Pressure bubbles stabilization features in the Stokes problem

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Abstract

The standard finite element approximation using equal-order-linear-continuous velocity-pressure variables is enriched with velocity and pressure bubble functions to model the Stokes problem. We show by static condensation that these bubble functions give rise to a stabilized method involving least-squares forms of the momentum and of the continuity equations. In particular, pressure bubbles play a key role in explaining the addition of the least-squares form of the continuity equation in a stabilized method for Stokes.

1 Introduction

Finite element approximations to the Stokes problem have been proposed and analyzed by several authors (see [5, 12] for an overview and references through 1990). The Galerkin method for this method is governed by the satisfaction of the Babuška-Brezzi condition [2, 3]. This condition limits the choice of subspaces to be used for velocity and pressure. In particular, equal-order linear piecewise polynomials are ruled out by this condition.

In order to be able to use the linear-velocity-pressure pair one can proceed to stabilize the Galerkin method via two routes: 1) To add bubble functions to the formulation; 2) To add perturbation terms which depend on the residuals of the governing equations. These strategies go back to the works [6] and [14] and have been generalized in various applications (see [10] for a review of stabilized methods for Stokes).

In [1] it was shown that a cubic bubble added to the piecewise-linear continuous velocity space rendered the equal-order linear velocity-pressure pair stable and convergent. This enriched pair was denoted by MINI element. Pierre [15] has shown that by eliminating the cubic bubble using static condensation, we recover the stabilized method proposed in [14].

This relation between the Galerkin method enriched with bubbles and stabilized methods has been extended to the advection-diffusion equation [4], providing us with an explanation on the legitimacy of appending perturbation terms to the Galerkin method.

When stabilized methods were being investigated for Stokes, besides the perturbation term involving the residual of the momentum equation, a perturbation term involving the residual of the continuity equation was also suggested to be added to the Galerkin method [7, 9]. Although this term is not needed to get a stable formulation, it becomes crucial when we extend the stabilized formulation to approximating incompressible Navier-Stokes equations [8].

The open question was whether we can go back to the Galerkin method, start with equal-order linears, enrich it as needed, and be able to reproduce the stabilized formulation with both perturbation terms. This work presents the missing link between both strategies. We will add two linearly independent bubbles to the velocity subspace and one bubble to the pressure subspace. Proceeding with static condensation we will be able to relate to the stabilized formulation containing both perturbation terms, shedding some light where least-squares of the continuity equation comes from.

Our study assumes: a) the differential operator acting on velocity is the vector Laplacian; b) the elements are allowed an angle-preserving stretch and/or rotation and/or translation from a parent domain element; c) the source is piecewise constant.

Despite these limitations, this work opens the door to enriching the pressure with special bubble functions that can be applied to Navier-Stokes equations (for example, residual-free bubbles [11]).

We organize the remainder of the paper as follows: In Section 2 the bubble selection is presented and the corresponding stabilized method is displayed; in Section 3 we make preliminary computations under the assumption of the meshes being allowed; in Section 4, we proceed with static condensation to obtain the link to this stabilized method with the least-squares of the continuity equation.

2 Enriching the velocity and pressure spaces with bubble functions

Let Ω be an open, bounded subset of $\mathbb{R}^2$ with smooth boundary Γ, $\mu > 0$ and $\boldsymbol{f} \in (L^2(\Omega))^2$. The Stokes problem is given by: find the velocity field $\boldsymbol{u}$ and the pressure field p such that

$$\begin{cases} -\mu\Delta\boldsymbol{u} + \nabla p = \boldsymbol{f} & \text{in } \Omega \\ \nabla\cdot\boldsymbol{u} = 0 & \text{in } \Omega \\ \boldsymbol{u} = \boldsymbol{0} & \text{on } \Gamma \end{cases} \tag{1}$$

Let V_h and Q_h be finite-dimensional subspaces of $V = (H_0^1(\Omega))^2$ and $Q = L_0^2(\Omega)$, respectively, with

$$L_0^2(\Omega) = \left\{ q \in L^2(\Omega)\ ;\ \int_\Omega q\, dx = 0 \right\} \tag{2}$$

