Analysis of 1-D Moment Equations for Immiscible Flow

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ABSTRACT. We derive and analytically and numerically solve statistical moment equations for immiscible flow in porous media in the limit of zero capillary pressure, with application to secondary oil recovery. Mean and variance of (water) saturation exhibit a bimodal character; two shocks replace the single shock front evident in the classical Buckley–Leverett saturation profile.

1. Introduction

Subsurface geologic properties at field scales are uncertain, and are often described statistically in practice. Flow profiles in such porous media are uncertain, and statistical flow outcomes are appropriate. We are primarily interested in mean behavior and a measure of the uncertainty about this mean.

A "zeroth-order" model of mean flow with averages of geologic properties ignores correlations between flow variables. Monte Carlo simulations of many realizations of geologic properties to estimate moments requires much computation time and careful sampling techniques [**8**], [**9**], [**28**]. Macrodispersion theories in contaminant transport capture a first-order effect of fluctuations via covariance functions in PDEs for the mean concentration [**5**], [**10**].

We derive second-order PDEs for the covariance functions and the mean flow, and solve for these moments simultaneously. The fundamental problem of closure of the system is addressed by a perturbation argument. The resulting *moment equations* directly approximate the local mean and covariance functions, for general boundary conditions and general stochastic geology [**31**].

1.1. Applications. A statistical description of subsurface flow is of particular interest for secondary oil recovery. The principal difficulty is a non-convex nonlinear flux function in an advection equation that leads to discontinuous solutions.

2000 *Mathematics Subject Classification.* Primary 76T99, 76S05; Secondary 35L60, 35L65.

Key words and phrases. porous media, stochastic, random fields, moment equations.

Dr. Jarman was supported by NSF Graduate Traineeship in Applied Mathematics (DMS-9208685), and DOE Laboratory Directed Research and Development. He would like to thank Dr. J. S. Oliveira for support and guidance.

Dr. Russell was supported in part by NSF Grant Nos. DMS-9706866 and DMS-0084438 and ARO Grant No. 37119-GS-AAS.

Standard pressure–saturation equations for 2-D horizontal flow of two immiscible fluids in porous media, in the limit of vanishing capillary pressure ($p_c = 0$), are

$$\phi\,\mathbf{v}(\mathbf{x}) = -\mathbf{K}(\mathbf{x})\,\nabla h(\mathbf{x}), \qquad \nabla\cdot\mathbf{v}(\mathbf{x}) = 0, \tag{1.1}$$

$$\partial_t s(\mathbf{x},t) + \nabla\cdot[f(s(\mathbf{x},t))\mathbf{v}(\mathbf{x})] = 0. \tag{1.2}$$

These are taken to be valid from *laboratory* (centimeters) to *field* scales of reservoir depth (10-100 meters) and length (100-1000 meters). Hydraulic conductivity $\mathbf{K}$ may be an anisotropic tensor; here, for simplicity, it will be an isotropic scalar K. Assume also that K depends weakly on (water) saturation s [**1**], [**3**]. Apply (1.1)–(1.2) to the flow of oil and water, for arbitrary fluid mobilities. Denote the *total velocity*, a scaled total volumetric flux of both fluids, by $\mathbf{v}$, hydraulic water head by h, and porosity (assumed constant) by ϕ. The *fractional flow function* $f(s)$, for $p_c = 0$, represents the fraction of $\mathbf{v}$ due to water. It is typically S-shaped; we use here a functional form arising from quadratic relative permeabilities (see [**1**]), though our method does not depend on any such specific choice.

Capillary pressure regularizes sharp fronts caused by the nonlinear advection term. To obtain a linear approximation to this effect, add $\epsilon_D\nabla^2 s(\mathbf{x},t)$ with $\epsilon_D > 0$ to the right side of (1.2). Letting $\epsilon_D \to 0$ defines the *vanishing-viscosity* solution [**25**], which is the one we seek.

As is standard in subsurface applications, let $Y = \ln K$ be a random field with prescribed mean and covariance functions; e.g., it is often claimed that Y is multivariate Gaussian, based on empirical observations [**6**] (our method does not depend on this). Through (1.1)–(1.2), $\mathbf{v}$ and s are thus random fields. No other underlying sources of uncertainty are considered in this study.

Under the assumptions stated above, with steady boundary conditions, a steady $\mathbf{v}$ can be determined from (1.1). We evolve s from the stochastic PDE (1.2), assuming $\mathbf{v}$ is known. Moments of $\mathbf{v}$ and h can be estimated from established theory ([**29**] and [**31**] use moment equations). We seek to combine analytical and numerical techniques to model the propagation of uncertainty from an underlying random field $Y(\mathbf{x})$, through $\mathbf{v}(\mathbf{x})$, to the solution $s(\mathbf{x},t)$.

1.2. Previous work. Existing work on moment differential equations (MDEs) focuses mostly on advection equations with linear flux functions [**11**], and some nonlinear subsurface flow equations of a form different from (1.2) [**2**], [**27**], [**29**], [**30**].

