j+1}) and (x_{j+1}, x_{j+2}) for every $j \in I_{N-2}\backslash\{k-1\}$, this implies that sign of $L_i(x_k^-)$ is different from sign of $L_i(x_k^+)$ for every i different from k. This, in turn, means that the sums in (4) are not equal to (5). Thus,

$$\frac{d}{dx}p_\alpha^-(x_k^-) \neq \frac{d}{dx}p_\alpha^-(x_k^+).$$

The same is true for $p_\alpha^+(x)$.□

The meaning of proposition 1 is that the fuzzy Lagrange interpolating polynomial construction of Kaleva[4] generates breaks in $p_\alpha^-(x)$ and $p_\alpha^+(x)$ at each nodes.

3.1.2 Fuzzy spline interpolation

As in the case of Lagrange interpolation, we induce a fuzzy-valued spline $\tilde{s}(x)$ of order k , again using the α-cut approach,

$$[s(x)]_\alpha = [s_\alpha^-(x), s_\alpha^+(x)] = \{z \in \mathbb{R} : z = s_{d\alpha}(x), d_{i\alpha} \in [z_i]_\alpha, \alpha \in I\},$$

where $s_{d\alpha}(x)$ is the spline of order k that interpolates the data points $\{(x_i, d_{i\alpha}), i \in I_N\}$.

If we define $\phi_i(x)$ as the spline of order k that interpolates the data $\{(x_j, \delta_{ij}),$ $j \in I_N\}$, where $\delta_{ij} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}$, then $s_{d\alpha}(x) = \sum_{i=0}^N d_{i\alpha}\phi_i(x)$ and

$$[s(x)]_\alpha = \sum_{i=0}^N [z_i]_\alpha \phi_i(x), \alpha \in I.$$

If $k = 2$ we obtain the piecewise linear interpolator. Since in this case $\phi_i(x)$ is always non-negative, the fuzzy spline is easy to construct for triangular fuzzy numbers. In particular,

$$\begin{aligned} s^-(x) &= z_i^-\phi_i(x) + z_{i+1}^-\phi_{i+1}(x), \\ s^1(x) &= z_i^1\phi_i(x) + z_{i+1}^1\phi_{i+1}(x), \\ s^+(x) &= z_i^+\phi_i(x) + z_{i+1}^+\phi_{i+1}(x), \end{aligned} \quad (6)$$

where $\phi_i(x) = \frac{(x_{i+1}-x)}{(x_{i+1}-x_i)}$ and $\phi_{i+1} = \frac{(x-x_i)}{(x_{i+1}-x_i)}$, $x \in [x_i, x_{i+1}]$.

It is trivial to see that fuzzy-valued linear splines satisfy both properties (1) and (2) for all α-cuts.

For $k = 4$ we have the case of cubic spline. Using the same procedure, the upper end $s_\alpha^+(x)$ of the resulting interval $[s(x)]_\alpha$ is determined choosing

$$d_{i\alpha} = \begin{cases} z_{i\alpha}^+ \text{ if } \phi_i(x) \geq 0 \\ z_{i\alpha}^- \text{ if } \phi_i(x) < 0 \end{cases},$$

and the lower $s_\alpha^-(x)$ taking

$$d_{i\alpha} = \begin{cases} z_{i\alpha}^- \text{ if } \phi_i(x) \geq 0 \\ z_{i\alpha}^+ \text{ if } \phi_i(x) < 0 \end{cases}.$$

The above is valid if we guarantee that $\phi_i(x)$ has the same sign for every interval (x_j, x_{j+1}). If we use the *not-a-knot* boundary conditions to evaluate $\phi_i(x)$, this condition is fulfilled (see page 68 of [4]). So, we can write

$$s_\alpha^-(x) = \sum_{\phi_i(x)\geq 0} z_{i\alpha}^- \phi_i(x) + \sum_{\phi_i(x)<0} z_{i\alpha}^+ \phi_i(x), \tag{7}$$

$$s_\alpha^+(x) = \sum_{\phi_i(x)\geq 0} z_{i\alpha}^+ \phi_i(x) + \sum_{\phi_i(x)<0} z_{i\alpha}^- \phi_i(x) \tag{8}$$

Restricting fuzzy numbers to triangular form, to obtain $\tilde{s}(x) = (s^-(x)/s^1(x)/s^+(x))$ from data points $\{(x_i, \tilde{z}_i)\}$, with $\tilde{z}_i = (z_i^-/z_i^1/z_i^+)$, $i \in I_N$, we obtain

$$\begin{aligned} s^-(x) &= \sum_{\phi_i(x)\geq 0} z_i^- \phi_i(x) + \sum_{\phi_i(x)<0} z_i^+ \phi_i(x), \\ s^1(x) &= \sum_{i=0}^{N} z_i^1 \phi_i(x), \\ s^+(x) &= \sum_{\phi_i(x)\geq 0} z_i^+ \phi_i(x) + \sum_{\phi_i(x)<0} z_i^- \phi_i(x). \end{aligned} \tag{9}$$

For quadratic and higher order splines, only $s^1(x)$ possesses the smoothness and continuity properties of splines. For all the other α-cuts, $s_\alpha^-(x)$ and $s_\alpha^+(x)$ have discontinuous derivatives at nodes.

The following proposition holds.

Proposition 2: Let $[s(x)]_\alpha \equiv [s_\alpha^-(x), s_\alpha^+(x)]$ be an α-cut of fuzzy-valued polynomial given by (6).The real-valued functions $s_\alpha^-(x)$ and $s_\alpha^+(x)$ have discontinuous first derivatives on the nodes of partition Δ.

Proof: The proof is similar to that of proposition 1, since the sign of $\phi_i(x)$ changes in the same way of sign of $L_i(x)$.□

3.1.3 Maintaining smoothness on fuzzy univariate interpolators

Since interpolators do not keep the same useful characteristics of the real-valued ones from which they were derived by applying the *extension principle* we develop an approach that remedies this.

