An Iterative Method for Generalized Set-Valued Nonlinear Mixed Quasi-Variational Inequalities

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December 26, 2001

Abstract

This paper presents an iterative method for solving the generalized nonlinear set-valued mixed quasi-variational inequality, a problem class that was introduced by Huang, Bai, and Kang, [Comp. Math. Appl., 40, 2000]. The method incorporates step size controls that enable application to problems where certain set-valued mappings do not always map to nonempty closed bounded sets.

1 Introduction

In recent years, a significant number of publications have appeared that define generalizations of the variational and quasi-variational inequality problems; see, for example, [1, 3, 4, 6, 8, 13, 9, 10, 11, 2, 12, 15], and references therein. One of the most general of these new problem classes is the generalized nonlinear set-valued mixed quasi-variational inequality (GNSVMQVI), which was introduced and studied in [5]. Before the GNSVMQVI can be defined, some definitions are needed. Let H be a real Hilbert space with norm $\|\cdot\|$ and inner product $\langle \cdot, \cdot \rangle$. Let 2^H represent the family of all subsets of H. A set-valued mapping $F : H \to 2^H$ is said to be *monotone* if for all $x_1, x_2 \in H$, $y_1 \in F(x_1), y_2 \in F(x_2)$,

$$\langle x_1 - x_2, y_1 - y_2 \rangle \geq 0.$$

F is said to be *maximal monotone* if its graph (i.e., the set $\{(x, y) \mid y \in F(x)\}$) is not properly contained in the graph of any other monotone mapping. The *effective domain* of F, denoted $\mathrm{dom}(F)$ is the set $\{x \mid F(x) \neq \emptyset\}$.

Let $G, S, T : H \to 2^H$ be set-valued mappings, and let $p : H \to H$ and $N : H \times H \to H$ be single-valued mappings. Suppose that $A : H \times H \to 2^H$

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is a set-valued mapping such that for each fixed $t \in H$, $A(\cdot, t) : H \to 2^H$ is a maximal monotone mapping and $\text{Range}(p) \bigcap \text{dom}(A(\cdot, t)) \neq \emptyset$ for each $t \in H$. The GNSVMQVI is to find $u \in H, x \in S(u), y \in T(u), z \in G(u)$ such that $p(u) \in \text{dom}\, A(\cdot, z)$ and

$$0 \in N(x, y) + A(p(u), z). \tag{1}$$

The above definition differs from the one given by Huang, et al. [5] in one important respect: Huang, et al. restricted 2^H to be the family of all *nonempty* subsets of H. In other words, they restricted the mappings G, S, T and A to map only to nonempty sets. This restriction is not at all unusual in the literature. In fact, many of the recent generalizations of quasi-variational inequalities have similar restrictions, apparently because the restriction is needed to make the algorithms work. However, this restriction is of considerable negative consequence because it prevents the application of the GNSVMQVI framework to certain problem classes.

As a simple example, let X be a convex subset of H and let $f : H \to H$ be a single-valued operator. The *variational inequality problem* (VI) is to

$$\text{find } x \in X \text{ such that } \langle f(x), z - x \rangle \geq 0 \text{ for all } z \in X.$$

It is well known (see, for example [14]) that this problem is equivalent to the *generalized equation*

$$\text{find } x \in H \text{ such that } 0 \in f(x) + N_X(x),$$

where $N_X : H \to 2^H$ is the normal cone operator to the set X, defined by

$$N_X(x) := \begin{cases} \{z \mid \langle z, y - x \rangle \leq 0 \text{ for all } y \in X\}, & x \in X \\ \emptyset, & x \notin X. \end{cases}$$

Note that depending on the choice of x, $N_X(x)$ is either the empty set, the singleton $\{0\}$, or an unbounded cone.

Since the normal cone operator is maximal monotone, the variational inequality problem is a special case of GNSVMQVI formed by choosing S,T,G,N,A and p by the relations $S(x) := x$, $T(x) := 0$, $G(x) := X$, $N(x, y) := f(x)$, $A(x, X) := N_X(x)$ and $p(x) := x$. However, because N_X can map to the empty set, the above formulation of VI as a special case of GNSVMQVI would be excluded from the framework of [5].