Langlo and Espedal [**16**], [**17**] presented a macrodispersion approach for the stochastic version of (1.2). The flux function is expanded in a Taylor series, and high-order terms are neglected; then standard techniques represent macrodispersivity as a function of flow velocity covariance. Zhang, Tchelepi, and Li take advantage of the steady velocity field, and transform 2-D flow to 1-D Lagrangian flow along streamlines [**32**], [**33**]. Then they formulate integral equations for moments from ensemble averages over the streamtubes.

An Eulerian MDE approach has been successful for single- and multiphase pressure and velocity equations [**29**], and a natural next step is to extend the theory from flux equations to transport equations. This framework differs from streamtubes not only in formulation, but also in that the MDEs need no velocity-distribution assumption, and an extension to transient velocity fields is relatively straightforward. The approach applies to any probability distribution of geologic properties and any correlation function, and does not require stationarity. Other stochastic theories generally require such restrictions.

Equations are derived in §2. In §3, we reduce the mean and variance equations to a simple form. These are shown to be strictly hyperbolic for 1-D flow, and an analytical solution is given in §4. Classification is briefly discussed for 2-D equations. Details of the results outlined here are presented in [**13**]. We conclude with an evaluation of our results and their practical implications, and a brief overview of additional questions that warrant future investigation.

2. Moment Equations

We compare two different approaches for statistical MDEs of 1-D flow. In §2.1, we expand fields in the form $u = \langle u\rangle + \delta u$ and the resulting MDEs are closed by neglecting products of 'δ' terms. Analogous equations in §2.3 result from a full asymptotic expansion. We show in §4 that the latter yields moments that violate physical constraints. In both cases, random fluctuations in $Y = \ln K$ are assumed small: $\sigma_Y \ll 1$. We can immediately generalize to higher dimensions via vector notation. Moments of h and $\mathbf{v}$ are assumed known from (1.1) and moments of Y.

The 1-D saturation equation (1.2), with initial data $s(x, 0) = g(x)$, is

$$\partial_t s + \partial_x (f(s)v) = 0, \tag{2.1}$$

We assume that g is known with certainty. Solutions are defined in terms of vanishing viscosity as in §1.1; henceforth, this is tacitly understood.

2.1. Two-term expansion. Moment equations may be derived in a number of ways. For examples of commonly used methods applied to various models in subsurface flow and transport, see [**5**], [**7**], [**10**], [**11**], [**24**], [**29**]. Here we apply a standard approach, separating mean fields from random fluctuations.

Let $\langle\cdot\rangle$ denote the expectation operator, defined by

$$\langle\psi\rangle \equiv \int_\Omega \psi(\omega)\, dP(\omega) \tag{2.2}$$

for any integrable function $\psi : \Omega \to \mathbb{R}$ on the sample space Ω with probability measure P. We omit reference to ω in what follows.

The random field Y is decomposed into deterministic mean plus random fluctuation: $Y = \langle Y\rangle + \delta Y$. Each field dependent on Y is represented similarly:

$$\begin{aligned} h(x) &= \langle h\rangle (x) + \delta h(x), \qquad v(x) = \langle v\rangle (x) + \delta v(x), \\ s(x,t) &= \langle s\rangle (x,t) + \delta s(x,t). \end{aligned} \tag{2.3}$$

Recall that we only need the decompositions of v and s here. Next, the fractional flow function is expanded in a Taylor series around $\langle s\rangle$:

$$f(s) = f(\langle s\rangle) + f'(\langle s\rangle)\delta s + \frac{1}{2} f''(\langle s\rangle)\delta s^2 + \cdots \tag{2.4}$$

So far, we make no assumption regarding the size of δs relative to $\langle s\rangle$.

To obtain the mean-saturation equation, apply the operator $\langle\cdot\rangle$ to (2.1):

$$\begin{aligned} \partial_t \langle s\rangle + \partial_x \Big[& f(\langle s\rangle) \langle v\rangle + f'(\langle s\rangle) \langle \delta s\, \delta v\rangle + \frac{1}{2} f''(\langle s\rangle) \langle v\rangle \langle \delta s^2\rangle \\ & + \frac{1}{2} f''(\langle s\rangle) \langle \delta s^2 \delta v\rangle + \frac{1}{3!} f'''(\langle s\rangle) \langle v\rangle \langle \delta s^3\rangle + \cdots \Big] = 0. \end{aligned} \tag{2.5}$$

The flux terms include a nonlinear advective mean, two covariances, and higher-order moments. For the saturation-fluctuation equation, subtract (2.5) from (2.1):

$$\begin{aligned} &\partial_t \delta s + \partial_x \Big[f(\langle s\rangle)\delta v + f'(\langle s\rangle)\delta s \langle v\rangle + f'(\langle s\rangle)\left(\delta s\, \delta v - \langle \delta s\, \delta v\rangle\right) \\ &+\frac{1}{2} f''(\langle s\rangle)\langle v\rangle \left(\delta s^2 - \left\langle \delta s^2\right\rangle\right) + \frac{1}{2} f''(\langle s\rangle)\left(\delta s^2 \delta v - \left\langle \delta s^2 \delta v\right\rangle\right) + \cdots \Big] = 0. \end{aligned} \tag{2.6}$$

We derive equations for the unknown covariance functions $\langle \delta s \delta v\rangle$ and $\left\langle \delta s^2\right\rangle$, using the following additional notation. The independent variables are x and t except where noted, and $\cdot|_y$ denotes the replacement of x by some y different from x. It is convenient, and useful, to derive equations for the more general two-point covariances $\langle \delta s \delta v|_y\rangle$ and $\langle \delta s \delta s|_y\rangle$ rather than for one-point covariances.