Let $\tilde{q}(x)$ be an interpolator of data $\{(x_i, \tilde{z}_i), i \in I_N\}$ derived from a real-valued one $q(x)$ through the *extension principle*; for example, $q(x)$ can be a fuzzy-valued Lagrange polynomial $\tilde{p}(x)$, a fuzzy-valued spline $\tilde{s}$ or any other interpolator such that $\tilde{q}(x_i) = \tilde{z}_i$, for every i in I_N. An obvious idea to keep the original properties of $q(x)$ is to approximate any α-cut of the fuzzy interpolator $\tilde{q}(x)$ by the interval

$$I_{q_\alpha} \equiv [q_{z_\alpha^-}(x), q_{z_\alpha^+}(x)],$$

where $q_{z_\alpha^-}(x)$ and $q_{z_\alpha^+}(x)$ are real-valued polynomials of the same type of $q(x)$ and interpolating $\{(x_i, z_{i\alpha}^-)\}$ and $\{(x_i, z_{i\alpha}^+)\}$,respectively.

The problem is that the width of the interval I_{q_α} can be large/small, with respect the distance measure compared to the width of $[q(x)]_\alpha$. This can result in oscillations with large amplitudes that sometimes cause condition $q_{z_\alpha^-}(x) \leq q_{z_\alpha^+}(x)$ to fail for some x in the interpolation interval thus violating the property (1) of consistent fuzzy surfaces (see figure 3).

To explain our idea, lets start by using the case of Kaleva Lagrange interpolation with triangular fuzzy numbers expressed by (3). Taking for example $p^+(x)$, the propose is to approximate it within the interpolation interval $[a, b]$ with a Lagrange polynomial $p_{\zeta^+}(x)$ such that $p_{\zeta^+}(x) \geq p^+(x)$ for all x in $[a, b]$, to guarantee property (1). Property (2) is obtained when a Lagrange polynomial is chosen to approximate $p^+(x)$. That polynomial, $p_{\zeta^+}(x)$, is defined by $p_{\zeta^+}(x'_i) = \zeta_i^+$, for all i in I_N. To improve the approximation, the partition $\Delta' : a = x'_0 < x'_1 < ... < x'_N = b$ can be different from the original Δ . The number N of nodes must be the same order of Lagrange polynomial but for splines that is not a restriction.

To obtain $p_{\zeta^+}(x)$ we must find $\zeta^+ \equiv (\zeta_0^+ ... \zeta_N^+)$ by solving the constrained optimization problem

$$\min_{p_{\zeta^+}(x)\geq p^+(x), x\in[a,b]} \int_a^b [p_{\zeta^+}(x) - p^+(x)]dx.$$

An analogous procedure can be applied to approximate $p^-(x)$ and also to other α-cuts of $\tilde{p}(x)$, when needed. These results are used in an algorithm to solve the general problem:

- **Algorithm 1** (Consistent univariate fuzzy interpolation)

Input: Data set $\{(x_i, \tilde{z}_i)\}$, where $\Delta : a = x_0 < x_1 < ... < x_N = b$ and every fuzzy number $\tilde{z}_i$ is given by its α-cuts. Interpolation polynomial type $q(x)$ to be used (Lagrange or spline of order k, for example).

Output: Fuzzy-valued polynomial of inputted type given by set $\{(x_i', [\zeta_{i\alpha}^-, \zeta_{i\alpha}^+])\}$ for every given α-cut.

Step 1. Construct the Kaleva interpolation polynomials using (1) or (6).

Step 2. For every inputted α solve the optimization problems

$$\min_{q_{\zeta_\alpha^+}(x)\geq q_\alpha^+(x)} \int_a^b (q_{\zeta_\alpha^+}(x) - q_\alpha^+(x))dx$$

and

$$\min_{q_{\zeta_\alpha^-}(x)\leq q_\alpha^-(x)} \int_a^b (q_\alpha^-(x) - q_{\zeta_\alpha^-}(x))dx,$$

where $q_{\zeta_\alpha^-}(x)$ and $q_{\zeta_\alpha^+}(x)$ are polynomials of same the type of $q(x)$ evaluated for $\{(x_i', \zeta_{i\alpha}^-)\}$ and $\{(x_i', \zeta_{i\alpha}^+)\}$, respectively.

The following theorem verifies that the algorithm 1 construction is consistent.

Theorem 1: Algorithm 1 produces a fuzzy consistent surface.

Proof: The generated surfaces possess the underlying smoothness by construction. Moreover, since $q_{\zeta_\alpha^-}(x)$ and $q_{\zeta_\alpha^+}(x)$ can approximate (from above (below) respectively) $q_\alpha^{+/-}(x)$ arbitrarily close and maintains $q_{\alpha 1}^+(x) \leq q_{\alpha 2}^+(x)$ and $q_{\alpha 1}^-(x) \geq q_{\alpha 2}^-(x)$ for $\alpha_1 \leq \alpha_2$ (by construction), the order (property (1)) of the generated surface with respect α is maintained.□

3.2 Bivariate case

For the bivariate case we begin with a grid Γ in $\mathbb{R}^2$ consisting of NM real values $\{(x_i, y_j) : a = x_0 < x_1 < ... < x_N = b \text{ and } c = y_0 < y_1 < ... < y_M = d\}$, where every pair (x_i, y_j) has an associated fuzzy number $\tilde{z}_{ij} \in \mathcal{F}(\mathbb{R})$, with $(i, j) \in I_{N\times M} = \{0, 1, ..., N\} \times \{0, 1, ..., M\}$. We construct a polynomial to interpolate the fuzzy function $\tilde{f} : \mathbb{R}^2 \to \mathcal{F}(\mathbb{R})$ such that $\tilde{f}(x_i, y_j) = \tilde{z}_{ij}$ for every (i, j) in $I_{N\times M}$, where $[\tilde{f}(x_i, y_j)]_\alpha \equiv [f^-_\alpha(x_i, y_j), f^+_\alpha(x_i, y_j)]$ is the α-cut of the fuzzy number $\tilde{f}(x_i, y_j)$.

3.2.1 Fuzzy interpolation over a triangulation

Before extending the univariate Kaleva fuzzy interpolation to the bivariate case, we introduce a simple fuzzy interpolation idea that can be used over an irregularly distributed set of N' data points $\{(x_i, y_i, \tilde{z}_i)\}$, $i \in I_{N'} = \{0, 1, ..., N'\}$.