The classical Galerkin formulation for (1) is: find $\boldsymbol{u}_h \in V_h$ and $p_h \in Q_h$ such that

$$\begin{cases} a(\boldsymbol{u}_h, \boldsymbol{v}_h) + b(\boldsymbol{v}_h, p_h) & = F(\boldsymbol{v}_h) & \forall\, \boldsymbol{v}_h \in V_h \\ -b(\boldsymbol{u}_h, q_h) & = 0 & \forall\, q_h \in Q_h \end{cases}, \tag{3}$$

where

$$a(\boldsymbol{u}, \boldsymbol{v}) = \mu \int_\Omega \nabla\boldsymbol{u} : \nabla\boldsymbol{v}\, dx \ \ ,\ \ b(\boldsymbol{v}, q) = -\int_\Omega q\nabla\cdot\boldsymbol{v}\, dx\ \ , \tag{4}$$

$$F(v) = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\, dx \tag{5}$$

The choice of V_h and Q_h is limited by the Babuška-Brezzi condition:

$$\exists C > 0\ ;\ \inf_{0\neq q\in Q_h}\ \sup_{0\neq v\in V_h} \frac{b(\boldsymbol{v}, q)}{\|v\|_1 \|p\|_0} \geq C \tag{6}$$

Condition (6) does not allow, for instance, the choice of equal low-order interpolation spaces [5, Sec. VI.3], unless we add stabilization terms to the variational formulation [13, 14] or enrich the space of velocities as done with the MINI element [1].

Let $\mathcal{T}_h$ be a family of triangulations of the domain Ω. Given an element $T \in \mathcal{T}_h$, let F_T be the linear transformation from the reference element $\hat{T}$ (Fig. 1) to T.

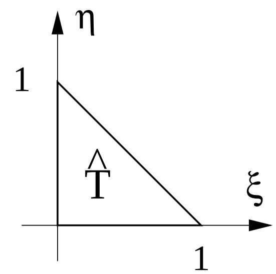

Figure 1: The reference triangular element.

Consider a Galerkin method employing piecewise linear shape functions for both velocity and pressure plus two polynomials bubbles ϕ, $\tilde{\phi}$ for velocity and one polynomial ψ for pressure within each element, that is,

$$V_h = (P_1)^2 \oplus \bigcup_T (B_V(T))^2 \tag{7}$$

$$Q_h = P_1 \oplus \bigcup_T B_Q(T) \tag{8}$$

$$B_V(T) = \{v \in H^1(T)\ ;\ v = \alpha\phi \circ F_T^{-1} + \beta\tilde{\phi} \circ F_T^{-1},\ \alpha,\ \beta \in \mathbb{R}\} \tag{9}$$

$$B_Q(T) = \{p \in H^1(T)\ ;\ p = \gamma\psi \circ F_T^{-1},\ \gamma \in \mathbb{R}\}\ , \tag{10}$$

where P_1 denotes the space of continuous, piecewise linear elements and ϕ, $\tilde{\phi}$ and ψ are defined in the reference element as follows:

$$\phi(\xi,\eta) = \xi\eta(1-\xi-\eta) \tag{11}$$

$$\tilde{\phi}(\xi,\eta) = \xi\eta(1-\xi-\eta)(\xi-\eta) \tag{12}$$

$$\psi(\xi,\eta) = (\xi-\eta)^2 \tag{13}$$

Remark 2.1 *If we only enrich with ϕ ($\beta = \gamma = 0$ in (9)-(10)), then we recover the MINI element.*

Remark 2.2 *The pressure bubble shape functions are not zero at element boundaries. This allows for discontinuous pressures when we take its linear part along with the bubble part. However, this does not pose a difficulty, since pressure of the continuous problem is a function in $L_0^2(\Omega)$.*