To obtain an equation for the saturation-velocity covariance $\langle \delta s \delta v|_y\rangle$, multiply (2.6) by $\delta v(y)$ and apply $\langle\cdot\rangle$. This results in:

$$\begin{aligned} &\partial_t \langle \delta s \delta v|_y\rangle + \partial_x \Big[f(\langle s\rangle)\langle \delta v\, \delta v|_y\rangle + f'(\langle s\rangle)\langle v\rangle \langle \delta s\, \delta v|_y\rangle + f'(\langle s\rangle)\langle \delta s \delta v\, \delta v|_y\rangle \\ &+\frac{1}{2} f''(\langle s\rangle)\langle v\rangle \left\langle \delta s^2 \delta v|_y\right\rangle + \frac{1}{2} f''(\langle s\rangle)\left\langle \delta s^2 \delta v\, \delta v|_y\right\rangle + \cdots \Big] = 0. \end{aligned} \tag{2.7}$$

Similarly, multiplying (2.6) by $\delta s(y,t)$, and using the identity[1] $\delta s \partial_t \delta s|_y + \delta s|_y \partial_t \delta s = \partial_t(\delta s \delta s|_y)$, yields this equation for the two-point saturation covariance:

$$\begin{aligned} &\partial_t \langle \delta s \delta s|_y\rangle + \partial_x \Big[f(\langle s\rangle)\langle \delta s|_y \delta v\rangle + f'(\langle s\rangle)\langle v\rangle \langle \delta s\, \delta s|_y\rangle + f'(\langle s\rangle)\langle \delta s\, \delta v\, \delta s|_y\rangle \\ &+\frac{1}{2} f''(\langle s\rangle)\langle v\rangle \left\langle \delta s^2 \delta s|_y\right\rangle + \frac{1}{2} f''(\langle s\rangle)\left\langle \delta s^2 \delta v\, \delta s|_y\right\rangle + \cdots \Big] \\ &+\partial_y \Big[f(\langle s|_y\rangle)\langle \delta s\, \delta v|_y\rangle + f'(\langle s|_y\rangle)\langle v|_y\rangle \langle \delta s|_y \delta s\rangle + f'(\langle s|_y\rangle)\langle (\delta s\, \delta v)|_y \delta s\rangle \\ &+\frac{1}{2} f''(\langle s|_y\rangle)\langle v|_y\rangle \left\langle \delta s^2|_y \delta s\right\rangle + \frac{1}{2} f''(\langle s|_y\rangle)\left\langle (\delta s^2 \delta v)|_y \delta s\right\rangle + \cdots \Big] = 0. \end{aligned} \tag{2.8}$$

2.2. Closure by perturbation argument. If $\sigma_Y \ll 1$ so that fluctuations and their derivatives may be assumed small relative to the means, and if f is smooth, then we can approximate (2.5)–(2.8) by a closed, coupled system. Defining $c_{sv}(x,y,t) = \langle \delta s \delta v|_y\rangle$, $c_s = \langle \delta s \delta s|_y\rangle$, $c_v = \langle \delta v \delta v|_y\rangle$, $\langle s\rangle|_y = \langle s\rangle(y,t)$, and $\widehat{c}_{sv}(x,y,t) = c_{sv}(y,x,t)$, the resulting system is

$$\partial_t \langle s\rangle + \partial_x \Big[f(\langle s\rangle)\langle v\rangle + f'(\langle s\rangle)\sigma_{sv} + \frac{1}{2} f''(\langle s\rangle)\sigma_s^2 \langle v\rangle \Big] = 0, \tag{2.9a}$$

$$\partial_t c_{sv} + \partial_x \Big[f(\langle s\rangle) c_v + f'(\langle s\rangle)\langle v\rangle c_{sv} \Big] = 0, \tag{2.9b}$$

$$\partial_t c_s + \partial_x \Big[f(\langle s\rangle)\widehat{c}_{sv} + f'(\langle s\rangle)\langle v\rangle c_s \Big] + \partial_y \Big[f(\langle s\rangle|_y) c_{sv} + f'(\langle s\rangle|_y)\langle v|_y\rangle c_s \Big] = 0. \tag{2.9c}$$

Initial data are $\langle s\rangle(x,0) = g(x)$, $c_{sv}(x,y,0) = c_s(x,y,0) = 0$; recall that $\langle v\rangle$ and $\langle \delta v \delta v|_y\rangle$ are assumed known. Both (2.9b) and (2.9c) have advective flux terms, are coupled to the mean equation (2.9a), and are first-order in σ_Y^2. This is consistent with the approximation to (2.9a), which is second-order in σ_Y.