Applying an appropriate triangulation algorithm (see, for example, [7]) to $\{(x_i, y_i), i \in I_{N'}\}$, we can obtain a set of triangular domains $\{T_k, k = 1, 2, ...\}$ in $\mathbb{R}^2$, defined by its vertices $(x_{k1}, y_{k1}), (x_{k2}, y_{k2})$ and (x_{k3}, y_{k3}) for every k.

A straightforward method to build a continuous interpolating surface is to draw a *fuzzy triangle* for every T_k. A *fuzzy triangle* $\tilde{T}$ can be defined by its α-cuts

$$[T(x, y)]_\alpha = \{z \in \mathbb{R} : z = T_{d\alpha}(x, y), d_{i\alpha} \in [z_i]_\alpha\},$$

where

$$T_{d\alpha}(x, y) = a_\alpha x + b_\alpha y + c_\alpha, (x, y) \in T_k$$

is the triangular planar region in $\mathbb{R}^3$ defined by point set $\{(x_i, y_i, d_{i\alpha}), i = 1, 2, 3\}$. This means that a_α, b_α and c_α are solution of the linear system

$$Ax = b, \text{ with } A = \begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix} \text{ and } b = \begin{bmatrix} d_{1\alpha} \\ d_{2\alpha} \\ d_{3\alpha} \end{bmatrix}.$$

For every (x, y) in T_k and α in I we have

$$T^-_\alpha(x, y) \le T_{d\alpha}(x, y) \le T^+_\alpha(x, y),$$

where $T^-_\alpha(x,y)$ and $T^+_\alpha(x,y)$ are the triangular planar regions for $\{(x_i, y_i, z^-_{i\alpha})\}$ and $\{(x_i, y_i, z^+_{i\alpha})\}$, respectively. That (10) is true can be seen geometrically using a plane π passing through 3 vertices $\{(x_l, y_l, z_l), l = 1, 2, 3\}$ of a triangle. If we draw other plane π' through (x_l, y_l, z'_l), where $z'_l \geq z_l$ then every point of π' within the triangle will have a bigger or equal z coordinate than the correspondent point in π with same x and y coordinates.

Thus we have verified the following theorem.

Theorem 3: A fuzzy triangle is obtained from 3 fuzzy points $\{(x_i, y_i, \tilde{z}_i), i = 1, 2, 3\}$ using

$$[T(x,y)]_\alpha = [T^-_\alpha(x,y), T^+_\alpha(x,y)].$$

For triangular fuzzy numbers, it is easy to see that a fuzzy triangle can be find through the combined solution of three (crisp) linear systems similar to (10) having the same matrix A.

- **Algorithm 2** (Consistent bivariate real-valued fuzzy surfaces from triangulations)

Input: Data set $\{(x_i, y_i, \tilde{z}_i)\}$, $i \in I_{N'}\}$, where every fuzzy number $\tilde{z}_i$ is given by its α-cuts.

Output: Fuzzy polyhedral surface given by set $\{[T^-_{k\alpha}(x,y), T^+_{k\alpha}(x,y)], (x,y) \in T_k, k = 1, 2, ...\}$ for every given α-cut, where $T^-_{k\alpha}(x,y) = a^-_{k\alpha}x + b^-_{k\alpha}y + c^-_{k\alpha}$ and $T^+_{k\alpha}(x,y) = a^+_{k\alpha}x + b^+_{k\alpha}y + c^+_{k\alpha}$.

Step 1. Apply an appropriate triangulation algorithm (for example from [7]) to projection set $\{(x_i, y_i)\}$. Obtain a set of triangular domains $\{T_k, k = 1, 2, ...\}$ defined by its vertices $(x_{k1}, y_{k1}), (x_{k2}, y_{k2})$ and (x_{k3}, y_{k3}), for every k.

Step 2. For every inputted α and for every k obtain the solution

$$\begin{bmatrix} a^-_{k\alpha} & a^+_{k\alpha} \\ b^-_{k\alpha} & b^+_{k\alpha} \\ c^-_{k\alpha} & c^+_{k\alpha} \end{bmatrix}$$

of a double linear system

$$AX = B, \text{ with } A = \begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix}, X = \begin{bmatrix} \xi^-_1 & \xi^+_1 \\ \xi^-_2 & \xi^+_2 \\ \xi^-_3 & \xi^+_3 \end{bmatrix} \text{ and } B = \begin{bmatrix} z^-_1 & z^+_1 \\ z^-_2 & z^+_2 \\ z^-_3 & z^+_3 \end{bmatrix}.$$

Note that: (i) X and B depend of α and k whereas A only depends of k and,

(ii) $a^-_{k\alpha} \leq a^+_{k\alpha}$ does not always hold (the same for b and c) which means that they do not represent $\alpha-$cut limits of fuzzy numbers.

Theorem 4: Algorithm 2 generates a consistent fuzzy polyhedral surface.

Proof: Property (1) of consistent fuzzy surfaces is guaranteed by (10). The second property is achieved because every surface corresponding to lower and upper limits of α-cuts of this fuzzy surface have the same characteristics of the crisp counterparts, being continuous but not smooth on the edges.

These fuzzy surfaces can be very useful when data points do not have an a-priori defined evenly-spaced (regular) grid arrangement, which often happens with geographic data.

3.2.2 Bivariate fuzzy Lagrange interpolation

Following the procedure used for the univariate case, we can write the bivariate fuzzy Lagrange polynomial $\tilde{p}(x, y)$ in terms of its α-cuts

$$[p(x, y)]_\alpha = \sum_{(i,j)\in I_{N\times M}} [z_{ij}]_\alpha L_{ij}(x, y), \alpha \in I,$$

where

$$L_{ij}(x, y) = L_i(x)L_j(y),$$

with

$$L_i(x) = \prod_{\substack{k=0\\k\neq i}}^{N} \frac{(x - x_k)}{(x_i - x_k)}, i \in I_N \quad \text{and} \quad L_j(y) = \prod_{\substack{l=0\\l\neq j}}^{M} \frac{(y - y_l)}{(y_j - y_l)}, j \in I_M,$$

The upper limit $p^+_\alpha(x, y)$ of the interval $[p(x, y)]_\alpha$ is the solution of the optimization problem

$$\max_{z^-_{ij\alpha} \leq d_{ij\alpha} \leq z^+_{ij\alpha}, (i,j)\in I_{N\times M}} p_{d_\alpha}(x, y)$$

where $[z_{ij}]_\alpha$ is denoted by $[z^-_{ij\alpha}, z^+_{ij\alpha}]$ and $p_{d\alpha}(x, y)$ is an interpolation polynomial such that $p_{d\alpha}(x_i, y_j) = d_{ij\alpha}$ for the same grid Γ.

