Areas of Fuzzy Geographical Entities

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Abstract.

The use of geographical entities with fuzzy spatial extent, called here fuzzy geographical entities (FGE's), in Geographical Information Systems requires the existence of operators capable of processing them. This paper focuses on the computation of areas of FGE's. Two methods are considered, one crisp due to Rosenfeld (1984), which will be shown to be limited in its applicability, and the other fuzzy that is a new approach. The new fuzzy area operator gives more information about the possible values of the area and enables the propagation of the fuzziness in the spatial extent of the entity. Crisp and fuzzy areas have different meanings and the use of one or the other depends not only on the purpose of the computation but also on the semantics of the membership functions. Since the fuzzy area operator generates fuzzy numbers when the FGE's are represented by normal fuzzy sets, arithmetic operations can be performed with these areas using fuzzy arithmetic. However, as will be shown, care must be taken when using the arithmetic operators because in some situations the usual fuzzy operators shouldn't be applied. Properties of the fuzzy area operators are analyzed, establishing a parallel with properties of the areas of crisp sets.

1 Introduction

Several authors have used fuzzy sets to represent the uncertainty associated to the attributes or the position of geographical entities, or to represent the transition between entities that don't have precise boundaries (Burrough 1996, Burrough and MacDonnell 1998, Cheng and Molenaar 1999, Edwards and Lowell 1996, Fonte and Lodwick 2001, Zhang and Kirby 1999). These approaches enable the generation of maps representing degrees of belonging of every point in the geographical space to the attributes or the geographical entities. A geographical entity is characterized by an attribute and a location in the geographical space, and its location corresponds to the position of the point, line or region obtained aggregating contiguous points or regions where the attribute characterizing it is considered to exist. Therefore, even when the membership grades represent degrees of membership to the attribute characterizing the geographical entity, they also reflect degrees of belonging to the entity itself and translate a degree of fuzziness in its position.

Keeping in mind that the fuzziness in the position of an entity can have several sources, and that the degrees of membership can have several meanings (Fonte and Lodwick 2001), we define fuzzy geographical entity as follows:

Definition: A *fuzzy geographical entity (FGE) E* is a geographical entity whose position in the geographical space is defined by the fuzzy set $E = \{\text{Points belonging to geographical entity E}\}$, with membership function $\mu_E(x,y) \in [0,1]$ defined for every point (x,y) in the space of interest. The membership value one represents full membership. The membership value zero represents no membership, and the values in between correspond to membership grades to *E*, decreasing from one to zero.

FGE's are an efficient way to represent the positional uncertainty of geographical entities or the gradual variation between entities, but their inclusion in Geographical Information Systems (GIS's) requires not only their construction but also the development of operators capable of processing them. Operators with crisp and fuzzy outputs can be considered. The operators with fuzzy output propagate the fuzziness in the input data to the results of the analyses operations. Some work has already been done in this area. Altman (1994) suggested a way to compute distances and directions between FGE's, Duff and Guesgen (2002) developed an operator to generate buffers of FGE's, and some authors have proposed the execution of, for example, intersection and union with the standard fuzzy operators (McBratney and Odeh 1997, Burrough and MacDonnell 1998, Fonte and Lodwick 2001).

An important attribute in a GIS that is associated with geographical entities is *area*. This paper focuses on the computation of crisp and fuzzy areas of FGE's. Since a FGE is represented by a fuzzy set, the operator proposed by Rosenfeld (1984) to compute the area of a fuzzy set can be applied to FGE's. This operator generates a crisp value for the area. In section 2 some properties of the Rosenfeld area are analyzed and some limitations of this crisp approach are shown. A new area operator that generates fuzzy areas is presented in section 3. Examples are given and the properties of this new fuzzy area operator are analyzed.

Geographical entities in a GIS may be represented considering a continuous or discrete geographical space. Continuous space is usually associated with the vector data structure while discrete space with the raster data structure. For simplicity, we restrict ourselves to the discrete space. However, the concepts presented herein can be easily extended to continuous space. The geographical entities used in this paper are represented using the raster data structure and the pixels are considered to have unitary area. Since this research focuses on the computation of areas, all geographical entities are considered to be of the type "area", and all FGE's considered are represented by normal fuzzy sets and are called *normal FGE's*. Throughout this paper it is assumed that the reader is familiar with the basic ideas of fuzzy set theory (Klir and Yuan, 1995).

2 Crisp area of fuzzy geographical entities

Rosenfeld (1984) introduced the notion of area (AR) of a fuzzy set. For a fuzzy set *E*, in the discrete case,

$$AR(E) = \sum_x \sum_y \mu(x, y). \qquad (2.1)$$

Since the geographical location of a FGE is represented by a fuzzy set, this concept can be applied to FGE's. This operator generates a crisp value for the area.

2.1 Properties of the Rosenfeld area operator

The areas of crisp sets E_c and F_c satisfy the following properties:

$$\forall E_C, F_C : \forall (x, y), \mu_{E_C}(x, y) \in \{0,1\} \wedge \mu_{F_C}(x, y) \in \{0,1\}$$

1. $Area(E_C) \geq 0$
2. $Area(E_C \cup F_C) = Area(E_C) + Area(F_C) - Area(E_C \cap F_C)$
3. If $F_C \subseteq E_C$ then $Area(E_C - F_C) = Area(E_C) - Area(F_C)$, where $E_C - F_C = E_C \cap \overline{F_C}$.

Some properties of the Rosenfeld area are now analysed considering the standard fuzzy intersection, union and complement operators (Klir and Yuan 1995) and establishing a comparison with the above three well known properties of areas of crisp regions. Proofs of all properties can be found in the appendix.

Property 2.1.1: For any FGE E, $AR(E) \geq 0$.

Property 2.1.2: Let E and F be two FGE's, then

$$AR(E \cup F) = AR(E) + AR(F) - AR(E \cap F).$$

Property 2.1.3: If $E \cap F = \varnothing$, that is, $\nexists (x, y) : \mu_E(x, y) > 0 \wedge \mu_F(x, y) > 0$, then

$$AR(E \cup F) = AR(E) + AR(F).$$

Property 2.1.4: Let E and F be two FGE's such that $F \subseteq E$. If $\forall (x, y) \in \text{support}(F)$, $\mu_E(x, y) = 1$ then

$$AR(E - F) = AR(E) - AR(F). \qquad (2.2)$$

Note that since

$$AR(E - F) = \sum_x \sum_y \min\left(\mu_E(x, y), 1 - \mu_F(x, y)\right) \qquad (2.3)$$

and

$$AR(E)-AR(F)=\sum_x\sum_y \mu_E(x,y)-\sum_x\sum_y \mu_F(x,y)=\sum_x\sum_y(\mu_E(x,y)-\mu_F(x,y)) \tag{2.4}$$

in the hypothesis of property 2.1.4, that is, if $support(F)\subseteq core(E)$, all points generate equal parcels in equations (2.3) and (2.4) (see proof of property 2.1.4 in the appendix). But if $\text{support}(F)\not\subset core(E)$, that is, $\exists(x,y)\in \text{support}(F):\mu_E(x,y)\neq 1$, the points in these conditions generate different parcels in equations (2.3) and (2.4). So if *E* and *F* are two FGE's such that $F\subseteq E$ and $\text{support}(F)\not\subset core(E)$, in general,

$$AR(E-F)\neq AR(E)-AR(F).$$

This means that, when $F\subseteq E$ the area of the FGE $E-F$ cannot be computed simply by subtracting the Rosenfeld areas of E and F. For example, if FGE's *F* and *E* are respectively the ones illustrated in Figure 1 and Figure 2 and $E-F$ the FGE represented in Figure 3, we have $AR(E)-AR(F)=15.6-9.5=6.1$ and $AR(E-F)=9$. Moreover, unless the hypotheses of 2.1.4 hold, if *G* is a FGE such that $AR(G)=AR(F)$ and $G\subseteq E$, the Rosenfeld areas of $E-F$ and $E-G$ may be different. For example, given *G* shown in Figure 4 it can be seen that $G\subseteq E$, $AR(G)=AR(F)=9.5$ and $AR(E-G)=9.9$ which is different from $AR(E-F)$. This happens because the fuzzy standard intersection is the minimum operator and its value may change if the membership values change position, even if their sum is the same.