It is a simple matter, as we have done in this paper, to change the definition of GNSVMQVI to remove the above difficulty. However, the algorithm proposed in [5] is not capable of solving the unrestricted problem. In fact, the main convergence theorem for that algorithm requires that S, T, and G map everywhere to nonempty, closed, bounded sets. Therefore, this paper proposes a new iterative method for solving GNSVMQVI that does not require the set-valued mappings to map everywhere to nonempty or bounded sets. To prove convergence to a solution, we do however, assume that S, G and T map only to closed sets. This algorithm is an adaptation of Algorithm 1 from [5]. The main change is to introduce step-size controls that enable the algorithm to ensure that the iterates stay in the effective domain of all of the set-valued mappings.

2 Preliminaries

Let $F : H \to 2^H$ be a maximal monotone mapping. Given a constant $\rho > 0$, the *resolvent operator* for F is defined by

$$J_\rho^F := (I + \rho F)^{-1}. \tag{2}$$

where I is the identity mapping on H and the inverse is the set-valued inverse defined by $F^{-1}(y) = \{x \mid y \in F(x)\}$. It is well-known that J_ρ^F is a single-valued, nonexpansive mapping [7].

We define a pseudo-metric $M : 2^H \times 2^H \to \mathbb{R} \bigcup \{\infty\}$ by

$$M(\Gamma, \Lambda) := \max \left\{ \sup_{u \in \Gamma} \mathrm{dist}\,(u \mid \Lambda)\,, \sup_{v \in \Lambda} \mathrm{dist}\,(v \mid \Gamma)) \right\}, \tag{3}$$

where $\mathrm{dist}\,(u \mid S) := \inf_{v \in S} \|u - v\|$. Note that if the domain of M is restricted to closed bounded sets, then M is the Hausdorff metric.

3 Iterative Algorithm

A key to solving (1) is the following lemma, which relates solutions of (1) to the resolvent operator for $A(\cdot, z)$:

Lemma 3.1 *([5, Lemma 3.1]) (u, x, y, z) is a solution of problem (1) if and only if (u, x, y, z) satisfies the relation*

$$p(u) = J_\rho^{A(\cdot,z)}(p(u) - \rho N(x, y))), \tag{4}$$

where $\rho > 0$ is a constant and $J_\rho^{A(\cdot,z)}$ is the resolvent operator defined by (2).

To develop a fixed point algorithm for (1), we rewrite (4) as follows:

$$u = u - p(u) + J_\rho^{A(\cdot,z)}\left(p(u) - \rho N(x, y)\right), \tag{5}$$

where $\rho > 0$ is a constant. This fixed point formulation allows us to suggest the following iterative algorithm.

Algorithm 1

Step 0 Let $\rho > 0$ be a constant. Choose $u_0 \in \mathrm{int}(\mathrm{dom}(S) \bigcap \mathrm{dom}(T) \bigcap \mathrm{dom}(G))$ and choose $x_0 \in S(u_0), y_0 \in T(u_0)$, and $z_0 \in G(u_0)$. Set $n = 0$.

Step 1 Let

$$u_{n+1} = u_n + \alpha_n \left(-p(u_n) + J_\rho^{A(\cdot,z_n)}(p(u_n) - \rho N(x_n, y_n) \right), \tag{6}$$

where $\alpha_n \in (0, 1]$ is chosen sufficiently small to ensure that $u_{n+1} \in \mathrm{int}(\mathrm{dom}(S) \bigcap \mathrm{dom}(T) \bigcap \mathrm{dom}(G))$.

Step 2 Choose $\epsilon_{n+1} \geq 0$, and choose $x_{n+1} \in S(u_{n+1}), y_{n+1} \in T(u_{n+1}), z_{n+1} \in G(u_{n+1})$ satisfying

$$\|x_{n+1} - x_n\| \leq (1 + \epsilon_{n+1})M(S(u_{n+1}), S(u_n)), \quad (7)$$
$$\|y_{n+1} - y_n\| \leq (1 + \epsilon_{n+1})M(T(u_{n+1}), T(u_n)), \quad (8)$$
$$\|z_{n+1} - z_n\| \leq (1 + \epsilon_{n+1})M(G(u_{n+1}), G(u_n)), \quad (9)$$

Step 3 If $u_{n+1}, x_{n+1}, y_{n+1}, z_{n+1}$ satisfy (4) to sufficient accuracy, stop; otherwise, set $n := n + 1$ and return to Step 1.

Discussion From the definition of M, (3), it is clear that the restrictions (7) – (9) imposed on the points x_n, y_n, and z_n can always be satisfied for any $\epsilon_n > 0$. If S, T, and G always map to closed bounded sets, then the restrictions can be satisfied with $\epsilon_n = 0$.