The signs of $L_i(x)$ and $L_j(y)$ are constant for all x in (x_k, x_{k+1}) and y in (y_l, y_{l+1}), respectively. So the sign of $L_{ij}(x, y) = L_i(x)L_j(y)$ will be constant in every rectangle $(x_k, x_{k+1}) \times (y_l, y_{l+1})$. Then, the optimal value is attained when

$$d_{ij\alpha} = \begin{cases} z^+_{ij\alpha} \text{ if } L_{ij}(x,y) \geq 0 \\ z^-_{ij\alpha} \text{ if } L_{ij}(x,y) < 0 \end{cases}, \text{ for all } (i,j) \in I_{N\times M}, \alpha \in I.$$

A similar procedure can be done for $p^-_\alpha(x,y)$, yielding

$$d_{ij\alpha} = \begin{cases} z^-_{ij\alpha} \text{ if } L_{ij}(x) \geq 0 \\ z^+_{ij\alpha} \text{ if } L_{ij}(x) < 0 \end{cases}, \text{ for all } (i,j) \in I_{N\times M}, \alpha \in I.$$

Using (10), we have a method to obtain $\tilde{p}(x,y)$.

Taking the particular case of triangular fuzzy numbers, so that the fuzzy data points are $\{(x_i, y_j, \tilde{z}_{ij})\}$ with $\tilde{z}_{ij} = (z^-_{ij}/z^1_{ij}/z^+_{ij})$, $(i,j) \in I_{N\times M}$, we can obtain $\tilde{p}(x,y) = (p^-(x,y)/p^1(x,y)/p^+(x,y))$ using

$$\begin{aligned} p^-(x,y) &= \sum_{L_{ij}(x,y)\geq 0} z^-_{ij} L_{ij}(x,y) + \sum_{L_{ij}(x,y)<0} z^+_{ij} L_{ij}(x,y), \\ p^1(x,y) &= \sum_{(i,j)\in I_{N\times M}} z^1_{ij} L_{ij}(x,y), \\ p^+(x,y) &= \sum_{L_{ij}(x,y)\geq 0} z^+_{ij} L_{ij}(x,y) + \sum_{L_{ij}(x,y)<0} z^-_{ij} L_{ij}(x,y). \end{aligned} \tag{10}$$

Analogously to univariate case, there is no smoothness along the grid cells' boundaries in the α-cuts of bivariate fuzzy Lagrange polynomial. Only when the core is a singleton does the smoothness holds for $p^1(x,y)$.

3.2.3 Bivariate fuzzy splines

Defining a two dimensional fuzzy spline $\tilde{s}(x,y)$ of order $k_x \times k_y$ by its α-cut we have

$$[s(x,y)]_\alpha = \{z \in \mathbb{R} : z = s_{d\alpha}(x,y), d_{ij\alpha} \in [z_{ij}]_\alpha, \alpha \in I\},$$

where $s_{d\alpha}(x,y)$ is the spline of order $k_x \times k_y$ which interpolates the data points $\{(x_i, y_j, d_{ij\alpha})\ ,\ (i,j) \in I_{N\times M}\}$. If we define ϕ_{ij} as the spline of order $k_x \times k_y$

that interpolates the data $\{(x_k, y_l, \delta_{kl})\ ,\ (k,l) \in I_{N\times M}\}$, with

$$\delta_{kl} = \begin{cases} 1 \text{ if } k = i \text{ and } l = j \\ 0 \qquad \text{otherwise} \end{cases},$$

then

$$s_{d\alpha}(x) = \sum_{(i,j)\in I_{N\times M}} d_{ij\alpha}\phi_{ij}(x,y)$$

and

$$[s(x,y)]_\alpha = \sum_{(i,j)\in I_{N\times M}} [z_{ij}]_\alpha \phi_{ij}(x,y), \alpha \in I.$$

We can write$\phi_{ij}(x,y)$ as the product $\phi_i(x)\phi_j(y)$, where $\phi_i(x)$ and $\phi_j(y)$ are like in the above univariate case for x and y, respectively.

If $k_x = k_y = 2$, we have a piecewise bilinear interpolator. In this bivariate case $\phi_{ij}(x,y)$ is also always non-negative. Then, for triangular fuzzy numbers and supposing (x,y) in the rectangle $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$ we have

$$\begin{aligned} s^-(x,y) &= z_{ij}^-\phi_{ij}(x) + z_{i+1,j}^-\phi_{i+1,j}(x,y) + z_{i,j+1}^-\phi_{i,j+1}(x,y) + z_{i+1,j+1}^-\phi_{i+1,j+1}(x,y), \\ s^1(x,y) &= z_{ij}^1\phi_{ij}(x) + z_{i+1,j}^1\phi_{i+1,j}(x,y) + z_{i,j+1}^1\phi_{i,j+1}(x,y) + z_{i+1,j+1}^1\phi_{i+1,j+1}(x,y), \\ s^+(x,y) &= z_{ij}^+\phi_{ij}(x) + z_{i+1,j}^+\phi_{i+1,j}(x,y) + z_{i,j+1}^+\phi_{i,j+1}(x,y) + z_{i+1,j+1}^+\phi_{i+1,j+1}(x,y), \end{aligned}$$

where

$$\phi_{ij}(x) = \frac{(x_{i+1}-x)}{(x_{i+1}-x_i)}\frac{(y_{j+1}-y)}{(y_{j+1}-y_j)}, \quad \phi_{i+1,j}(x,y) = \frac{(x-x_i)}{(x_{i+1}-x_i)}\frac{(y_{j+1}-y)}{(y_{j+1}-y_j)},$$

$$\phi_{ij+1}(x) = \frac{(x_{i+1}-x)}{(x_{i+1}-x_i)}\frac{(y-y_j)}{(y_{j+1}-y_j)}, \quad \phi_{i+1,j+1}(x,y) = \frac{(x-x_i)}{(x_{i+1}-x_i)}\frac{(y-y_j)}{(y_{j+1}-y_j)}.$$