0	0	0	0	0	0	0	0
0	0	0	0.5	0	0	0	0
0	0	0.6	0.9	0.8	0.7	0.3	0
0	0.3	0.8	1	1	0.8	0.4	0
0	0	0.4	1	1	0.9	0.5	0
0	0	0.2	0.6	0.7	0.9	0.4	0
0	0	0.1	0.3	0.2	0.3	0	0
0	0	0	0	0	0	0	0

Figure 1 – Fuzzy geographical entity E.

0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0.1	0.5	0.3	0.5	0	0
0	0	0.5	0.9	1	0.5	0.1	0
0	0	0.4	0.9	1	0.7	0.2	0
0	0	0.1	0.2	0.6	0.7	0.2	0
0	0	0	0.1	0	0	0	0
0	0	0	0	0	0	0	0

Figure 2 - Fuzzy geographical entity F.

0	0	0	0	0	0	0	0
0	0	0	0.5	0	0	0	0
0	0	0.6	0.5	0.7	0.5	0.3	0
0	0.3	0.5	0.1	0	0.5	0.4	0
0	0	0.4	0.1	0	0.3	0.5	0
0	0	0.2	0.6	0.4	0.3	0.4	0
0	0	0.1	0.3	0.2	0.3	0	0
0	0	0	0	0	0	0	0

Figure 3 – Fuzzy geographical entity *E-F*.

0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0.7	0.5	0.2	0
0	0	0	0	1	0.7	0.3	0
0	0	0.1	0.5	1	0.9	0.5	0
0	0	0.1	0.5	0.6	0.9	0.4	0
0	0	0.1	0.2	0.1	0.2	0	0
0	0	0	0	0	0	0	0

Figure 4 - Fuzzy geographical entity G.

Even though for the general case $AR(E-F)\neq AR(E)-AR(F)$, a relation between them exists as stated in the next property.

Property 2.1.5: For any FGE's *E* and *F* such that $F\subseteq E$

$$AR(E-F)>AR(E)-AR(F)$$

There is another interesting characteristic associated with the computation of the difference of two FGE's with the standard fuzzy intersection. If two FGE's *E* and *H* are such that $H=E$, then $E-H$ is not an empty set. For example if both *E* and *H* are the FGE shown in Figure 1, $E-H$ is the FGE shown in Figure 5. Since $E-H$ is not an empty set then

$AR(E-H) \neq 0$. For this particular example $AR(E-H) = 6.2$. Note that in these cases we have $AR(E) - AR(H) = 0$.

0	0	0	0	0	0	0	0
0	0	0	0.5	0	0	0	0
0	0	0.4	0.1	0.2	0.3	0.3	0
0	0.3	0.2	0	0	0.2	0.4	0
0	0	0.4	0	0	0.1	0.5	0
0	0	0.2	0.4	0.3	0.1	0.4	0
0	0	0.1	0.3	0.2	0.3	0	0
0	0	0	0	0	0	0	0

Figure 5 - Fuzzy geographical entity *E-H*, *when H=E*, and *E* is the FGE in Figure 1.

3 Fuzzy area of fuzzy geographical entities

The method proposed by Rosenfeld to compute the area of a FGE considers that the contribution of the area of each pixel to the total area is proportional to the membership function value assigned to it. So, this operator is appropriate when the concept of FGE arises from mixed pixels and the membership function values correspond to the percentage of the pixel area occupied by the attribute characterizing the FGE. For example, if the geographical entity *A* shown in Figure 6 is represented in the raster data structure as a FGE assigning to each pixel the percentage of the pixel area occupied by the entity, the FGE $\tilde{A}$ shown in Figure 7 is obtained. The area of entity *A*, computed with the coordinates of the points defining the border of *A*, is equal to the Rosenfeld area of $\tilde{A}$. Both have the value $138.14\ m^2$.

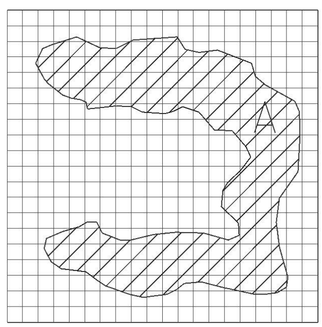

Figure 6 – Geographical entity *A*, represented in the vector data structure, overlaid with a grid of cells.

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0.03	0.19	0	0	0	0.11	0.18	0.25	0.02	0	0	0	0	0	0	0	0
0	0	0.56	0.95	1	0.81	0.57	0.78	1	1	1	0.61	0.33	0.38	0.09	0	0	0	0	0
0	0.05	0.97	1	1	1	1	1	1	1	1	1	1	1	0.98	0.60	0	0	0	0
0	0	0.48	1	1	1	1	1	1	1	1	1	1	1	1	0.99	0.34	0	0	0
0	0	0	0.34	0.71	0.99	1	1	1	1	1	1	1	1	1	1	1	0.73	0.17	0
0	0	0	0	0	0.25	0.20	0.16	0.41	0.59	0.53	0.29	0.72	1	1	1	1	1	0.77	0
0	0	0	0	0	0	0	0	0	0	0	0	0.03	0.62	0.83	1	1	1	0.83	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.04	0.81	1	1	0.83	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.48	1	1	0.78	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.26	0.96	1	1	0.59	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0.06	0.95	1	1	0.87	0.06	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0.13	0.98	1	1	0.48	0	0
0	0	0	0	0.10	0.35	0	0	0	0	0	0	0	0	0.32	1	1	0.37	0	0
0	0	0.22	0.75	0.99	1	0.57	0.27	0.38	0.60	0.72	0.74	0.69	0.47	0.42	1	1	0.45	0	0
0	0	0.47	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0.62	0	0
0	0	0.04	0.52	0.71	0.95	1	1	1	1	1	1	1	1	1	1	1	0.90	0	0
0	0	0	0	0	0.15	0.72	0.99	1	1	0.98	0.58	0.47	0.69	0.92	1	1	0.95	0.03	0
0	0	0	0	0	0	0	0.10	0.36	0.33	0.13	0	0	0	0	0.13	0.21	0.07	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Figure 7 – Fuzzy geographical entity $\tilde{A}$ obtained from entity *A* in Figure 6, considering the membership function value equal to the percentage of the pixel area occupied by the entity.