We will show that the Galerkin method defined by (3), (7)–(8) is related to the Galerkin least-squares (GLS) method involving the residual of both the momentum and the continuity equations [9] using equal-order piecewise linear variables: find $(\boldsymbol{u_1}, p_1) \in (P_1)^2 \times P_1$ such that

$$B_{GLS}(\boldsymbol{u_1}, p_1; \boldsymbol{v_1}, q_1) = F_{GLS}(\boldsymbol{v_1}, q_1) \;\; \forall\, (\boldsymbol{v_1}, q_1) \in (P_1)^2 \times P_1 \;\; , \tag{14}$$

$$\begin{aligned} B_{GLS}(\boldsymbol{u}, p; \boldsymbol{v}, q) &= a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) + b(\boldsymbol{u}, q) \\ &- \sum_T \delta_1 h^2 \int_T (-\mu \Delta \boldsymbol{u} + \nabla p) \cdot (-\mu \Delta \boldsymbol{v} + \nabla q)\, dx \qquad (15) \\ &+ \sum_T \delta_2 \int_T (\nabla \cdot \boldsymbol{u}) \cdot (\nabla \cdot \boldsymbol{v})\, dx \;\; , \\ F_{GLS}(\boldsymbol{v}, q) &= F(\boldsymbol{v}) - \sum_T \delta_1 h^2 \int_T \boldsymbol{f} \cdot (-\mu \Delta \boldsymbol{v} + \nabla q)\, dx \qquad (16) \end{aligned}$$

3 Preliminary computations

For simplicity, we assume that the domain Ω, the source term $\boldsymbol{f}$ and the partition $\mathcal{T}_h$ satisfy the following properties:

(i) $\boldsymbol{f}$ is piecewise constant;

(ii) Ω is a convex, polygonal domain

(iii) $\mathcal{T}_h$ is uniform and for each $T \in \mathcal{T}_h$ the transformation F_T is limited to angle-preserving stretching and/or rotating and/or translating the reference element $\hat{T}$.

From (iii), we have that F_T can be written as follows (Fig.2):

$$\begin{cases} x = h(\xi \cos\theta - \eta \sin\theta) + a \\ y = h(\xi \sin\theta + \eta \cos\theta) + b \end{cases} \;\; , \tag{17}$$

where h does not depend on T. The Jacobian of F_T is

$$\boldsymbol{J} = h \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \tag{18}$$

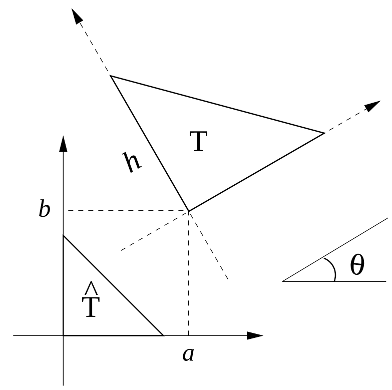

Figure 2: An admissible transformation of $\hat{T}$.

Note that $|\boldsymbol{J}| = h^2$ and $\boldsymbol{J}^{-1} = h^{-2}\boldsymbol{J}^t$. Given the functions $\hat{v} \in L^2(\hat{T})$ and $v = \hat{v} \circ F_T^{-1} \in L^2(T)$, we have the following integration formula:

$$\iint_T v \, dxdy = \iint_{\hat{T}} \hat{v} \, |\boldsymbol{J}| \, d\xi d\eta = h^2 \iint_{\hat{T}} \hat{v} \, d\xi d\eta \tag{19}$$

Moreover, $\nabla v = h^{-2}\boldsymbol{J}\nabla\hat{v}$, that is,

$$\begin{pmatrix} \dfrac{\partial v}{\partial x} \\ \\ \dfrac{\partial v}{\partial y} \end{pmatrix} = \frac{1}{h} \begin{pmatrix} \dfrac{\partial \hat{v}}{\partial \xi}\cos\theta - \dfrac{\partial \hat{v}}{\partial \eta}\sin\theta \\ \\ \dfrac{\partial \hat{v}}{\partial \xi}\sin\theta + \dfrac{\partial \hat{v}}{\partial \eta}\cos\theta \end{pmatrix} \tag{20}$$