For $k = 4$ we have the case of bicubic spline. Using the same procedure, the upper end $s_\alpha^+(x,y)$ of the resulting interval $[s(x,y)]_\alpha$ is determined by choosing

$$d_{ij\alpha} = \begin{cases} z_{ij\alpha}^+ \text{ if } \phi_{ij}(x,y) \geq 0 \\ z_{ij\alpha}^- \text{ if } \phi_i j(x,y) < 0 \end{cases}, \text{ for all } (i,j) \in I_{N\times M}, \alpha \in I,$$

and the lower end $s^-_\alpha(x,y)$ by

$$d_{ij\alpha} = \begin{cases} z^-_{ij\alpha} \text{ if } \phi_{ij}(x,y) \geq 0 \\ z^+_{ij\alpha} \text{ if } \phi_i j(x,y) < 0 \end{cases}, \text{ for all } (i,j) \in I_{N\times M}, \alpha \in I.$$

However, to guarantee the same sign behavior in $\phi_{ij}(x,y)$ like that of the Lagrange polynomial, we have to use again the *not-a-knot* boundary conditions for the elementary splines $\phi_i(x)$ and $\phi_j(y)$.

Particularizing for triangular fuzzy numbers, to obtain $\tilde{s}(x,y) \equiv (s^-(x,y)/s^1(x,y)/s^+(x,y))$ from data points $\{(x_i, y_j, \tilde{z}_{ij})\}$, with $\tilde{z}_{ij} = (z^-_{ij}/z^1_{ij}/z^+_{ij})$, we obtain

$$\begin{aligned} s^-(x,y) &= \sum_{\phi_{ij}(x,y)\geq 0} z^-_{ij}\phi_{ij}(x,y) + \sum_{\phi_{ij}(x,y)<0} z^+_{ij}\phi_{ij}(x,y), \\ s^1(x,y) &= \sum_{i=0}^{N} z^1_{ij}\phi_{ij}(x,y), \\ s^+(x,y) &= \sum_{\phi_{ij}(x,y)\geq 0} z^+_{ij}\phi_{ij}(x,y) + \sum_{\phi_{ij}(x,y)<0} z^-_{ij}\phi_{ij}(x,y). \end{aligned} \tag{11}$$

Again, like in the previous cases, only $s^1(x,y)$ retains the characteristics of a classic bicubic spline, at all the other α-cuts, s^-_α and s^+_α have discontinuous derivatives at cells' boundaries.

3.2.4 Maintaining smoothness on fuzzy bivariate interpolators

The procedure to achieve property (2) for univariate interpolators can be extended to bivariate ones.

Let $\tilde{q}(x,y)$ be a fuzzy interpolator of a gridded data set $\{(x_i, y_j, \tilde{z}_{ij}), (i,j) \in I_{N\times M}\}$ derived from a real-valued one $q(x,y)$ through the *extension principle* in the way we have done above.

From the above development, we have the following algorithm:

- **Algorithm 3** (General consistent bivariate fuzzy interpolators)

Input: Data set $\{(x_i, y_j, \tilde{z}_{ij}), (i,j) \in I_{N\times M}\}$, where every fuzzy number $\tilde{z}_{ij}$ is given by its α-cuts. Interpolation polynomial type $q(x,y)$ to be used (Lagrange or spline of order $k_x \times k_y$).

Output: Fuzzy-valued polynomial of inputted type given by set $\{(x'_i, y'_j, [\zeta^-_{ij\alpha}, \zeta^+_{ij\alpha}])\}$ for every given α-cut.

Step 1. Find interpolation polynomial using (10) or (11).

Step 2. For every inputted α solve the optimization problems

$$\min_{q_{\zeta_\alpha^+}(x,y)\geq q_\alpha^+(x,y)} \int_a^b \int_c^d (q_{\zeta_\alpha^+}(x,y) - q_\alpha^+(x,y))dydx$$

and

$$\min_{q_{\zeta_\alpha^-}(x,y)\leq q_\alpha^-(x,y)} \int_a^b \int_c^d (q_\alpha^-(x,y) - q_{\zeta_\alpha^-}(x,y))dydx.$$

The consequence of algorithm 3 is the following theorem.

Theorem 4: Algorithm 3 produces a fuzzy consistent surface.

Proof: The arguments is analogous to theorem 3.□

4 Numerical Examples

4.1 Univariate case

To give an example of our algorithms we used the following data set

x_i	0	15	25	50	90	121	143	165	200
z_i^-	19.5	14.9	5.8	−3.9	39.0	22.3	32.1	29.4	2.5
z_i^1	20.0	15.0	6.0	−4.0	40.0	23.0	33.0	30.0	3.0
z_i^+	20.3	15.6	6.3	−4.2	41.2	23.7	34.0	30.1	3.2

where attributes $\tilde{z}_i$ are triangular fuzzy numbers given by $(z_i^-/z_i^1/z_i^+)$.

First we determine the fuzzy-valued Lagrange polynomial (see Kaleva[4]) which graphic is represented in figure 2.

To illustrate our statement about property (1) in 3.1.3, we draw two real-valued Lagrange polynomials passing through lower and upper limits, respectively, of an α-cut. They do not guarantee property (1), since the graphs cross each other. See, for example, the case of 0-cut in figure 3.

For the same data set we show in figure 4 the fuzzy-valued linear spline and the respective cubic spline in figure 5.