Since u_n is always in the interior of the intersections of the domains of S, T, and G, it is always possible to choose positive values of α_n that ensure that u_{n+1} remains in the interior of the intersections of the domains of S, T, and G.

In order to ensure convergence, we will need to make the additional assumption that $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Note that for $\alpha_n = 1$, Algorithm 1 collapses to Algorithm 3.1 of Huang, et al. [5].

4 Existence and Convergence Theorems

This section proves, under similar conditions used in [5], that the iterates produced by Algorithm 1 converge to a solution of problem (1). Note, however, that we do not require that S, G, T and A map to nonempty, or bounded sets.

Definition 4.1 *A mapping $g : H \to H$ is said to be* strongly monotone *if there exists some $\delta > 0$ such that*

$$\langle g(u_1) - g(u_2), u_1 - u_2 \rangle \geq \delta \|u_1 - u_2\|^2,$$

for all $u_1, u_2 \in H$. g is Lipschitz continuous *if there exists some $\sigma > 0$ such that*

$$\|g(u_1) - g(u_2)\| \leq \sigma \|u_1 - u_2\|,$$

for all $u_1, u_2 \in H$.

Definition 4.2 *A set-valued mapping $S : H \to 2^H$ is said to be M-*Lipschitz continuous *if there exists a constant $\eta > 0$ such that*

$$M(S(u_1), S(u_2)) \leq \eta \|u_1 - u_2\|,$$

for all $u_1, u_2 \in H$. S is strongly monotone with respect to the first argument of $N(\cdot, \cdot) : H \times H \to H$*, if there exists a constant $\zeta > 0$ such that*

$$\langle N(x_1, \cdot) - N(x_2, \cdot), u_1 - u_2 \rangle \geq \zeta \|u_1 - u_2\|^2$$

for all $x_1 \in S(u_1), x_2 \in S(u_2)$.

Definition 4.3 *The operator* $N : H \times H \to H$ *is said to be* Lipschitz continuous with respect to the first argument *if there exists a constant* $\beta > 0$ *such that*

$$\|N(u_1, \cdot) - N(u_2, \cdot)\| \leq \beta \|u_1 - u_2\|$$

for all $u_1, u_2 \in H$*. Similarly,* N *is Lipschitz continuous with respect to the second argument if there exists* $\xi > 0$ *such that*

$$\|N(\cdot, v_1) - N(\cdot, v_2)\| \leq \xi \|v_1 - v_2\|$$

for all $v_1, v_2 \in H$*.*

Lipschitz continuity of N with respect to the second argument is defined similarly.

The following technical lemma will be needed in the proof of our main theorem.

Lemma 4.4 *Let* δ *and* σ *be positive scalars with* $\delta \leq \sigma$*. Then for all* $\alpha \in [0, 1]$,

$$1 - 2\delta\alpha + \sigma^2\alpha^2 \leq \left(1 - \alpha + \alpha\sqrt{1 - 2\delta + \sigma^2}\right)^2.$$

Proof Since $\sigma \geq \delta > 0$, we have

$$\begin{aligned} 1 - 2\delta + \sigma^2 &= (1-\delta)^2 + \sigma^2 - \delta^2 \\ &\geq (1-\delta)^2 \geq 0. \end{aligned}$$

Thus,

$$\begin{aligned} 1 - \delta &\leq \sqrt{1 - 2\delta + \sigma^2} \\ \text{so,} \quad 2\alpha(1-\alpha)(1-\delta) &\leq 2\alpha(1-\alpha)\sqrt{1 - 2\delta + \sigma^2}. \end{aligned}$$

Adding $(1-\alpha)^2 + \alpha^2(1 - 2\delta + \sigma^2)$ to both sides and simplifying yields the desired result. □

For the following theorem, define $C(H)$ to be the collection of all closed subsets of H.

Theorem 4.5 *Let* N *be Lipschitz continuous with respect to the first and second arguments with constants* β *and* ξ *respectively. Let* $S, T, G : H \to C(H)$ *be* M*-Lipschitz with constants* η, γ *and* μ*, respectively; suppose that* S *is strongly monotone with respect to the first argument of* $N(\cdot, \cdot)$ *with constant* ζ*; and suppose that* $\text{int}(\text{dom}(S) \bigcap \text{dom}(T) \bigcap \text{dom}(G)) \neq \emptyset$*. Let* $p : H \to H$ *be strongly monotone and Lipschitz continuous with constants* δ *and* σ*, respectively. Suppose that there exist constants* $\lambda > 0$ *and* $\rho > 0$ *such that, for each* $x, y, z \in H$*,*

$$\left\| J_\rho^{A(\cdot,x)}(z) - J_\rho^{A(\cdot,y)}(z) \right\| \leq \lambda \|x - y\| \tag{10}$$

and

$$\theta := 1 - 2\sqrt{1-2\delta+\sigma^2} - \sqrt{1-2\rho\xi+\rho^2\beta^2\eta^2} - \lambda\mu - \rho\xi\gamma > 0. \tag{11}$$