Another common closure argument, which may be called a Gaussian assumption, might be applied here. For example, for the linear case ($f(s) = s$), (2.9) is exact if one assumes that velocity and saturation are jointly multivariate normal [**7**].

[1]The identity is not valid in a strong sense for discontinuous solutions. Recall, however, that we define solutions in terms of the (smooth) viscous solution, in the limit $\epsilon_D \to 0$.

This follows without a perturbation argument. The linear case under this assumption is studied in [**11**], where the effect of small-scale diffusion is included. Using simulations, we found that the Gaussian assumption, even for transformations of saturation and velocity fields, is inappropriate for this problem [**12**].

Note that the mean equation (2.9a) contains the functions $\sigma_{sv} = c_{sv}(x,x,t)$ and $\sigma_s^2 = c_s(x,x,t)$ rather than $c_{sv}(x,y,t)$ and $c_s(x,y,t)$. This mix of one-point and two-point covariance functions prevents us from immediately treating the MDEs as classically hyperbolic, even though they can be put in conservation-law form. Also, independent variables x and y are permuted in $\langle s\rangle$ and c_{sv} in (2.9c). In general, $\langle s\rangle|_y \neq \langle s\rangle$, and $\widehat{c}_{sv} \neq c_{sv}$. We address these issues in §3.

We refer to this problem as "1-D," even though the covariance functions involve two spatial variables, and the saturation covariance has fluxes in both directions. The two variables represent two different points in the same 1-D domain. Similarly, the "2-D" problem has four spatial coordinates.

2.3. Infinite expansion. In this alternative derivation, flow variables head $h(x)$, velocity $v(x)$ and saturation $s(x,t)$ are represented by formal infinite perturbation series expansions in powers of a parameter ϵ:

$$h = \sum_{n=0}^{\infty} \epsilon^n h_n(x), \quad v = \sum_{n=0}^{\infty} \epsilon^n v_n(x), \quad s = \sum_{n=0}^{\infty} \epsilon^n s_n(x,t). \tag{2.10}$$

The expansion parameter $\epsilon = \sigma_Y$ is shown to be appropriate within the context of flow equations (1.1) [**5**, pp. 184–190], [**31**]. For example, for single phase, stationary uniform mean flow in 1-D, σ_v^2 is approximated by $\epsilon^2 \langle v_1^2\rangle = v_0^2\sigma_Y^2$.

Moment equations analogous to (2.9) can be derived [**13**]:

$$\partial_t s_0 + \partial_x\Big[f(s_0)v_0\Big] = 0, \qquad \partial_t\langle s_1\rangle + \partial_x\Big[v_0 f'(s_0)\langle s_1\rangle\Big] = 0, \tag{2.11a}$$

$$\partial_t\langle s_2\rangle + \partial_x\Big[f(s_0)\langle v_2\rangle + f'(s_0)\sigma_{sv} + f'(s_0)\langle s_2\rangle v_0 + \frac{1}{2}f''(s_0)\sigma_s^2 v_0\Big] = 0, \tag{2.11b}$$

$$\partial_t c_{sv} + \partial_x\Big[f(s_0)c_v + f'(s_0)v_0 c_{sv}\Big] = 0, \tag{2.11c}$$

$$\partial_t c_s + \partial_x\Big[f(s_0)\widehat{c}_{sv} + f'(s_0)v_0 c_s\Big] + \partial_y\Big[f(s_0|_y)c_{sv} + f'(s_0|_y)v_0|_y c_s\Big] = 0. \tag{2.11d}$$

Initial data are given by $s_0(x,0) = g(x)$, $\langle s_1\rangle(x,0) = \langle s_2\rangle(x,0) = c_{sv}(x,y,0) = c_s(x,y,0) = 0$. The second-order mean is $s_0 + \epsilon\langle s_1\rangle + \epsilon^2\langle s_2\rangle$. Note that the argument of $f^{(k)}$ is s_0, the zeroth-order mean. The system is in fact closed again using a perturbation argument, but now this argument is contained in the assumption that the formal power series in ϵ converges. Thus, throughout §2, second-order equations are closed by assuming that heterogeneity is weak ($\sigma_Y \ll 1$).

3. Classification of reduced equations

We exploit special structure in the 1-D equations that allows simplification and classification of (2.9) and (2.11). For classification, we need the following

DEFINITION 3.1. Let $\mathbf{u}(x,y,t) : \mathbb{R}^2 \times \mathbb{R}_+ \to \mathbb{R}^n$, $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$, and $\mathbf{G} : \mathbb{R}^n \to \mathbb{R}^n$. A system of equations in *conservation-law* form is given by

$$\partial_t\mathbf{u} + \partial_x\mathbf{F}(\mathbf{u}) + \partial_y\mathbf{G}(\mathbf{u}) = 0, \quad \mathbf{u}(x,0) = \mathbf{u}_0(x). \tag{3.1}$$

This system is *hyperbolic* if the eigenvalues of $c_1\,DF(\mathbf{u}) + c_2\,DG(\mathbf{u})$ are real for all $c_1, c_2 \in \mathbb{R}$, where DF and DG are the Jacobian matrices of $\mathbf{F}$ and $\mathbf{G}$ [**20**].