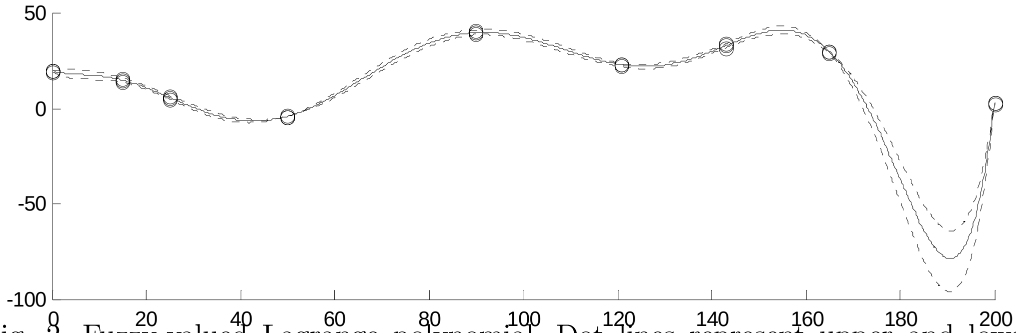

Fig. 2. Fuzzy-valued Lagrange polynomial. Dot lines represent upper and lower limits of fuzzy support (*zero*-cut) and solid line represents modal value (*one*-cut). Small circles are data points.

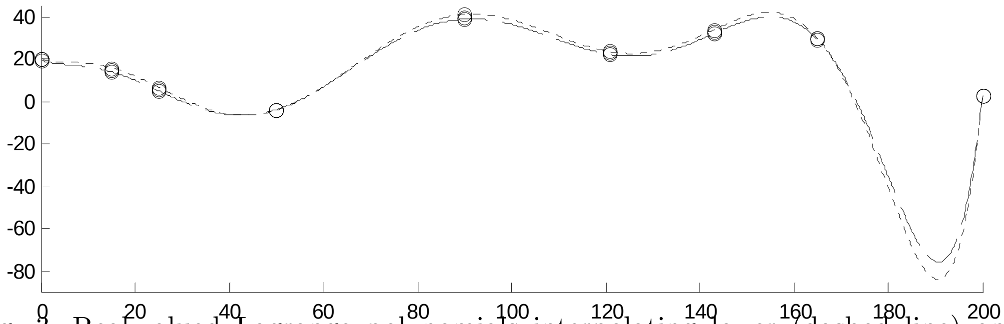

Fig. 3. Real-valued Lagrange polynomials interpolating lower (dashed line) and upper (dot line) limits, respectively, of fuzzy support of data points.

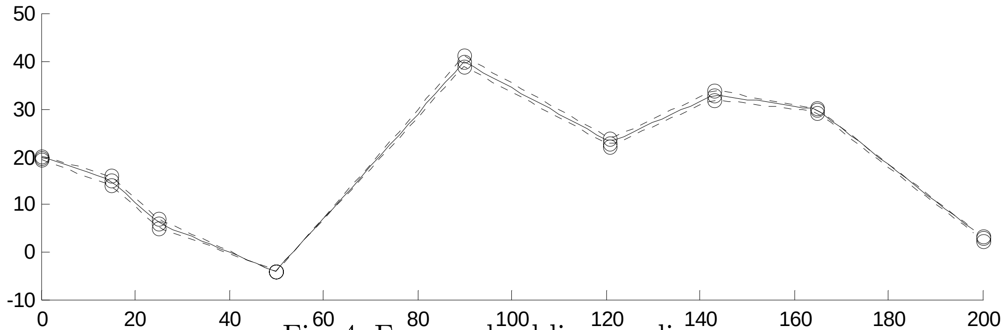

Fig. 4. Fuzzy-valued linear spline.

Using algorithm 1 we obtain a consistent fuzzy surface is illustrated in figures 6 and 7.

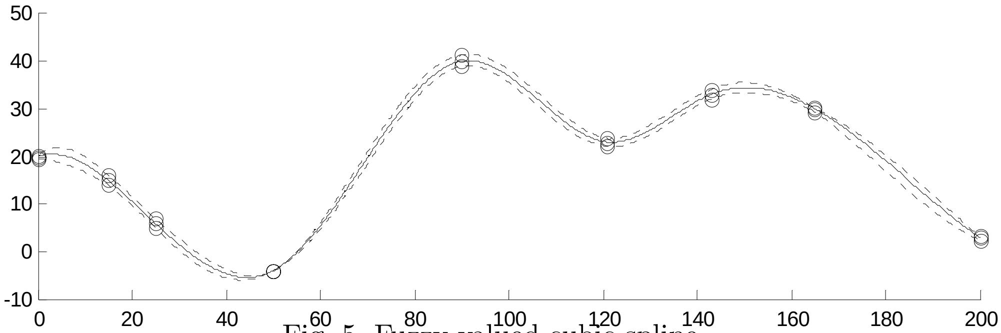

Fig. 5. Fuzzy-valued cubic spline.

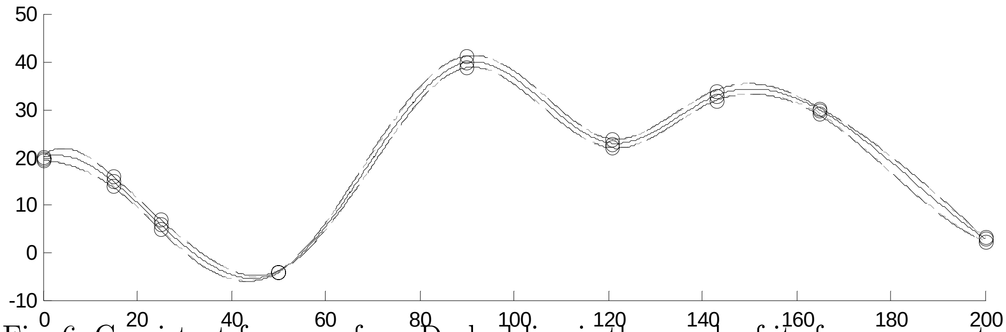

Fig. 6. Consistent fuzzy surface. Dashed line is the graph of its fuzzy support.