If $\epsilon_n \to 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$, then there exist $u \in H$, $x \in S(u)$, $y \in T(u)$, and $z \in G(u)$ satisfying problem (1), and the sequences $\{u_n\}$,$\{x_n\}$,$\{y_n\}$,$\{z_n\}$, generated by Algorithm 1 converge strongly in H to u, x, y and z, respectively.

Proof For $n = 0, 1, \ldots$, define

$$\Gamma_n := -p(u_n) + J_\rho^{A(\cdot,z_n)}\left(p(u_n) - \rho N(x_n, y_n)\right) \tag{12}$$

and note that

$$u_{n+1} = u_n + \alpha_n \Gamma_n. \tag{13}$$

We will first establish a bound on $\|\Gamma_n\|$. From (12) and (13), we have

$$\begin{aligned}
\|\Gamma_n\| &= \|\Gamma_{n-1} + \Gamma_n - \Gamma_{n-1}\| = \|(u_n - u_{n-1})/\alpha_{n-1} + \Gamma_n - \Gamma_{n-1}\| \\
&\le \|(u_n - u_{n-1})/\alpha_{n-1} - (p(u_n) - p(u_{n-1}))\| \quad + \\
&\qquad \left\| J_\rho^{A(\cdot,z_n)}\left(p(u_n) - \rho N(x_n, y_n)\right) - J_\rho^{A(\cdot,z_{n-1})}\left(p(u_{n-1}) - \rho N(x_{n-1}, y_{n-1})\right) \right\| \\
&\le \|(u_n - u_{n-1})/\alpha_{n-1} - (p(u_n) - p(u_{n-1}))\| \\
&\quad + \left\| J_\rho^{A(\cdot,z_n)}\left(p(u_n) - \rho N(x_n, y_n)\right) - J_\rho^{A(\cdot,z_{n-1})}\left(p(u_n) - \rho N(x_n, y_n)\right) \right\| \\
&\quad + \Big\| J_\rho^{A(\cdot,z_{n-1})}\left(p(u_n) - \rho N(x_n, y_n)\right) \\
&\qquad\qquad - J_\rho^{A(\cdot,z_{n-1})}\left(p(u_{n-1}) - \rho N(x_{n-1}, y_{n-1})\right) \Big\|.
\end{aligned} \tag{14}$$

By (10), (9), and the M-Lipschitz continuity of G,

$$\begin{aligned}
&\left\| J_\rho^{A(\cdot,z_n)}\left(p(u_n) - \rho N(x_n, y_n)\right) - J_\rho^{A(\cdot,z_{n-1})}\left(p(u_n) - \rho N(x_n, y_n)\right) \right\| \\
&\qquad \le \lambda \|z_n - z_{n-1}\| \\
&\qquad \le \lambda(1+\epsilon_n) M(G(u_n), G(u_{n-1})) \\
&\qquad \le \lambda\mu(1+\epsilon_n)\|u_n - u_{n-1}\|.
\end{aligned} \tag{15}$$

Since $J_\rho^{A(\cdot,z_{n-1})}$ is nonexpansive, the last term in (14) is bounded by

$$\begin{aligned}
&\Big\| J_\rho^{A(\cdot,z_{n-1})}\left(p(u_n) - \rho N(x_n, y_n)\right) \\
&\qquad\qquad - J_\rho^{A(\cdot,z_{n-1})}\left(p(u_{n-1}) - \rho N(x_{n-1}, y_{n-1})\right) \Big\| \\
&\le \|p(u_n) - \rho N(x_n, y_n) - (p(u_{n-1}) - \rho N(x_{n-1}, y_{n-1}))\| \\
&\le \|u_n - u_{n-1} - (p(u_n) - p(u_{n-1}))\| \\
&\quad + \|u_n - u_{n-1} - \rho\left(N(x_n, y_n) - N(x_{n-1}, y_n)\right)\| \\
&\quad + \rho\|N(x_{n-1}, y_{n-1}) - N(x_{n-1}, y_n)\|.
\end{aligned} \tag{16}$$