We expect the equations to be nearly hyperbolic since the original deterministic PDE (1.2) is hyperbolic, but the systems (2.9) or (2.11) as given cannot even be written in conservation-law form due to the inconsistencies mentioned above. However, we show that both sets of moment equations reduce to hyperbolic systems of conservation laws and yield analytic solutions. The key discovery that allows this reduction is a relationship between covariance functions in 1-D.

Observe that velocity $\mathbf{v}$ is constant in 1-D, so it is merely a random variable rather than a random field. Thus, velocities at any two points in space are perfectly correlated (they are the same random variable). Equivalently, the velocity correlation length is infinite. This infinite correlation length characterizes the principal difference between stochastic subsurface flow in one and two space dimensions [**12**].

3.1. Two-term expansion. The MDEs (2.9) reduce to a system of hyperbolic PDEs. The first step to this end addresses the inconsistencies mentioned earlier.

Both expansion terms $\langle v\rangle$ and δv are constant: by applying the expectation operator to $\partial_x v = 0$ we obtain $\partial_x \langle v\rangle = 0$, so that

$$0 = \partial_x v = \partial_x \langle v\rangle + \partial_x \delta v \quad \Rightarrow \quad \partial_x \delta v = 0.$$

This implies that $c_v(x,y)$ is also constant, and that $c_{sv}(x,y,t)$ is independent of its second argument. Consequently, c_v is identical to the second-order approximation to velocity variance σ_v^2, and $c_{sv}(x,y,t)$ is identical to $\sigma_{sv}(x,t)$.

This last identity removes the inconsistency of having σ_{sv} instead of c_{sv}. We still have σ_s^2 in (2.9a), instead of c_s, and we have $\langle s\rangle|_y$ and $\widehat{c}_{sv}$ in (2.9c). A key variance-covariance relationship, $\sigma_{sv} = \sigma_s\,\sigma_v$, follows: in 1-D, the saturation profile is completely determined by knowledge of the velocity, for any positive time. Thus, saturation and velocity are perfectly correlated.

We divide (2.9b) by $\sigma_v > 0$, and retain only the first two equations in (2.9) in the following. Replace c_v by σ_v^2 and c_{sv} by $\sigma_v\sigma_s$, to reduce the system to the following new equations for mean and standard deviation of saturation:

$$\partial_t \begin{pmatrix} \langle s\rangle \\ \sigma_s \end{pmatrix} + \partial_x \begin{pmatrix} \langle v\rangle f(\langle s\rangle) + \sigma_v f'(\langle s\rangle)\sigma_s + \frac{1}{2}\langle v\rangle f''(\langle s\rangle)\sigma_s^2 \\ \sigma_v f(\langle s\rangle) + \langle v\rangle f'(\langle s\rangle)\sigma_s \end{pmatrix} = 0. \tag{3.2}$$

Dependence on the second space variable y has been eliminated. Thus, (3.2) is in conservation-law form, with $\mathbf{u}(x,t) = (\langle s\rangle, \sigma_s)$, and flux function

$$\mathbf{F}(\mathbf{u}) = \begin{pmatrix} \langle v\rangle f(\langle s\rangle) + \sigma_v f'(\langle s\rangle)\sigma_s + \frac{1}{2}\langle v\rangle f''(\langle s\rangle)\sigma_s^2 \\ \sigma_v f(\langle s\rangle) + \langle v\rangle f'(\langle s\rangle)\sigma_s \end{pmatrix}.$$

LEMMA 3.2. *The moment equations* (2.9) *reduce to a hyperbolic system.*

PROOF. It remains to show that (3.2) is strictly hyperbolic. The Jacobian matrix $DF(\langle s\rangle, \sigma_s) = \begin{pmatrix} j_{11} & j_{12} \\ j_{12} & j_{22} \end{pmatrix}$ has entries

$$\begin{aligned} j_{11} &= \langle v\rangle f'(\langle s\rangle) + \sigma_v f''(\langle s\rangle)\sigma_s + \frac{1}{2}\langle v\rangle f^{(3)}(\langle s\rangle)\sigma_s^2, \\ j_{12} &= \sigma_v f'(\langle s\rangle) + \langle v\rangle f''(\langle s\rangle)\sigma_s, \qquad j_{22} = \langle v\rangle f'(\langle s\rangle). \end{aligned}$$

DF is symmetric; thus the eigenvalues of DF are real [**26**], hence (3.2) is hyperbolic. In fact, we show that (3.2) is strictly hyperbolic; i.e., DF has a complete set of linearly independent eigenvectors [**20**]. If the eigenvalues are distinct, then they must correspond to independent eigenvectors. Thus, we may assume that DF

has only one distinct eigenvalue. This occurs when the discriminant of the characteristic polynomial vanishes: $(j_{11}-j_{22})^2+4j_{12}^2=0$. But this would imply that $j_{12}=0$, hence DF is a multiple of the identity, and it again has a complete set of eigenvectors. The conclusion follows. □

In addition, the eigenvalues are distinct unless $(\langle s\rangle, \sigma_s) \in \{(0,0),\ (1,0)\}$ [**12**].