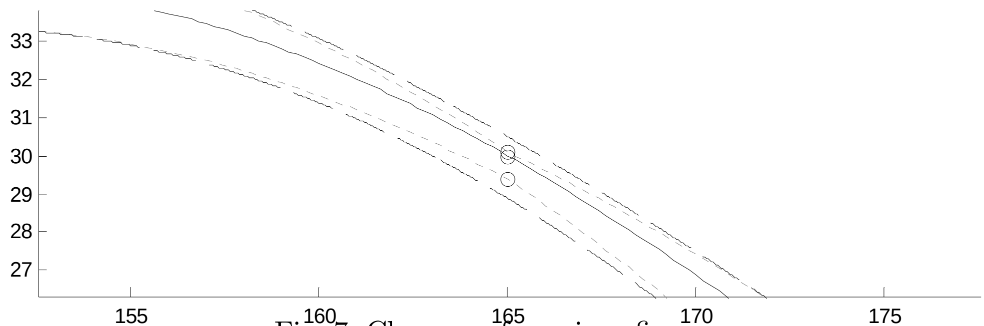

Fig. 7. Close up of previous figure.

4.2 Bivariate case

We used an irregularly distributed set of 40 data points $\{(x_i, y_i, \tilde{z}_i)\}$ for this example, where $\tilde{z}_i$ are again triangular fuzzy numbers. Data set is plotted in figure 8.

First we generate a consistent fuzzy polyhedral surface using algorithm 2. The surface is plotted in figure 9.

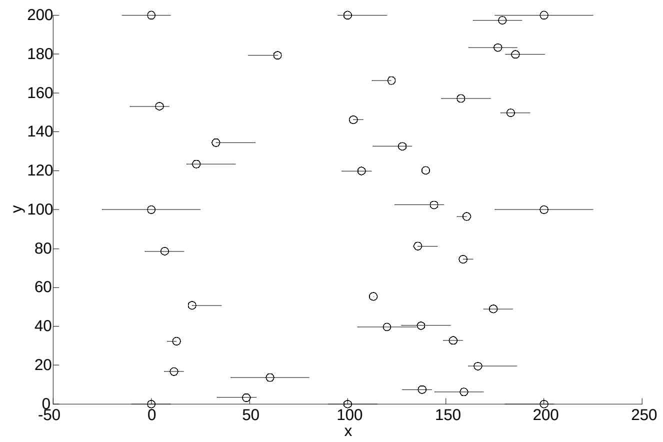

Fig. 8. Initial irregularly distributed set of 40 data points $\{(x_i, y_i, \tilde{z}_i)\}$. The bars attached to every point is a representation of the fuzzy support (amplified 50 times). Circles are centered in (x_i, y_i, z_i^1).

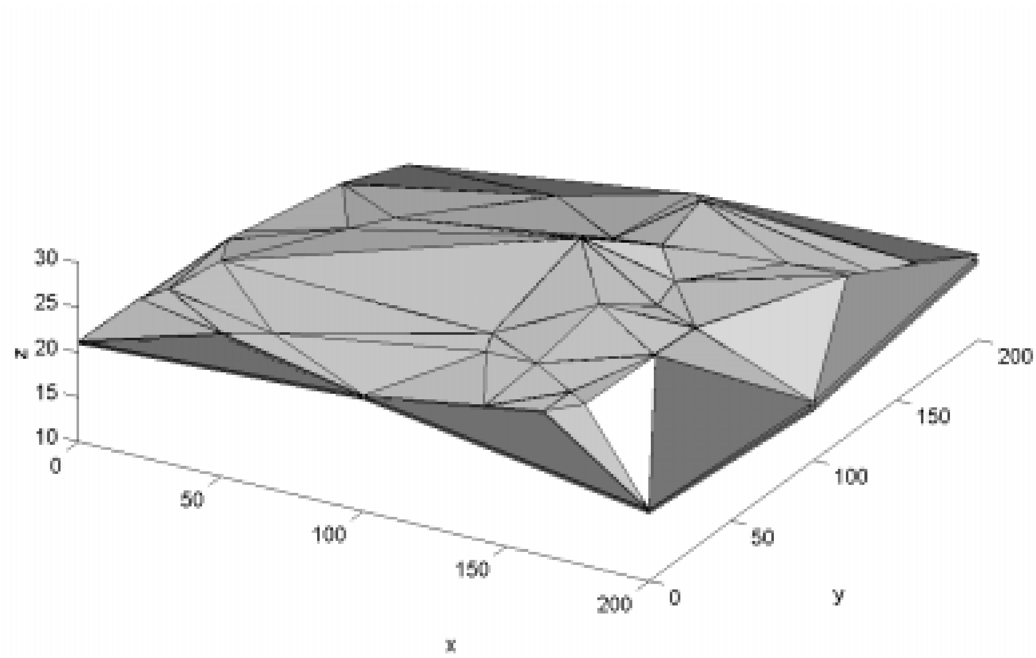

Fig. 9. Consistent fuzzy polyhedral surface obtained from previous data points.

Using the previous surface we derive a gridded data set $\{(x_i, y_j, \tilde{z}_{ij}), (i, j) \in I_{9\times 11}\}$. With that data we produce a bivariate fuzzy-valued Lagrange polynomial which is plotted in figure 10 for a 101×101 square grid.

Bivariate fuzzy-valued linear and cubic splines are also generated from that data set and plotted in figures 11 and 12, respectively.