Let T be an arbitrary element, $u^b = \phi \circ F_T^{-1}$, $v^b = \tilde{\phi} \circ F_T^{-1}$ and $p^b = \psi \circ F_T^{-1}$. We have that u^b, v^b and p^b satisfy the following:

$$\iint_T u^b \, dxdy = h^2 \iint_{\hat{T}} \phi \, d\xi d\eta = \frac{h^2}{120} \tag{21}$$

$$\iint_T v^b \, dxdy = 0 \tag{22}$$

$$\iint_T p^b \, dxdy = \frac{h^2}{12} \tag{23}$$

$$\iint_T \nabla u^b \cdot \nabla v^b \, dxdy = h^2 \iint_{\hat{T}} (\nabla\phi)^t \boldsymbol{J}^{-1} \boldsymbol{J}^{-t} (\nabla\tilde{\phi}) \, d\xi d\eta \tag{24}$$

$$
\begin{aligned}
&= \iint_{\hat{T}} (\nabla\phi)^t(\nabla\tilde{\phi})\, d\xi d\eta = 0 \\
\iint_T p^b \frac{\partial u^b}{\partial x}\, dxdy &= -\iint_T u^b \frac{\partial p^b}{\partial x}\, dxdy \\
&= -h \iint_{\hat{T}} \phi \left(\cos\theta \frac{\partial \psi}{\partial \xi} - \sin\theta \frac{\partial \psi}{\partial \eta} \right) d\xi d\eta \qquad (25) \\
&= -2h(\cos\theta + \sin\theta) \iint_{\hat{T}} (\xi - \eta)\phi\, d\xi d\eta = 0 \\
\iint_T p^b \frac{\partial u^b}{\partial y}\, dxdy &= -2h(\sin\theta - \cos\theta) \iint_{\hat{T}} (\xi - \eta)\phi\, d\xi d\eta = 0 \qquad (26)
\end{aligned}
$$

$$
\begin{aligned}
\iint_T \nabla u^b \cdot \nabla u^b\, dxdy &= \frac{1}{90} \qquad (27) \\
\iint_T \nabla v^b \cdot \nabla v^b\, dxdy &= \frac{1}{630} \qquad (28)
\end{aligned}
$$

$$
\begin{aligned}
\iint_T p^b \frac{\partial v^b}{\partial x}\, dxdy &= -2h(\cos\theta + \sin\theta) \iint_{\hat{T}} (\xi - \eta)\tilde{\phi}\, d\xi d\eta \\
&= -h \frac{\cos\theta + \sin\theta}{630} \qquad (29) \\
\iint_T p^b \frac{\partial v^b}{\partial y}\, dxdy &= -h \frac{\sin\theta - \cos\theta}{630} \qquad (30)
\end{aligned}
$$

Furthermore, if w is piecewise linear, we have

$$
\begin{aligned}
\iint_T \nabla u^b \cdot \nabla w\, dxdy &= \int_{\partial T} u^b \nabla w \cdot \boldsymbol{n}\, ds - \iint_T u^b \cdot \Delta w\, dxdy = 0 \qquad (31) \\
\iint_T \nabla v^b \cdot \nabla w\, dxdy &= \int_{\partial T} v^b \nabla w \cdot \boldsymbol{n}\, ds - \iint_T v^b \cdot \Delta w\, dxdy = 0 \qquad (32)
\end{aligned}
$$

4 Static condensation

Let $T \in \mathcal{T}_h$ an arbitrary element and u^b, v^b and p^b be as in the former section. We denote the restriction of the approximate solution $\{\boldsymbol{u}_h, p_h\}$ and the right-hand side $\boldsymbol{f}$ of (3) to T as follows:

$$
\boldsymbol{u}_h|_T = \begin{pmatrix} u_1^1 \\ u_2^1 \end{pmatrix} + \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} u^b + \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} v^b \qquad (33)
$$

$$p_h|_T = p^1 + \gamma p^b \tag{34}$$

$$\boldsymbol{f}|_T = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \tag{35}$$

Consider a function b such that $b|_T = u^b$ and $b = 0$ elsewhere. Choosing $\boldsymbol{v}_h = (b, 0)^t \in V_h$ in the first equation of (3) yields

$$\mu \iint_T \nabla(u_1^1 + \alpha_1 u^b + \beta_1 v^b) \cdot \nabla u^b dxdy - \iint_T (p^1 + \gamma p^b) \frac{\partial u^b}{\partial x} dxdy = \iint_T f_1 u^b dxdy \tag{36}$$

Substituting (24), (25)–(26) and (31) into (36) yields

$$\alpha_1 \mu \iint_T \nabla u^b \cdot \nabla u^b \, dxdy = \left(f_1 - \frac{\partial p^1}{\partial x} \right) \iint_T u^b \, dxdy \tag{37}$$