However, if the membership function values represent the degree of uncertainty of whether the pixels can be classified as belonging to a geographical entity characterized by a certain attribute, the spatial extent of the geographical entity is not known. In this case its corresponding area cannot be known either. For this situation the value of the Rosenfeld area operator is just a crisp approximate value of the entity's area, and no information regarding the other possible values is given. The Rosenfeld area operator is not equipped to give this type of information. For example, consider a FGE that translates the danger of air pollution with respect to a particular polluting agent. The degree of membership indicates the possibility of being affected by the pollutant (Usery (1996) presents a simple example for the construction of the membership function). Since any point with $\mu > 0$ can belong to the affected region, the area of this entity can vary between the area of the region formed by the pixels that have membership value $\mu > 0$ and the area of the region formed by the pixels that have $\mu = 1$. If the area of all regions that can be affected is to be determined, or the area of the regions that have a possibility of being affected higher than 0.5, the Rosenfeld area operator is of no use. Moreover, the Rosenfeld area operator gives no information about the variation of the area when different membership function values are considered.

If the membership function values translate a degree of belonging based on similarity to the attribute, the area of the geographical entity mainly depends on the degree of acceptable similarity. For example, if the membership function represents degrees of membership to a certain soil type, based on the similarity of the characteristics observed on the ground and what is considered to be the typical characteristics of that soils type, as in the previous case, the entity area can take values between the area of the region corresponding to $\mu > 0$ and the one corresponding to $\mu = 1$. If for a certain application just the degrees of similarity corresponding to the typical characteristics are considered acceptable, the area of the entity is the area corresponding to $\mu = 1$, and for this application the Rosenfeld area operator is not useful.

The previous examples show that the information provided by the Rosenfeld area operator is limited and will turn out to be insufficient for many applications. To overcome the limitations of the Rosenfeld area operator a new area operator with a fuzzy output, called the *fuzzy area operator* (*AF*), is developed.

A FGE is characterized by a fuzzy set. A normal fuzzy set can be represented in a unique way by a family of alpha levels, for $\alpha \in [0,1]$. Since an alpha level is a crisp set $E_\alpha = \{(x,y) : \mu_E(x,y) \ge \alpha\}$, its area, denoted here by $Area(E_\alpha)$, is the sum of the areas of the pixels belonging to the alpha level. That is, for a FGE *E*:

$$Area_E : [0,1] \to \mathbb{R}_0^+$$
$$Area_E(\alpha) = Area(E_\alpha) = z$$

Since the alpha levels of a fuzzy set are nested, that is $\beta < \gamma \Rightarrow E_\beta \supseteq E_\gamma$, the function *Area* is decreasing because $E_\beta \supseteq E_\gamma \Rightarrow Area(E_\beta) \ge Area(E_\gamma)$.

Let's now denote by z_i, $i = 1,...,n$, a set of values of $\mathbb{R}_0^+$ such that $\exists \alpha_i : z_i = Area_E(\alpha_i)$, where $0 < \alpha_i < \alpha_{i+1} \le 1$.

Definition: The *fuzzy area* of a FGE *E*, is the fuzzy set $AF(E)=\{(z,\mu_{AF(E)}(z))\}$

where

$$\mu_{AF(E)}: \mathbb{R}_0^+ \to [0,1]$$

$$\mu_{AF(E)}(z)=\begin{cases} \max\limits_{z_i=Area_E(\alpha_i)} \alpha_i & when \quad \exists i: z=z_i \\ \dfrac{z-z_k}{z_{k+1}-z_k}(\alpha_{k+1}-\alpha_k)+\alpha_k, \text{ with } z_k=\max\limits_{z_i\le z}(z_i) & when \quad \nexists i: z=z_i \\ 0 & when \quad z\notin [Area_E(1), Area_E(0)) \end{cases}$$

(3.1)

To compute the fuzzy area of the FGE E represented in Figure 1 some alpha levels α_i have to be considered and their areas computed. Table 1 shows the areas z_i obtained for the alpha levels α_i. Figure 8 shows a plot of the fuzzy area of E obtained using the values of Table 1. The same procedure was used to compute the fuzzy area of the FGE *F* represented in Figure 2, and the obtained fuzzy area is plotted in Figure 9.

i	α_i	$z_i = Area(E_{\alpha_i})$
1	0.001	26
2	0.1	26
3	0.2	25
4	0.3	23
5	0.4	19
6	0.5	16
7	0.6	14
8	0.7	12
9	0.8	10
10	0.9	7
11	1	4

Table 1 - Area z_i of the alpha levels of *E*.

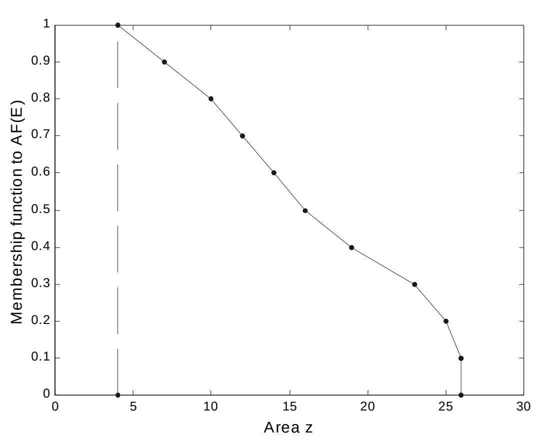

Figure 8 - Fuzzy area of the geographical entity *E*.

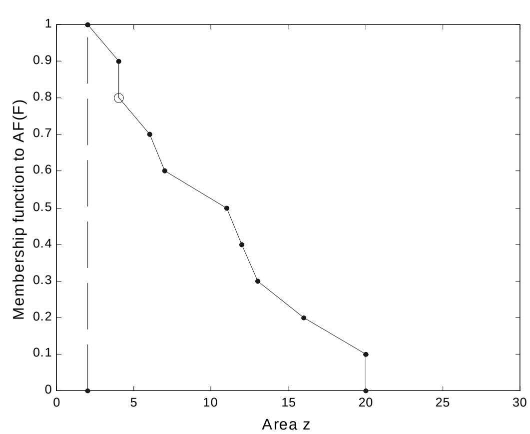

Figure 9 - Fuzzy area of the geographical entity *F*.

3.1 Properties of the fuzzy area

We next verify properties of fuzzy area that are similar to those presented in section 2.1. The proofs are found in the appendix.

Property 3.1.1: $\forall E$, the support of $AF(E)$ is a subset of $\mathbb{R}_0^+$.

Property 3.1.2: $\forall E, \mu_{AF(E)}(z)$ is a decreasing left continuous function over $z \in [Area(1), Area(0)]$.

Property 3.1.3: If E is a normal FGE, then $AF(E)$ is a fuzzy number.

Fuzzy areas of normal FGE's are fuzzy numbers so fuzzy arithmetic can be applied. It is then necessary to analyse which and when these operations are meaningful, and identify useful properties.

Consider the FGE's *A* and *B* shown in Figure 10 and Figure 11. The union of these two FGE's is the entity shown in Figure 12. The fuzzy areas of *A* and *B* are equal to the fuzzy areas of *E* and *F* and can be seen in Figure 8 and Figure 9. The fuzzy area of $A \cup B$ as well as $AF(A) + AF(B)$ can be seen in Figure 13. Note that the fuzzy numbers obtained for $AF(A \cup B)$ and $AF(A) + AF(B)$ are equal.

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0.5	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0.6	0.9	0.8	0.7	0.3	0	0	0	0	0	0	0	0	0
0	0.3	0.8	1	1	0.8	0.4	0	0	0	0	0	0	0	0	0
0	0	0.4	1	1	0.9	0.5	0	0	0	0	0	0	0	0	0
0	0	0.2	0.6	0.7	0.9	0.4	0	0	0	0	0	0	0	0	0
0	0	0.1	0.3	0.2	0.3	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Figure 10 – Fuzzy geographical entity *A*.