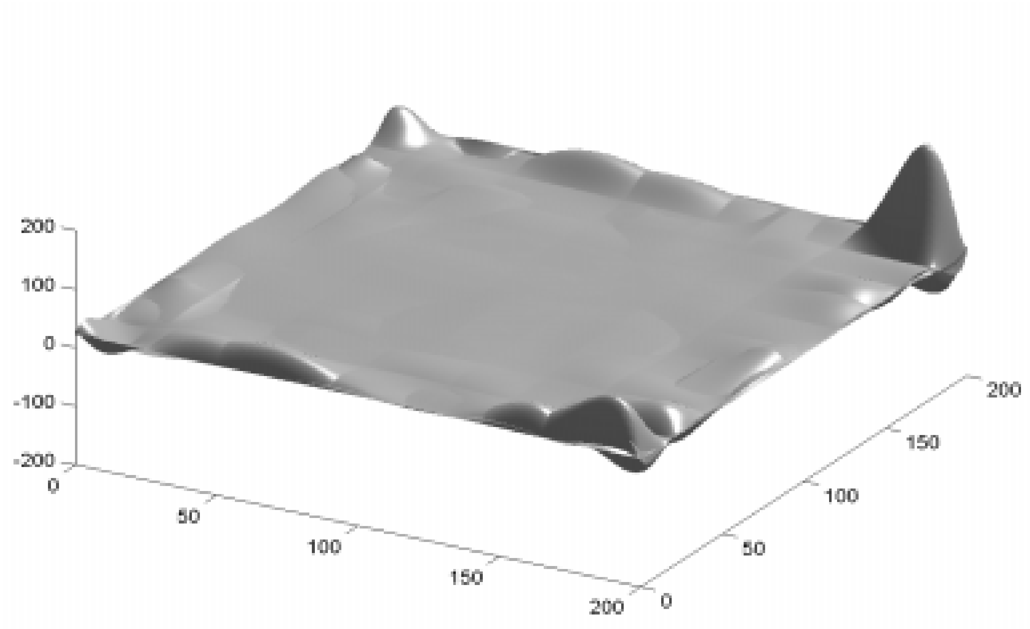

Fig. 10. Bivariate fuzzy-valued Lagrange polynomial.

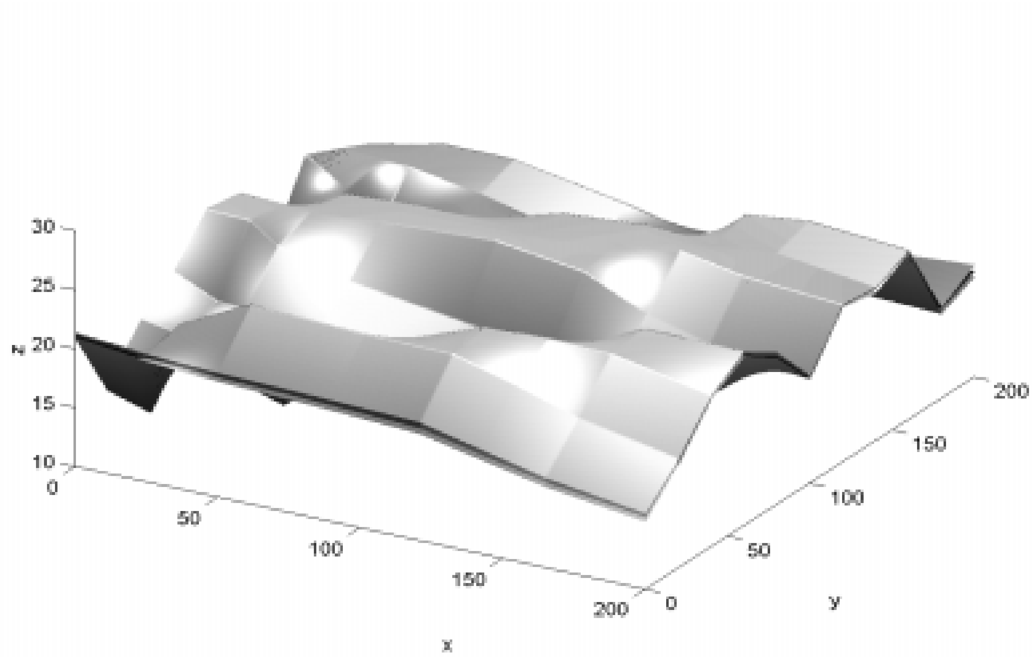

Fig. 11. Bivariate fuzzy-valued linear spline.

5 Conclusion

We have shown how to construct consistent surfaces from fuzzy data. The importance of these methods are that they capture the underlying uncertainty in the data and generate a set of surfaces faithful to this underlying uncertainty. If the data is derived from more qualitative assessments, then the suite

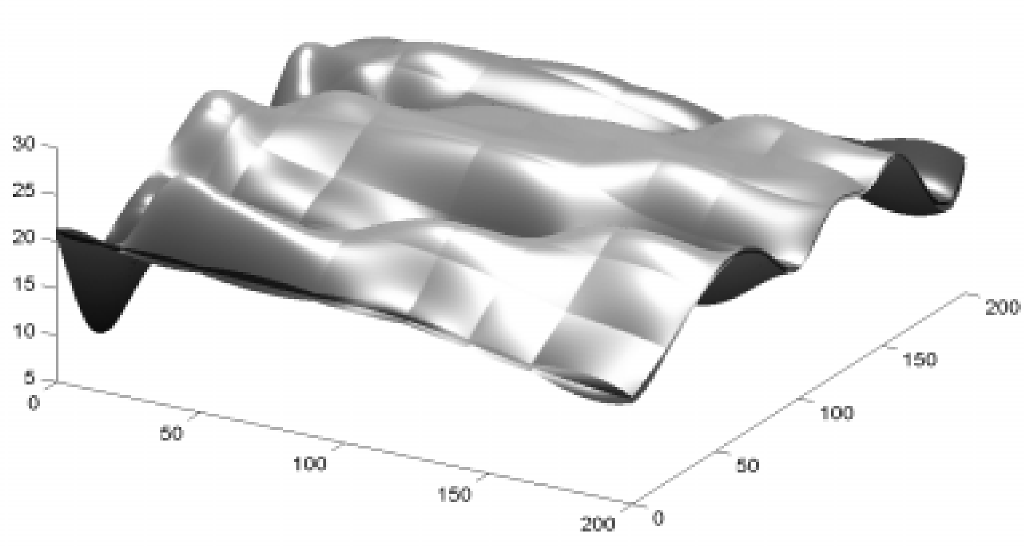

Fig. 12. Bivariate fuzzy-valued cubic spline.

of surfaces generated give an indication of the range of possible generated surfaces. This would be important in assessments of the possible scenarios for studies such as mud slide analysis and flash flood predictions. An interesting question is whether or not we will be able to generate meaning surfaces from consistent possibility and necessity measures according to [3] from probabilistic uncertainties in the data set. If this could be done, we then would be able to generate two bounding surfaces that represent the upper and lower bounds on the probability. Moreover, the more likely surfaces would be generated in-between the bounding surfaces. The bounding surfaces are derived from the probability data and surfaces generated from the possibility distribution consistent with the probability uncertainty in the data represents the upper limit of generated surfaces.

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