The integrals above are given by (21) and (27). Thus,

$$\alpha_1 = \frac{3}{4\mu} h^2 \left(f_1 - \frac{\partial p^1}{\partial x} \right) \tag{38}$$

Analogously, choosing $\boldsymbol{v}_h = (0, b)^t \in V_h$ yields

$$\alpha_2 = \frac{3}{4\mu} h^2 \left(f_2 - \frac{\partial p^1}{\partial y} \right) \tag{39}$$

Now let $b|_T = v^b$, $b = 0$ elsewhere and choose $\boldsymbol{v}_h = (b, 0)^t \in V_h$. We have

$$\mu \iint_T \nabla(u_1^1 + \alpha_1 u^b + \beta_1 v^b) \cdot \nabla v^b dxdy - \iint_T (p^1 + \gamma p^b) \frac{\partial v^b}{\partial x} dxdy = \iint_T f_1 v^b dxdy \tag{40}$$

We have, as in (36),

$$\beta_1 \mu \iint_T \nabla v^b \cdot \nabla v^b \, dxdy - \gamma \iint_T p^b \frac{\partial v^b}{\partial x} \, dxdy = \left(f_1 - \frac{\partial p^1}{\partial x} \right) \iint_T v^b \, dxdy \tag{41}$$

Substituting (22), (28) and (29) above yields

$$\beta_1 \frac{\mu}{630} + \gamma h \frac{\cos\theta + \sin\theta}{630} = 0 \tag{42}$$

Choosing $\boldsymbol{v}_h = (b, 0)^t$ and proceeding as in (40)–(42), we have

$$\beta_2 \frac{\mu}{630} + \gamma h \frac{\sin\theta - \cos\theta}{630} = 0 \tag{43}$$

Select $q \in Q_h$ such that $q|_T = p^b$ and $q = 0$ in the second equation of (3):

$$\iint_T p^b \frac{\partial}{\partial x}(u_1^1 + \alpha_1 u^b + \beta_1 v^b)\, dxdy + \iint_T p^b \frac{\partial}{\partial y}(u_2^1 + \alpha_2 u^b + \beta_2 v^b)\, dxdy = 0 \tag{44}$$

Substitute (25)–(26) into (44) and rearrange the terms:

$$-\beta_1 \iint_T p^b \frac{\partial v^b}{\partial x}\, dxdy - \beta_2 \iint_T p^b \frac{\partial v^b}{\partial y}\, dxdy = \nabla \cdot \boldsymbol{u^1} \iint_T p^b\, dxdy \tag{45}$$

Substituting (23) and (29)–(30) into (45) yields

$$\beta_1 h \frac{\cos\theta + \sin\theta}{630} + \beta_2 h \frac{\sin\theta - \cos\theta}{630} = \frac{h^2}{12} \nabla \cdot \boldsymbol{u^1} \tag{46}$$

Equations (42), (43) and(46) yield a linear system in the following form:

$$\begin{bmatrix} A & 0 & B \\ 0 & A & C \\ B & C & 0 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \gamma \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ F \end{bmatrix} \tag{47}$$

The solution of (47) is

$$\begin{aligned} \beta_1 &= \frac{BF}{B^2 + C^2} = \frac{105h}{4}(\cos\theta + \sin\theta)\nabla \cdot \boldsymbol{u^1} && (48) \\ \beta_2 &= \frac{CF}{B^2 + C^2} = \frac{105h}{4}(\sin\theta - \cos\theta)\nabla \cdot \boldsymbol{u^1} && (49) \\ \gamma &= \frac{-AF}{B^2 + C^2} = -\frac{105\mu}{4}\nabla \cdot \boldsymbol{u^1} && (50) \end{aligned}$$

Let $\boldsymbol{v}^1 = (v_1^1, v_2^1)$ and q^1 be arbitrary functions in $(P_1)^2$ and P_1, respectively. Substituting $\boldsymbol{v}^1$ into (3) yields

$$\begin{aligned} a(\boldsymbol{u}^1, \boldsymbol{v}^1) &+ \sum_T \iint_T \mu[\nabla(\alpha_1 u^b + \beta_1 v^b) \cdot \nabla v_1^1 + \nabla(\alpha_2 u^b + \beta_2 v^b) \cdot \nabla v_2^1]\, dxdy \\ &+ b(\boldsymbol{v}^1, p^1) - \sum_T \gamma \iint_T p^b \nabla \cdot \boldsymbol{v^1}\, dxdy = F(\boldsymbol{v}^1) \qquad (51) \end{aligned}$$