3.2. Infinite expansion. The evolution of moments in this case is given by the system (2.11). If velocity is constant then v_j is constant for each j, so long as the perturbation series can be differentiated termwise. Thus, $c_v(x,y)$ is also constant, and $c_{sv}(x,y,t)$ is independent of its second argument. Consequently, c_v is identical to the second-order approximation to velocity variance σ_v^2, and $c_{sv}(x,y,t)$ is identical to $\sigma_{sv}(x,t)$. Finally, the relationship $\sigma_{sv}=\sigma_v\,\sigma_s$ holds as before. We obtain these equations analogous to (3.2):

$$\partial_t \begin{pmatrix} s_0 \\ \epsilon\,\langle s_1\rangle \\ \epsilon^2\,\langle s_2\rangle \\ \\ \sigma_s \end{pmatrix} + \partial_x \begin{pmatrix} v_0\, f(s_0) \\ v_0\, f'(s_0)\,\epsilon\,\langle s_1\rangle \\ \epsilon^2\,\langle v_2\rangle\, f(s_0) + \sigma_v\, f'(s_0)\sigma_s + v_0\, f'(s_0)\,\epsilon^2\,\langle s_2\rangle \\ \qquad + \tfrac{1}{2} v_0\, f''(s_0)\sigma_s^2 \\ \sigma_v f(s_0) + v_0\, f'(s_0)\sigma_s \end{pmatrix} = 0. \tag{3.3}$$

We have shown that this system is (not strictly) hyperbolic [**13**]. The Jacobian matrix in general does not have a full set of linearly independent eigenvectors. This degeneracy leads to secular terms in the solution. Use of the second-order mean as the argument of $f^{(k)}$ in §2.3 yields a modified infinite expansion that does not have this drawback, and is a fourth-order correction to the two-term expansion [**12**].

3.3. Uniqueness, and an additional analytical result. Because (2.9) and (2.11) are nearly hyperbolic systems, one might expect to extend uniqueness methods from the theory of such systems. The viscosity method is an appealing way to prove uniqueness [**25**]. Variations on this approach generally require systems that are *genuinely nonlinear* or *linearly degenerate* ([**25**]; see [**4**] for a more recent result and additional references). Neither the deterministic version of equation (2.1) nor the reduced systems above possess either of these properties. Therefore we cannot apply existing uniqueness arguments to (2.9). A review of the literature does not reveal a uniqueness result general enough to guarantee uniqueness for (2.9). Thus uniqueness remains an open question; however, physical and mathematical arguments suggest that such results can eventually be obtained. For now, we must be satisfied with

CONJECTURE 3.3 (uniqueness). *Moment equations* (2.9) *or* (2.11) *have at most one vanishing-viscosity solution for uniformly bounded, measurable initial data.*

Now we may obtain a more general form of the covariance relationship stated earlier, directly from the moment equations.

LEMMA 3.4. *If there exists a unique solution to* (2.9) *with bounded, measurable initial data, then*

$$c_s\, c_v = c_{sv}\, \widehat{c}_{sv}. \tag{3.4}$$

The proof in [**13**] uses the fact that the same PDE is satisfied by both $c_{sv}\,\widehat{c}_{sv}$ and $c_s\,c_v$. Again $\sigma_{sv}=\sigma_s\,\sigma_v$ follows, by letting $y\to x$ so that $c_s\to\sigma_s^2$ and $\widehat{c}_{sv}\to\sigma_{sv}$. It is important to recognize that this covariance relationship follows

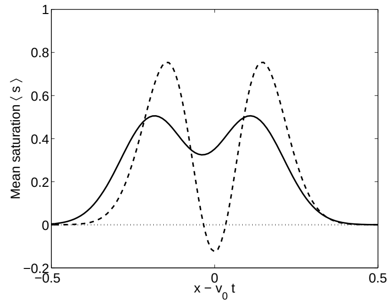

FIGURE 1. Mean saturation, linear flux with Gaussian initial profile for uniform mean flow. Both the two-term (solid) and infinite-term (dotted) moments exhibit bimodal behavior. The latter violates physical bounds on saturation.

from observation. That it is also a consequence of the moment equations (subject to uniqueness) shows that the MDEs are consistent with this intuitive result.

4. Solution

Solutions of nonlinear hyperbolic conservation laws consist of shock and rarefaction curves in phase space [**23**], [**25**]. We present results for the linear advection case first, then a solution for nonlinear advection with the two-term expansion.

4.1. Linear advection. Figure 1 shows mean saturation obtained from (3.2) and (3.3) for linear fractional flow ($f(s) = s$) with Gaussian initial profile. This linear-flux case represents the pure-advection form of conservative solute transport in single-phase flow, with s representing concentration of solute. Both solutions are bimodal, and the solution to (3.3) violates physical bounds on saturation.

This bimodal profile is a non-physical result; we do not expect an initial localized pulse of solute to have bimodal behavior in the mean. Adding a linear diffusive effect will not eliminate bimodality, as is evident in the nonlinear case (§4.2), where numerical and artificial diffusion are present in our numerical scheme.