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0.1	0.5	0.3	0.5	0	0
0	0	0	0	0	0	0	0	0	0	0.5	0.9	1	0.5	0.1	0
0	0	0	0	0	0	0	0	0	0	0.4	0.9	1	0.7	0.2	0
0	0	0	0	0	0	0	0	0	0	0.1	0.2	0.6	0.7	0.2	0
0	0	0	0	0	0	0	0	0	0	0	0.1	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Figure 11 - Fuzzy geographical entity *B*.

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0.5	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0.6	0.9	0.8	0.7	0.3	0	0	0	0.1	0.5	0.3	0.5	0	0
0	0.3	0.8	1	1	0.8	0.4	0	0	0	0.5	0.9	1	0.5	0.1	0
0	0	0.4	1	1	0.9	0.5	0	0	0	0.4	0.9	1	0.7	0.2	0
0	0	0.2	0.6	0.7	0.9	0.4	0	0	0	0.1	0.2	0.6	0.7	0.2	0
0	0	0.1	0.3	0.2	0.3	0	0	0	0	0	0.1	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Figure 12 - Fuzzy geographical entity $A \cup B$

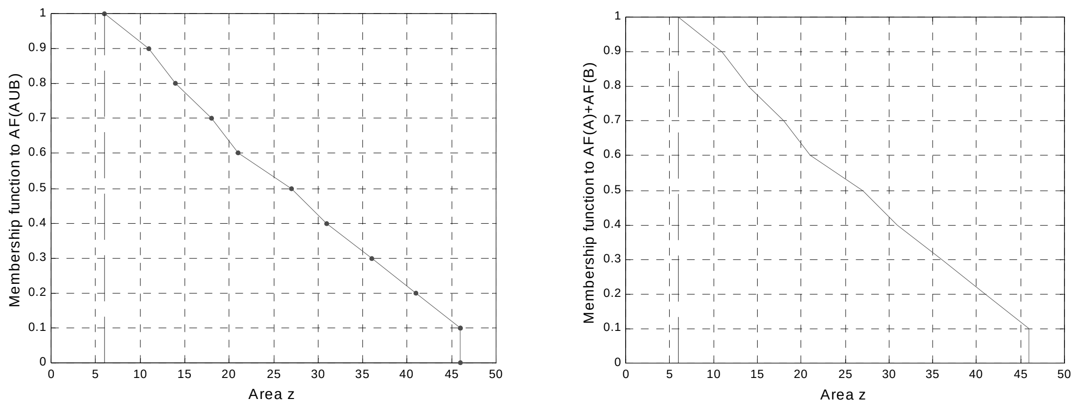

Figure 13 - $AF\left(A\cup B\right)$ **and** $AF\left(A\right)+AF\left(B\right)$**.**

This example illustrates the following property:

Property 3.1.4 - If $\nexists(x,y):\mu_E(x,y)>0\wedge\mu_F(x,y)>0$ (that is, $E\cap F=\varnothing$), then

$$AF\left(E\cup F\right)=AF\left(E\right)+AF\left(F\right).$$

When there are points that belong simultaneously to geographical entities *E* and *F* the equivalent to property 2.1.2 for the Rosenfeld area is not satisfied. For example, consider geographical entities *C* and *D* shown respectively in Figure 14 and Figure 15. The fuzzy area of fuzzy entities *C* and *D* are such that $AF(C)=AF(E)$ and $AF(D)=AF(F)$, $C\cup D$ is shown in Figure 16 and $C\cap D$ in Figure 17. Note that, as there are no pixels having simultaneously full membership to *C* and *D*, $C\cap D$ is not a normal fuzzy set. Therefore $AF\left(C\cap D\right)$ is not a fuzzy number. Since fuzzy arithmetic operators are only applied to fuzzy numbers they cannot be applied in this case, and $AF\left(C\right)+AF\left(D\right)-AF\left(C\cap D\right)$ cannot be computed. This case requires the use of arithmetic operations performed with subnormal fuzzy sets, and won't be addressed in this paper. Consider now FGE's C_1 and D_1 shown respectively in Figure 14 and Figure 15. These entities correspond to *C* and *D* moved in such a way that their intersection is a normal fuzzy set. The fuzzy entities corresponding to their union and intersection are shown respectively in Figure 16 and Figure 17. In Figure 19 are plotted $AF\left(C_1\right)+AF\left(D_1\right)-AF\left(C_1\cap D_1\right)$ and $AF\left(C_1\cup D_1\right)$, and it can be easily seen that they are not equal. The main difference is that the left part of the fuzzy number obtained with fuzzy arithmetic is not vertical and doesn't have the characteristics of a fuzzy area. Note also that the computation of $AF\left(C_1\right)+AF\left(D_1\right)-AF\left(C_1\cap D_1\right)$ using the usual fuzzy difference operator generates negative values for the area. This happens because the difference of the two fuzzy numbers is computed using the extension principle, and therefore all combinations of values of the alpha cuts of both fuzzy numbers are considered possible. So it is considered possible that $AF\left(C_1\cap D_1\right)$ has values larger than $AF\left(C_1\right)+AF\left(D_1\right)$. That is, the area of the intersection of the two sets is larger than the sum of their areas, which of course does not make physical sense. The problem with this computation is that when the variables are interactive (the equivalent to dependency in probability and statistics) fuzzy arithmetic overestimates the result. In this case $C_1\cap D_1$ is linked with C_1 and D_1, and therefore $AF\left(C_1\right)+AF\left(D_1\right)-AF\left(C_1\cap D_1\right)$ will be overestimated. So, the use of the usual fuzzy arithmetic may raise problems in the computation of the fuzzy area of the union of two FGE's. To overcome these problems a relation between the areas of C_1, D_1 and $C_1\cap D_1$ can be considered, allowing only the combination of values corresponding to the same alpha-cuts. That is, the computation is done alpha cut by alpha cut. Considering this relation $AF\left(C_1 \cup D_1\right)$ can be computed using $AF\left(C_1\right)$, $AF\left(D_1\right)$ and $AF\left(C_1 \cap D_1\right)$, and properties similar to property 2 of the areas of crisp sets and property 2.1.2 of the Rosenfeld areas can be identified.

0	0	0	0	0	0	0	0	0	0	0
0	0	0	0.5	0	0	0	0	0	0	0
0	0	0.6	0.9	0.8	0.7	0.3	0	0	0	0
0	0.3	0.8	1	1	0.8	0.4	0	0	0	0
0	0	0.4	1	1	0.9	0.5	0	0	0	0
0	0	0.2	0.6	0.7	0.9	0.4	0	0	0	0
0	0	0.1	0.3	0.2	0.3	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0

C

0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0.5	0	0	0	0
0	0	0	0	0	0.6	0.9	0.8	0.7	0.3	0
0	0	0	0	0.3	0.8	1	1	0.8	0.4	0
0	0	0	0	0	0.4	1	1	0.9	0.5	0
0	0	0	0	0	0.2	0.6	0.7	0.9	0.4	0
0	0	0	0	0	0.1	0.3	0.2	0.3	0	0
0	0	0	0	0	0	0	0	0	0	0

C_1

Figure 14 – Fuzzy geographical entities *C* and C_1.

0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0.1	0.5	0.3	0.5	0	0
0	0	0	0	0	0.5	0.9	1	0.5	0.1	0
0	0	0	0	0	0.4	0.9	1	0.7	0.2	0
0	0	0	0	0	0.1	0.2	0.6	0.7	0.2	0
0	0	0	0	0	0	0.1	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0

D

0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0.1	0.5	0.3	0.5	0	0	0
0	0	0	0	0.5	0.9	1	0.5	0.1	0	0
0	0	0	0	0.4	0.9	1	0.7	0.2	0	0
0	0	0	0	0.1	0.2	0.6	0.7	0.2	0	0
0	0	0	0	0	0.1	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0

D_1

Figure 15 - Fuzzy geographical entity *D* and D_1.