From (31)–(32), the first sum in (51) vanishes. From (23) and since $\nabla \cdot \boldsymbol{v^1}$ is constant within T, we have

$$-\gamma \iint_T p^b \nabla \cdot \boldsymbol{v^1}\, dxdy = \frac{105\mu}{4} \nabla \cdot \boldsymbol{u^1} \nabla \cdot \boldsymbol{v^1} \iint_T p^b\, dxdy$$

$$
\begin{aligned}
&= \frac{35h^2\mu}{16}\nabla\cdot\boldsymbol{u}^{\mathbf{1}}\nabla\cdot\boldsymbol{v}^{\mathbf{1}} \\
&= \frac{35\mu}{8}\iint_T \nabla\cdot\boldsymbol{u}^{\mathbf{1}}\nabla\cdot\boldsymbol{v}^{\mathbf{1}}\,dxdy
\end{aligned} \tag{52}
$$

Therefore, (51) reduces to

$$
a(\boldsymbol{u}^1,\boldsymbol{v}^1)+b(\boldsymbol{v}^1,p^1)+\sum_T\frac{35\mu}{8}\iint_T \nabla\cdot\boldsymbol{u}^{\mathbf{1}}\nabla\cdot\boldsymbol{v}^{\mathbf{1}}\,dxdy = F(\boldsymbol{v}^1) \tag{53}
$$

Let us substitute q^1 into (3):

$$
-b(\boldsymbol{u}^1,q^1)+\sum_T\iint_T q^1\left[\frac{\partial}{\partial x}(\alpha_1 u^b+\beta_1 v^b)+\frac{\partial}{\partial y}(\alpha_2 u^b+\beta_2 v^b)\right]dxdy=0 \tag{54}
$$

Integrating by parts each term of the sum above and taking into account (21)–(22), we have

$$
\begin{aligned}
\iint_T q^1\frac{\partial}{\partial x}(\alpha_1 u^b+\beta_1 v^b)\,dxdy &= -\frac{\partial q^1}{\partial x}\alpha_1\iint_T u^b\,dxdy \\
&= -\frac{\partial q^1}{\partial x}\frac{3}{4\mu}h^2\left(f_1-\frac{\partial p^1}{\partial x}\right)\frac{h^2}{120} \\
&= -\frac{h^2}{80\mu}\iint_T\left(f_1-\frac{\partial p^1}{\partial x}\right)\frac{\partial q^1}{\partial x}\,dxdy
\end{aligned} \tag{55}
$$

$$
\begin{aligned}
\iint_T q^1\frac{\partial}{\partial y}(\alpha_2 u^b+\beta_2 v^b)\,dxdy &= -\frac{\partial q^1}{\partial y}\alpha_2\iint_T u^b\,dxdy \\
&= -\frac{\partial q^1}{\partial y}\frac{3}{4\mu}h^2\left(f_2-\frac{\partial p^1}{\partial y}\right)\frac{h^2}{120} \\
&= -\frac{h^2}{80\mu}\iint_T\left(f_2-\frac{\partial p^1}{\partial y}\right)\frac{\partial q^1}{\partial y}\,dxdy
\end{aligned} \tag{56}
$$

Note that (54) reduces to

$$
-b(\boldsymbol{u}^1,q^1)-\sum_T\frac{h^2}{80\mu}\iint_T(\boldsymbol{f}-\nabla p^1)\cdot\nabla q^1\,dxdy=0 \tag{57}
$$

Since $-\mu\Delta\boldsymbol{u}^{\mathbf{1}} = -\mu\Delta\boldsymbol{v}^{\mathbf{1}} = \mathbf{0}$, we have that equations (53) and (57) recover (14) with

$$
\delta_1=\frac{1}{80\mu}\text{ and }\delta_2=\frac{35\mu}{8} \tag{58}
$$

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