Since p is strongly monotone and Lipschitz continuous,

$$\begin{aligned}
&\|(u_n - u_{n-1}) - (p(u_n) - p(u_{n-1}))\|^2 \\
&= \|u_n - u_{n-1}\|^2 - 2\langle u_n - u_{n-1}, p(u_n) - p(u_{n-1})\rangle + \|p(u_n) - p(u_{n-1})\|^2 \\
&\leq (1 - 2\delta + \sigma^2)\|u_n - u_{n-1}\|^2. \qquad (17)
\end{aligned}$$

Similarly, since S is strongly monotone with respect to the first argument of N, and N is Lipschitz continuous with respect to the first argument, then

$$\begin{aligned}
&\|u_n - u_{n-1} - \rho(N(x_n, y_n) - N(x_{n-1}, y_n))\|^2 \\
&= \|u_n - u_{n-1}\|^2 - 2\rho\langle N(x_n, y_n) - N(x_{n-1}, y_n), u_n - u_{n-1}\rangle \\
&\quad + \rho^2\|N(x_n, y_n) - N(x_{n-1}, y_n)\|^2 \\
&\leq (1 - 2\rho\zeta + \rho^2\beta^2\eta^2)\|u_n - u_{n-1}\|^2. \qquad (18)
\end{aligned}$$

Using the Lipschitz continuity of the operator $N(\cdot,\cdot)$ with respect to the second argument and M-Lipschitz continuity of T for all $y \in T(u)$, we have:

$$\begin{aligned}
\|N(x_{n-1}, y_n) - N(x_{n-1}, y_{n-1})\| &\leq \xi\|y_n - y_{n-1}\| \\
&\leq \xi(1 + \epsilon_n)M(T(u_n), T(u_{n-1})) \\
&\leq \xi\gamma(1 + \epsilon_n)\|u_n - u_{n-1}\|. \qquad (19)
\end{aligned}$$

Finally, by similar arguments to the derivation of (17), we have

$$\begin{aligned}
&\|(u_n - u_{n-1})/\alpha_{n-1} - (p(u_n) - p(u_{n-1}))\|^2 \\
&\leq \frac{1}{\alpha_{n-1}^2}(1 - 2\delta\alpha_{n-1} + \sigma^2\alpha_{n-1}^2)\|u_n - u_{n-1}\|^2 \qquad (20) \\
&\leq \frac{1}{\alpha_{n-1}^2}\left(1 - \alpha_{n-1} + \alpha_{n-1}\sqrt{1 - 2\delta + \sigma^2}\right)^2\|u_n - u_{n-1}\|^2, \qquad (21)
\end{aligned}$$

where the last inequality follows from Lemma 4.4 and the fact that the Lipschitz constant σ of p must be at least as large as the constant of monotonicity δ.

This, together with (14)–(19), yields

$$\|\Gamma_n\| \leq (1 - \alpha_{n-1}\theta_n)\|u_n - u_{n-1}\|/\alpha_{n-1} = (1 - \alpha_{n-1}\theta_n)\|\Gamma_{n-1}\|, \qquad (22)$$

where $\theta_n := 1 - \left\{2\sqrt{1 - 2\delta + \sigma^2} + (1 + \epsilon_n)(\lambda\mu + \rho\xi\gamma) + \sqrt{1 - 2\rho\zeta + \rho^2\beta^2\eta^2}\right\}$. Since $\epsilon_n \to 0$, then $\theta_n \to \theta$. By (11), $\theta > 0$. Thus, for all n sufficiently large, $\theta_n \geq \theta/2 > 0$. Define $\Phi := \theta/2$. Without loss of generality, we can assume $\theta_n \geq \Phi > 0$ for all n. It follows that

$$\|\Gamma_n\| \leq \|\Gamma_0\| \prod_{i=0}^{n-1}(1 - \alpha_i\Phi).$$

Since $\sum_{n=0}^{\infty}\alpha_n = \infty$, we conclude that $\lim_{n\to\infty}\|\Gamma_n\| = 0$ and therefore

$$\lim_{n\to\infty}\|u_n - u_{n-1}\| = 0.$$

Next, we show that $\{u_n\}$ converges. Let m be an arbitrary index. Since $\sum_{i=0}^{\infty} \alpha_i = \infty$ and $\alpha_i \leq 1$, there exists a sequence $\{k_j\}$ of indices, with $k_0 = m$ such that

$$1 \leq \sum_{i=k_j}^{k_{j+1}-1} \alpha_i < 2. \tag{23}$$

Define

$$\begin{aligned} \kappa_j &:= \left(\prod_{i=k_j}^{k_{j+1}-1} (1 - \alpha_i \Phi) \right)^{1/(k_{j+1}-k_j)} \qquad \text{and} \\ \tau_j &:= \left(\sum_{i=k_j}^{k_{j+1}-1} (1 - \alpha_i \Phi) \right) / (k_{j+1} - k_j). \end{aligned}$$