0	0	0	0	0	0	0	0	0	0	0
0	0	0	0.5	0	0	0	0	0	0	0
0	0	0.6	0.9	0.8	0.7	0.5	0.3	0.5	0	0
0	0.3	0.8	1	1	0.8	0.9	1	0.5	0.1	0
0	0	0.4	1	1	0.9	0.9	1	0.7	0.2	0
0	0	0.2	0.6	0.7	0.9	0.4	0.6	0.7	0.2	0
0	0	0.1	0.3	0.2	0.3	0.1	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0

$C \cup D$

0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0.5	0	0	0	0
0	0	0	0	0.1	0.6	0.9	0.8	0.7	0.3	0
0	0	0	0	0.5	0.9	1	1	0.8	0.4	0
0	0	0	0	0.4	0.9	1	1	0.9	0.5	0
0	0	0	0	0.1	0.4	0.6	0.7	0.9	0.4	0
0	0	0	0	0	0.1	0.3	0.2	0.3	0	0
0	0	0	0	0	0	0	0	0	0	0

$C_1 \cup D_1$

Figure 16 - Fuzzy geographical entity $C \cup D$ and $C_1 \cup D_1$

0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0.1	0.3	0	0	0	0
0	0	0	0	0	0.5	0.4	0	0	0	0
0	0	0	0	0	0.4	0.5	0	0	0	0
0	0	0	0	0	0.1	0.2	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0

$C \cap D$

0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0.5	0.3	0.5	0	0	0
0	0	0	0	0.3	0.8	1	0.5	0.1	0	0
0	0	0	0	0	0.4	1	0.7	0.2	0	0
0	0	0	0	0	0.2	0.6	0.7	0.2	0	0
0	0	0	0	0	0.1	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0

$C_1 \cap D_1$

Figure 17 - Fuzzy geographical entity $C \cap D$ and $C_1 \cap D_1$

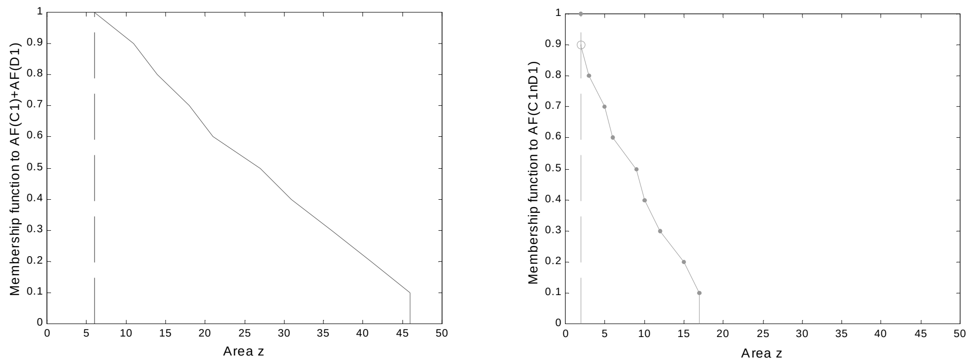

Figure 18 – Plot of $AF\left(C_1\right)+AF\left(D_1\right)$ and of $AF\left(C_1 \cap D_1\right)$.

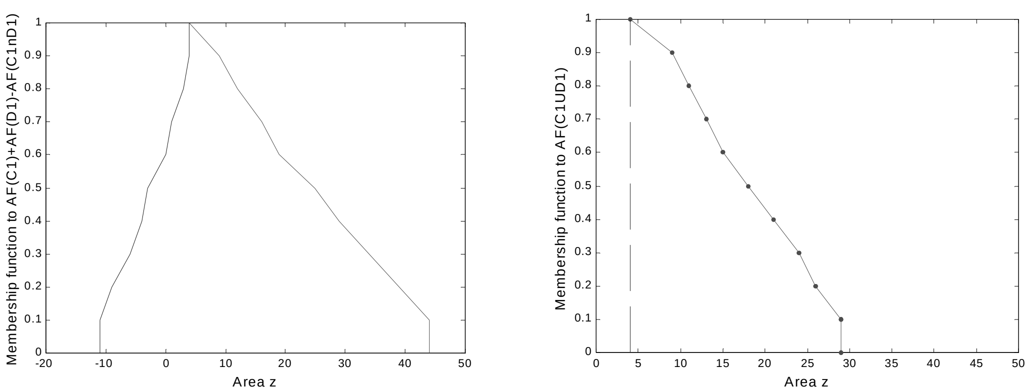

Figure 19 - Plot of $AF\left(C_1\right)+AF\left(D_1\right)-AF\left(C_1 \cap D_1\right)$ and of $AF\left(C_1 \cup D_1\right)$.

For a fuzzy set S with alpha cuts $S_\alpha = \left[s_\alpha^-, s_\alpha^+\right]$, the notation $S_\alpha^+ = s_\alpha^+$ and $S_\alpha^- = s_\alpha^-$ will be used.

Property 3.1.5 - Let E and F be two FGE's, such that $\exists(x,y): \mu_E(x,y)=1 \wedge \mu_F(x,y)=1$, then

$$AF(E \cup F)_\alpha^+ = AF(E)_\alpha^+ + AF(F)_\alpha^+ - AF(E \cap F)_\alpha^+ \leq \left[AF(E)+AF(F)-AF(E \cap F)\right]_\alpha^+$$

Property 3.1.6 - Let E and F be two FGE's, such that $\exists(x,y): \mu_E(x,y)=1 \wedge \mu_F(x,y)=1$, then

$$AF(E \cup F)_\alpha^- = AF(E)_\alpha^- + AF(F)_\alpha^- - AF(E \cap F)_\alpha^- \geq \left[AF(E)+AF(F)-AF(E \cap F)\right]_\alpha^-$$

Note that properties 3.1.5 and 3.1.6 show that

$$AF(E \cup F) \subseteq AF(E)+AF(F)-AF(E \cap F)$$

confirming that $AF(E)+AF(F)-AF(E \cap F)$ overestimates $AF(E \cup F)$.

It was already stated in section 2 that the difference of two fuzzy sets is sensitive to the position of the smaller set and consequently, in general, it is not possible to compute the Rosenfeld area of the difference without computing the set $E-F$. The same happens with the fuzzy area, but in some circumstances the difference of the fuzzy areas of *E* and *F* can still give some useful information, as stated in the following properties.

Property 3.1.7 – If *E* and *F* are FGE's such that $F \subseteq E$, then $AF(E-F)_\alpha^+ \leq \left[AF(E)-AF(F)\right]_\alpha^+$.

For example, for the FGE's *E* and *G* of respectively Figure 1 and Figure 4, $AF(E-G)$ is plotted in Figure 20 with the continuous thick line, $\left[AF(E)-AF(G)\right]_\alpha^+$ with the continuous crossed line, and $\left[AF(E)-AF(G)\right]_\alpha^-$ with the dashed line. Note that property 3.1.7 is satisfied. Note also that in this example no similar property is satisfied for $\left[AF(E)-AF(G)\right]_\alpha^-$. The plots of $\left[AF(E)-AF(G)\right]_\alpha^-$ and $AF(E-G)$ even crosses each other. Only when $F \subseteq core(E)$ it is possible to establish a relation between $AF(E-G)$ and $\left[AF(E)-AF(G)\right]_\alpha^-$, as stated in the property 3.1.8.