We have used these standard results for uniform mean flow for $x \in [0, L]$, with stationary log hydraulic conductivity ($\langle Y \rangle$ and σ_Y do not depend on x) [**5**], [**10**]:

$$\langle v(x) \rangle = \frac{K_G J}{\phi}\left(1 - \frac{\sigma_Y^2}{2}\right), \qquad \sigma_v^2 = \left(\frac{K_G J}{\phi}\right)^2 \sigma_Y^2,$$

where $K_G = \exp(\langle Y \rangle)$ and J is the negative mean head gradient. We used parameters $K_G\, J = 0.5$, $\phi = 0.2$, and $\sigma_Y = 0.5$.

REMARK 4.1. Bimodal mean concentration (or saturation, in our case) is noted in [**15**], [**18**], and [**19**]. All use methods to derive mean transport equations that do not involve second-order corrections. Further details and comparison of our work to these results can be found in [**12**] and [**13**].

4.2. Nonlinear advection. Equation (3.2) is in the form (3.1) with $\mathbf{G} \equiv 0$. For nonlinear $\mathbf{F}(\mathbf{u})$, a solution consists of a sequence of shock waves, constant states, and rarefaction waves. We will omit considerable detail that requires an appeal to the elegant theory of hyperbolic conservation laws. For much more detail, see [**13**].

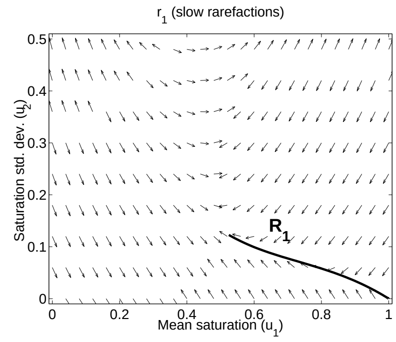

FIGURE 2. Construction of the slow rarefaction R_1 from $(1, 0)$.

We refer to the space with coordinates $\mathbf{u} = (u_1,\, u_2)$ as *phase space*. Rarefaction waves are integral curves of the vector fields defined by eigenvectors of DF. Shocks are allowed if we consider the weak form of (3.1). Then the *Rankine–Hugoniot* condition gives an expression for the speed of the shock, and entropy conditions are imposed to capture the vanishing-viscosity solution. In particular, we impose entropy conditions of Lax [**25**] and an extension due to Liu [**21**], [**22**].

We consider sure (deterministic) initial saturation given by the Heaviside function $s(x, 0) = H[-x]$, so that σ_s is initially zero. First we simplify the notation of (3.2) as follows. Set $u_1 = \langle s \rangle$ and $u_2 = \sigma_s$, and introduce the scaling $\tau = \langle v \rangle\, t$, so that $\partial_t = \langle v \rangle\, \partial_\tau$. Let $\epsilon \equiv \sigma_v / \langle v \rangle$. This ratio is consistent with the ϵ previously used as the expansion parameter. The scaled equations are

$$\partial_\tau \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \partial_x \begin{pmatrix} f + \epsilon f' u_2 + \frac{1}{2} f'' u_2^2 \\ \epsilon f + f' u_2 \end{pmatrix} = 0. \tag{4.1}$$

Note that the argument of $f^{(k)}$, the kth derivative of f, is always u_1. Let $\mathbf{u} = (u_1,\, u_2)$. The Jacobian matrix is

$$DF(\mathbf{u}) = \begin{pmatrix} f' + \epsilon f'' u_2 + \frac{1}{2} f^{(3)} u_2^2 & \epsilon f' + f'' u_2 \\ \epsilon f' + f'' u_2 & f' \end{pmatrix}.$$

From §3.1, its eigenvalues are real and may be ordered $\lambda_1 < \lambda_2$ except at $\mathbf{u} \in \{(0, 0),\, (1, 0)\}$, where both are zero. The associated eigenvectors are $\mathbf{r}_k(\mathbf{u})$, $k = 1, 2$.

We look for solutions to (4.1) in the phase space for $(u_1,\, u_2)$. The points $\mathbf{u} = (1, 0)$ and $\mathbf{u} = (0, 0)$ are endpoints of the solution to (4.1). They represent a boundary condition and the initial condition within the spatial domain, respectively. We find that a rarefaction connects to $(1, 0)$. This curve R_1 is shown in figure 2 to be an integral curve of the eigenvector associated with the smaller eigenvalue (a "slow" rarefaction). A shock must connect $(0, 0)$. Using the Rankine–Hugoniot and entropy conditions, we construct the shock curve S_2 (see figure 3).

The simplest connection between R_1 and S_2 is a single slow shock, denoted S_1 (see figure 3). The complete solution constructed in this manner is shown to satisfy entropy conditions in [**13**]. The solution in physical space is shown in figure 4, and is compared to the solution obtained from our numerical PDE scheme, for $L = 2,\ m = 1/2,\ \langle v \rangle = 5/2,\ \sigma_v = 5/4$ at a fixed time $t = 0.2$.