Note that κ_j and τ_j are the geometric and arithmetic means, respectively, of $(1 - \alpha_{k_j}\Phi), (1 - \alpha_{k_j+1}\Phi), \ldots, (1 - \alpha_{k_{j+1}-1})$; so $\kappa_j \leq \tau_j$. Thus,

$$\begin{aligned} \prod_{i=k_j}^{k_{j+1}-1} (1 - \alpha_i \Phi) &= \kappa_j^{(k_{j+1}-k_j)} \\ &\leq \tau_j^{(k_{j+1}-k_j)} \\ &= \left(\frac{\sum_{i=k_j}^{k_{j+1}-1} (1 - \alpha_i \Phi)}{k_{j+1} - k_j} \right)^{(k_{j+1}-k_j)} \\ &= \left(1 - \frac{\Phi \sum_{i=k_j}^{k_{j+1}-1} \alpha_i}{k_{j+1} - k_j} \right)^{(k_{j+1}-k_j)} \\ &\leq \left(1 - \frac{\Phi}{k_{j+1} - k_j} \right)^{(k_{j+1}-k_j)} \qquad \text{(using (23))} \\ &\leq e^{-\Phi}. \end{aligned} \tag{24}$$

It follows that

$$\left\| \Gamma_{k_{j+1}} \right\| \leq e^{-\Phi} \left\| \Gamma_{k_j} \right\| \leq (e^{-\Phi})^{j+1} \left\| \Gamma_m \right\|. \tag{25}$$

Thus,

$$\begin{aligned} \lim_{n \to \infty} \|u_n - u_m\| &\leq \sum_{i=0}^{\infty} \alpha_i \|\Gamma_i\| \\ &= \sum_{j=0}^{\infty} \sum_{i=k_j}^{k_{j+1}-1} \alpha_i \|\Gamma_i\| \end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{j=0}^{\infty} 2 \left\| \Gamma_{k_j} \right\| \quad \text{(by (23))} \\
&\leq 2 \left\| \Gamma_m \right\| \sum_{j=0}^{\infty} (e^{-\Phi})^j \quad \text{(by (25))} \\
&= 2 \left\| \Gamma_m \right\| / (1 - e^{-\Phi}).
\end{aligned}
$$

Since $\lim_{m\to\infty} \|\Gamma_m\| = 0$, it follows that $\lim_{n,m\to\infty} \|u_n - u_m\| = 0$, so $\{u_n\}$ converges strongly to some fixed $u \in H$.

Now we prove that $x_n \to x \in S(u)$. From (7), we have:

$$
\|x_n - x_{n-1}\| \leq (1 + \epsilon_n) M(S(u_n), S(u_{n-1})) \leq 2\eta \|u_n - u_{n-1}\|
$$

which implies that $\{x_n\}$ is a Cauchy sequence in H, so there exists $x \in H$ such that $x_n \to x$. Further,

$$
\begin{aligned}
d(x, S(u)) &= \inf \{ \|x - t\| \mid t \in S(u) \} \leq \|x - x_n\| + d(x_n, S(u)) \\
&\leq \|x - x_n\| + M(S(u_n), S(u)) \leq \|x - x_n\| + \eta \|u_n - u\| \to 0.
\end{aligned}
$$

Hence, since $S(u)$ is closed, we have $x \in S(u)$. Similarly, $\{y_n\}$ converges to some fixed $y \in T(u)$ and $\{z_n\}$ converges to some fixed $z \in G(u)$.

By continuity, u, x, y, z satisfy (4) and is therefore solve (1). □

5 Summary

The above theorem shows that Algorithm 1 converges to a solution under conditions similar to those used in Huang et al. [5]; however it does not require that the set-valued mappings map only to nonempty or bounded sets. Such an advance is important because it enables some well known problems to be solved as instances of GNSVMQVI. The key to this advance is the introduction of the stepsize α_n in Step 1 of the algorithm. The ideas behind this algorithm and the convergence proof are applicable to many other generalizations of quasi-variational inequalities. We developed them for the GNSVMQVI because it is among the most general such problem classes studied to date.

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