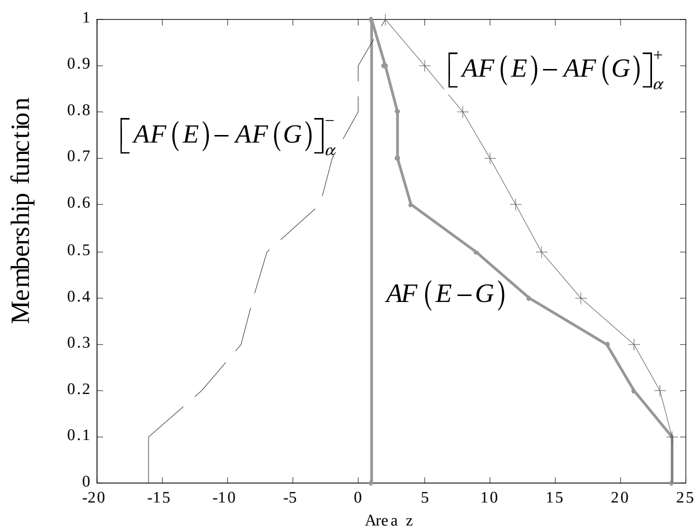

Figure 20 – Plot of $AF(E-G)$, $\left[AF(E)-AF(G)\right]_\alpha^+$ and $\left[AF(E)-AF(G)\right]_\alpha^-$, when E and G are respectively the FGE's represented in Figure 1 and Figure 4.

Property 3.1.8 - If *E* and *F* are FGE's such that $F \subseteq core(E)$ and *E-F* is a normal FGE, then

$$AF(E-F)_\alpha^- = \min\left(AF(E)-AF(F)\right)_\alpha^- = AF(E)_1^- - AF(F)_{0^+}^+$$

where $AF(F)_{0^+}$ represents the strong alpha cut of $AF(F)$ for $\alpha = 0$.

4 Conclusions

The inclusion of FGE's in a GIS requires operators capable of computing their areas. The computation of areas of fuzzy sets, and therefore of FGE's, can be done with the area operator developed by Rosenfeld (1984). This operator generates a crisp value for the area. In some situations this is appropriate, such as when the degrees of belonging to the geographical entity represent the percentage of the pixel area where the attribute characterizing the geographical entity exists. But when the degrees of belonging to the geographical entity represent the uncertainty of whether the pixels belong or not to the entity, or a degree of similarity between what exists in the pixels and the attribute characterizing the geographical entity, Rosenfeld's approach to compute the area of the geographical entity is not appropriate. The Rosenfeld area does not capture the fuzziness in the location or the spatial extent of the geographical entity. In these cases the fuzzy area operator presented herein has more convenient characteristics. This operator, when applied to normal FGE's, generates not only fuzzy sets but fuzzy numbers. Their support is the set of all values the area can take, and the degrees of belonging to the fuzzy area represent, in the cases of the two semantics considered above, the degree of certainty that the area takes at least that value, or the accepted degree of similarity between the attribute characterizing the geographical entity and what exists in the pixels. Therefore, fuzzy areas incorporate much more information about the area of the FGE than Rosenfeld areas.

It is possible to perform arithmetic operations with the new fuzzy area operator using fuzzy arithmetic since fuzzy areas are fuzzy numbers. However, care must be taken because in some situations arithmetic operations shouldn't be done in the usual way.

Some properties of both types of areas are analyzed to evaluate if the area of the union and difference of two FGE's can be computed using the areas of the entities and the area of their intersection. It can be concluded that the computation of the Rosenfeld area of the union of two fuzzy entities can be done adding the Rosenfeld areas of both entities and subtracting the Rosenfeld area of their intersection, as stated in property 2.1.2. This property is equivalent to the property satisfied by the area operator of crisp sets. On the other hand, the computation of the fuzzy area of the union of two fuzzy entities shouldn't be done by adding the two fuzzy numbers obtained for the area of each fuzzy entity and subtracting the fuzzy area of their intersection using the arithmetic operators based on the extension principle. But, as stated in properties 3.1.5 and 3.1.6, if a relation between the alpha cuts of the fuzzy sets is considered, that is, the computation is done alpha cut by alpha cut, the fuzzy area of the union of the two fuzzy sets is obtained as a function of the fuzzy areas of both sets and of their intersection.

Regarding the computation of the area of the difference of two FGE's (when one set is included in the other) using the areas of the fuzzy entities, both the Rosenfeld area operator and the fuzzy area operator show more differences in relation to the property satisfied by the areas of crisp sets. However, these differences are caused by the fact that the difference of the two FGE's is sensitive to their relative position. That is, if FGE E contains FGE F, then, in general, E-F depends on the position of F in relation to E, and therefore both the Rosenfeld and the fuzzy area of E-F also depend on the relative position of the entities. Then, in general, it is not possible to compute the area of $E-F$ knowing just the Rosenfeld or fuzzy areas of E and F. Even though, for the Rosenfeld area this is possible when F is inside the core of E, as expressed in property 2.1.4. For the fuzzy area it is shown in property 3.1.7 that, using the fuzzy areas of E and F, values always larger than each alpha cut of the fuzzy area of $E-F$ can be computed, and when $E-F$ is a normal set, also the smaller value of each alpha level can be computed (property 3.1.8).

The union of normal fuzzy entities generates normal fuzzy entities. However, the intersection of normal fuzzy entities can generate subnormal fuzzy entities, which have subnormal fuzzy area. In this case, there are some restrictions to the use of fuzzy arithmetic since fuzzy arithmetic was developed for fuzzy numbers and fuzzy numbers are by definition normal. This situation will be analyzed in future work.

This paper focused only on the computation of areas of FGE's, but in future work other operators will be developed and analyzed. The evaluation of the real potential of FGE's in GIS will only be possible when operators capable of processing these entities become available.

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Appendix

Proof of property 2.1.1: $AR(E)$ is a sum of parcels having values between zero and one, therefore it's minimum value is zero.

Proof of property 2.1.2:

$$AR(E \cup F) = \sum_x \sum_y \max(\mu_E(x,y), \mu_F(x,y))$$

$$AR(E) = \sum_x \sum_y \mu_E(x,y)$$

$$AR(F) = \sum_x \sum_y \mu_F(x,y)$$

$$AR(E \cap F) = \sum_x \sum_y \min(\mu_E(x,y), \mu_F(x,y))$$

We want to prove that

$$\sum_x \sum_y \max(\mu_E(x,y), \mu_F(x,y)) = \sum_x \sum_y \left(\mu_E(x,y) + \mu_F(x,y) - \min(\mu_E(x,y), \mu_F(x,y))\right) \quad \text{(A.1)}$$

For pixels (x,y) such that $\mu_E(x,y) \le \mu_F(x,y)$, we have that $\max(\mu_E(x,y), \mu_F(x,y)) = \mu_F(x,y)$, and $\min(\mu_E(x,y), \mu_F(x,y)) = \mu_E(x,y)$. Therefore, the pixels (x,y) satisfying $\mu_E(x,y) \le \mu_F(x,y)$ will generate equal quantities in both sides of equation (A.1). For pixels such that $\mu_E(x,y) > \mu_F(x,y)$, we have $\max(\mu_E(x,y), \mu_F(x,y)) = \mu_E(x,y)$ and $\min(\mu_E(x,y), \mu_F(x,y)) = \mu_F(x,y)$, and, as in the first situation, equal quantities will be generated in both sides of equation (A.1). Then, if for every pixel (x,y) equal parcels are generated in both sides of the equation, the sum of these parcels will also be equal, and therefore equality (A.1) holds. ■