Uniqueness of the solution remains in question. Most results, again, require genuine nonlinearity or linear degeneracy, and our system does not satisfy these

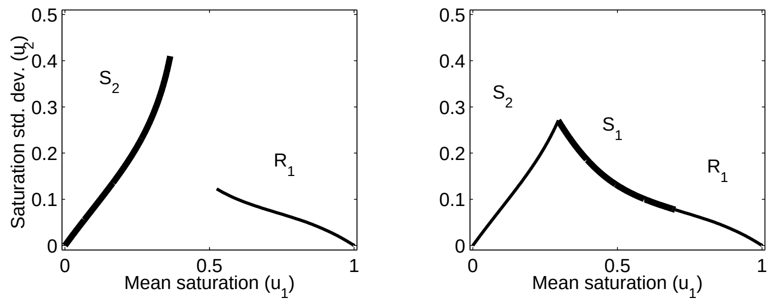

FIGURE 3. Construction of shocks connecting to rarefaction R_1 and $(0,0)$. First, a fast shock S_2 connecting to $(0,0)$ is constructed. Then a slow shock is introduced to connect R_1 and S_2.

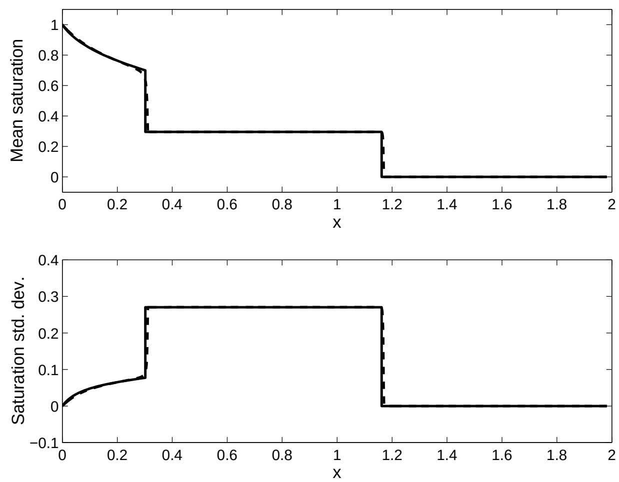

FIGURE 4. Mean and standard deviation of saturation. The solid curve is obtained from semi-analytical construction of shocks and rarefaction curves in phase space; dashed curve is obtained from numerical PDE scheme applied directly to (4.1).

conditions. Liu extended existence and uniqueness results to more general systems [**21**], [**22**], but the restrictions he places on flux functions are not met by F_1 and F_2. However, it is clear in figure 4 that this solution matches the solution obtained from our numerical PDE scheme, applied directly to (4.1). Our upwind PDE scheme is conservative, and includes numerical and artificial diffusion. The solution obtained using this scheme is therefore an approximation to the viscous solution. The diffusion coefficient is roughly three orders of magnitude smaller than the jumps in solution values. Thus the numerical solution is near the vanishing-viscosity limit. Furthermore, this shows that linear diffusion terms do not eliminate bimodality in the solution. Such terms only smooth out mean saturation fronts.

The saturation variance is supported primarily on an uncertainty interval between fronts. Physically, the solution represents two zones containing mixtures of the two fluid phases (for example, water and oil), and a third containing only the

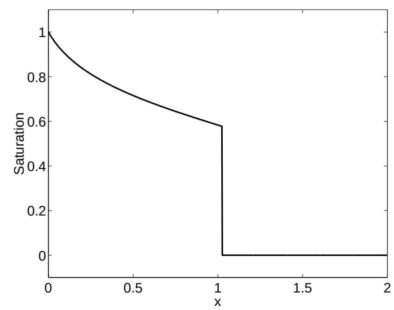

FIGURE 5. Buckley–Leverett saturation profile at $t > 0$.

oil phase. In the first zone, we have a smoothly varying mixture from the injection boundary ($x = 0$), where the mean oil content tends to zero, down to a constant mixture just left of a shock. In the second zone, we have a constant mix of oil and water. The solution does not represent physical reality. Rather than two shock waves, the true mean saturation is more likely to have a smooth, diffuse profile. Taken with the profile of σ_s, however, these second-order solutions provide some insight into the propagation of uncertainty in two-phase flow.

In the limit $\sigma_v \to 0$, the mean saturation tends to the classic Buckley–Leverett profile shown in figure 5. In fact, the construction of the rarefaction and shock profile is analogous to the solution of the scalar deterministic saturation equation. For that solution, an initial rarefaction is followed forward in x, up to a point where the shock speed matches the characteristic speed.

5. Conclusions

For spatial resolution of uncertainty, these bimodal results suggest that second-order Eulerian MDEs may be inappropriate for 1-D immiscible flow. Unlike 1-D, $\mathbf{v}$ has finite correlation length in 2-D, and macrodispersion models are based on correlations in $\delta\mathbf{v}$. In results for analogous MDEs for passive 2-D solute transport with diffusion, bimodality was not observed [**11**]; one might expect macrodispersion to lead to a similar result for immiscible flow. Nevertheless, in a forthcoming submission we show that somewhat mitigated bimodality does persist in 2-D, even with diffusion terms [**14**]. More positively, for spatial averages such as oil-production curves, good matches to Monte Carlo simulations are found.

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