Proof of property 2.1.3:

When $\nexists (x,y) : \mu_E(x,y) > 0 \wedge \mu_F(x,y) > 0$, $\min(\mu_E(x,y), \mu_F(x,y)) = 0$ for all (x,y). Therefore $AR(E \cap F) = 0$ for all (x,y) and property 3.1.2 will be simplified to $AR(E \cup F) = AR(E) + AR(F)$.■

Proof of property 2.1.4:

Since $E - F = E \cap \overline{F}$ and the standard fuzzy intersection and complement are being considered,

$$AR(E - F) = \sum_x \sum_y \min(\mu_E(x,y), 1 - \mu_F(x,y)) \quad \text{(A.2)}$$

$$AR(E) - AR(F) = \sum_x \sum_y \mu_E(x,y) - \sum_x \sum_y \mu_F(x,y) = \sum_x \sum_y (\mu_E(x,y) - \mu_F(x,y)) \quad \text{(A.3)}$$

<u>Case 1</u>: $(x,y) \in \text{support}(F)$

As $\forall (x,y) \in \text{support}(F)$, $\mu_E(x,y) = 1$,

$$\min(\mu_E(x,y), 1 - \mu_F(x,y)) = 1 - \mu_F(x,y)$$

$$\mu_E(x,y) - \mu_F(x,y) = 1 - \mu_F(x,y).$$

That is, points $(x,y) \in \text{support}(F)$ generate equal parcels in (A.2) and (A.3).

Case 2: $(x,y) \not\subset \text{support}(F)$

If $(x,y) \not\subset \text{support}(F)$ then $\mu_F(x,y)=0$, and

$$\min(\mu_E(x,y),1-\mu_F(x,y)) = \mu_E(x,y)$$

$$\mu_E(x,y) - \mu_F(x,y) = \mu_E(x,y)$$

So, (A.2) and (A.3) are sums of equal parcels and are therefore equal.■

Proof of property 2.1.5:

Since $E-F = E\cap\overline{F}$ and the standard fuzzy intersection and complement are being considered,

$$AR(E-F) = \sum_x\sum_y \min(\mu_E(x,y),1-\mu_F(x,y)) \quad \text{(A.4)}$$

$$AR(E)-AR(F) = \sum_x\sum_y \mu_E(x,y) - \sum_x\sum_y \mu_F(x,y) = \sum_x\sum_y(\mu_E(x,y)-\mu_F(x,y)) \quad \text{(A.5)}$$

Case 1: $(x,y)\in \text{support}(F)$ and $\mu_E(x,y)=1$

It was shown in the proof of Proposition 2.1.4 that in this case each point (x,y) generates equal parcels in equations (A.4) and (A.5).

Case 2: $(x,y)\in \text{support}(F)$ and $\mu_E(x,y)<1$

If $\min(\mu_E(x,y),1-\mu_F(x,y)) = 1-\mu_F(x,y)$, then, since $\mu_E(x,y)<1$, $\mu_E(x,y)-\mu_F(x,y) < 1-\mu_F(x,y)$ and the parcel obtained in (A.4) is always larger than the parcel obtained in (A.5).

If $\min(\mu_E(x,y),1-\mu_F(x,y)) = \mu_E(x,y)$, then, since $0<\mu_F(x,y)<1$

$$\mu_E(x,y) - \mu_F(x,y) < \mu_E(x,y)$$

and therefore the parcel obtained in (A.4) is always larger than the parcel obtained in (A.5).

Case 3: $(x,y) \not\subset \text{support}(F)$

If $(x,y) \not\subset \text{support}(F)$ then $\mu_F(x,y)=0$, and

$$\min(\mu_E(x,y),1-\mu_F(x,y)) = \mu_E(x,y)$$

$$\mu_E(x,y) - \mu_F(x,y) = \mu_E(x,y)$$

So it can be stated that all parcels obtained in (A.4) are larger than the parcels obtained in (A.5) and therefore $AR(E-F) > AR(E)-AR(F)$.■

Proof of property 3.1.1: The definition guarantees this property.■

Proof of property 3.1.2: This is true by the definition of $\mu_{AF(E)}(z)$.■

Proof of property 3.1.3:

A fuzzy set is a fuzzy number if 1) it is normal; 2) every alpha cut is a closed interval for $\alpha \in (0,1]$; 3) its support is bounded (Klir and Yuan, 1995).

1) If the fuzzy entity *E* is normal there is a region *C*, corresponding to the core of *E*, such that $\forall (x,y) \in C$, $\mu_E(x,y)=1$. The z value corresponding to the area of that region will have membership function to the fuzzy area equal to 1, and therefore the fuzzy area is a normal fuzzy set.

2) As $\mu_{AF(E)}(z)$ is a decreasing function continuous to the left, the alpha cuts $AF(E)_\alpha$ for $\alpha \in (0,1]$ are the closed intervals:

$$\begin{cases} \left[Area(1), z\right] & \Leftarrow \quad \exists z : \mu_{AF(E)}(z) = \alpha \\ \left[Area(1), z_s\right] & \Leftarrow \quad \nexists z : \mu_{AF(E)}(z) = \alpha \end{cases}$$

where z_s is such that $\mu_{AF(E)}(z_s) = \min\left\{\beta : \left(\beta > \alpha \wedge \exists z : \mu_{AF(E)}(z) = \beta\right)\right\}$.

3) The support of $AF(E)$ is the interval $\left[Area(1), Area(0)\right)$, which is bounded because we are only considering geographical entities with finite areas.■

Proof of property 3.1.4:

Since a fuzzy set can be uniquely represented by its alpha cuts, to prove that $AF(E \cup F) = AF(E) + AF(F)$ we must prove that $\forall \alpha \in [0,1]$, $\left[AF(E \cup F)\right]_\alpha = \left[AF(E) + AF(F)\right]_\alpha$.

Let

$AF(E)_\alpha = \left[e_\alpha^-, e_\alpha^+\right]$, $AF(F)_\alpha = \left[f_\alpha^-, f_\alpha^+\right]$ and $\left[AF(E \cup F)\right]_\alpha = \left[u_\alpha^-, u_\alpha^+\right]_\alpha$. Since for any fuzzy numbers N and M, $(N+M)_\alpha = N_\alpha + M_\alpha$ (Klir and Yuan, 1995),

$$\left[AF(E) + AF(F)\right]_\alpha = \left[AF(E)\right]_\alpha + \left[AF(F)\right]_\alpha = \left[e_\alpha^-, e_\alpha^+\right] + \left[f_\alpha^-, f_\alpha^+\right] = \left[e_\alpha^- + f_\alpha^-, e_\alpha^+ + f_\alpha^+\right].$$

So we need to prove that $u_\alpha^- = e_\alpha^- + f_\alpha^-$ and $u_\alpha^+ = e_\alpha^+ + f_\alpha^+$.

Since *E* and *F* are normal FGE's, by equation (3.1)

$$\forall \alpha \in (0,1], \begin{cases} e_\alpha^- = e_1^- = Area(E_1) \\ f_\alpha^- = f_1^- = Area(F_1) \\ u_\alpha^- = u_1^- = Area\left((E \cup F)_1\right) \end{cases}$$

Then, $e_\alpha^- + f_\alpha^- = Area(E_1) + Area(F_1)$.

According to theorem 2.1 in Klir and Yuan (1995) $(E \cup F)_\alpha = E_\alpha \cup F_\alpha$, so $u_\alpha^- = Area\left((E \cup F)_1\right) = Area(E_1 \cup F_1)$. Because $\nexists (x,y) : \mu_E(x,y) > 0 \wedge \mu_F(x,y) > 0$, E_1 and F_1 are disjoint sets and $Area(E_1 \cup F_1) = Area(E_1) + Area(F_1) = e_1^- + f_1^-$. That is, $u_\alpha^- = e_\alpha^- + f_\alpha^-$.

In a similar way, by equation (3.1) we have

$$\forall\alpha\in(0,1],\ \begin{cases} e_\alpha^+ = Area(E_\alpha) \\ f_\alpha^+ = Area(F_\alpha) \\ u_\alpha^+ = Area((E\cup F)_\alpha) \end{cases}$$

Then

$$u_\alpha^+ = Area((E\cup F)_\alpha) = Area((E)_\alpha \cup (F)_\alpha) = Area(E)_\alpha + Area(F)_\alpha = e_\alpha^+ + f_\alpha^+ .\blacksquare$$

Proof of property 3.1.5:

Let's consider $AF(E)_\alpha = [e_\alpha^-, e_\alpha^+]$, $AF(F)_\alpha = [f_\alpha^-, f_\alpha^+]$, $AF(E\cup F)_\alpha = [u_\alpha^-, u_\alpha^+]$ and $AF(E\cap F)_\alpha = [i_\alpha^-, i_\alpha^+]$. So, $AF(E)_\alpha^+ = e_\alpha^+$, $AF(F)_\alpha^+ = f_\alpha^+$, $AF(E\cup F)_\alpha^+ = u_\alpha^+$ and $AF(E\cap F)_\alpha^+ = i_\alpha^+$.

As $\forall\alpha\in(0,1],\ \begin{cases} e_\alpha^+ = Area(E_\alpha) \\ f_\alpha^+ = Area(F_\alpha) \\ u_\alpha^+ = Area((E\cup F)_\alpha) \\ i_\alpha^+ = Area((E\cap F)_\alpha) \end{cases}$

then

$$\begin{aligned} u_\alpha^+ &= Area((E\cup F)_\alpha) = Area((E)_\alpha \cup (F)_\alpha) = \\ &= Area(E)_\alpha + Area(F)_\alpha - Area((E)_\alpha \cap (F)_\alpha) = \\ &= Area(E)_\alpha + Area(F)_\alpha - Area(E\cap F)_\alpha = \\ &= e_\alpha^+ + f_\alpha^+ - i_\alpha^+ \end{aligned}$$

and the equality in Property 3.1.5 is proven. To prove the inequality note that:

$$\begin{aligned} &[AF(E) + AF(F) - AF(E\cap F)]_\alpha = \\ &[AF(E)]_\alpha + [AF(F)]_\alpha - [AF(E\cap F)]_\alpha = \\ &[e_\alpha^-, e_\alpha^+] + [f_\alpha^-, f_\alpha^+] - [i_\alpha^-, i_\alpha^+] = \\ &[e_\alpha^- + f_\alpha^-, e_\alpha^+ + f_\alpha^+] - [i_\alpha^-, i_\alpha^+] = \\ &[e_\alpha^- + f_\alpha^- - i_\alpha^+, e_\alpha^+ + f_\alpha^+ - i_\alpha^-] \end{aligned}$$

and therefore $[AF(E) + AF(F) - AF(E\cap F)]_\alpha^+ = e_\alpha^+ + f_\alpha^+ - i_\alpha^-$. Then, as $\forall\alpha,\ \ i_\alpha^- \le i_\alpha^+$, it can be stated that $[AF(E) + AF(F) - AF(E\cap F)]_\alpha^+ = e_\alpha^+ + f_\alpha^+ - i_\alpha^- \ge e_\alpha^+ + f_\alpha^+ - i_\alpha^+ = AF(E\cap F)_\alpha^+ .\blacksquare$

Proof of property 3.1.6:

Considering again $AF(E)_\alpha = [e_\alpha^-, e_\alpha^+]$, $AF(F)_\alpha = [f_\alpha^-, f_\alpha^+]$, $AF(E\cup F)_\alpha = [u_\alpha^-, u_\alpha^+]$ and $AF(E\cap F)_\alpha = [i_\alpha^-, i_\alpha^+]$, then $AF(E)_\alpha^- = e_\alpha^-$, $AF(F)_\alpha^- = f_\alpha^-$, $AF(E\cup F)_\alpha^- = u_\alpha^-$ and $AF(E\cap F)_\alpha^- = i_\alpha^-$.

As $\forall \alpha \in (0,1]$, $\begin{cases} e_\alpha^- = Area(E_1) \\ f_\alpha^- = Area(F_1) \\ u_\alpha^- = Area((E\cup F)_1) \\ i_\alpha^- = Area((E\cap F)_1) \end{cases}$

then

$$\begin{aligned} u_\alpha^- &= Area((E\cup F)_1) = Area((E)_1 \cup (F)_1) = \\ &= Area(E)_1 + Area(F)_1 - Area((E)_1 \cap (F)_1) = \\ &= Area(E)_1 + Area(F)_1 - Area(E\cap F)_1 = \\ &= e_\alpha^- + f_\alpha^- - i_\alpha^- \end{aligned}$$

To prove the inequality, it was already shown in the proof of property 3.1.5 that

$$[AF(E) + AF(F) - AF(E\cap F)]_\alpha = [e_\alpha^- + f_\alpha^- - i_\alpha^+, e_\alpha^+ + f_\alpha^+ - i_\alpha^-]$$

therefore,

$$[AF(E) + AF(F) - AF(E\cap F)]_\alpha^- = e_\alpha^- + f_\alpha^- - i_\alpha^+$$

Then, as $\forall \alpha$, $i_\alpha^- \le i_\alpha^+$, it can be stated that $[AF(E) + AF(F) - AF(E\cap F)]_\alpha^- = e_\alpha^- + f_\alpha^- - i_\alpha^+ \le e_\alpha^- + f_\alpha^- - i_\alpha^- = AF(E\cap F)_\alpha^-$.■

Proof of property 3.1.7:

The inclusion $F \subseteq E$ means that $\forall (x,y) \in U, \mu_F(x,y) \le \mu_E(x,y)$.

Let's consider $\alpha \in [0,1]$, $AF(E)_\alpha = [e_\alpha^-, e_\alpha^+]$ and $AF(F)_\alpha = [f_\alpha^-, f_\alpha^+]$. Then,

$$[AF(E) - AF(F)]_\alpha = AF(E)_\alpha - AF(F)_\alpha = [e_\alpha^-, e_\alpha^+] - [f_\alpha^-, f_\alpha^+] = [e_\alpha^- - f_\alpha^+, e_\alpha^+ - f_\alpha^-]$$

and $[AF(E) - AF(F)]_\alpha^+ = e_\alpha^+ - f_\alpha^-$.

As $\forall \alpha$ f_α^- is the area of the core of F and, as $F \subseteq E$, $core(F) \subseteq core(E)$, that is, $\{(x,y) : \mu_F(x,y) = 1\} \subseteq \{(x,y) : \mu_E(x,y) = 1\}$, then $[AF(E) - AF(F)]_\alpha^+$ is obtained subtracting the area of the core of F to the $Area(E_\alpha)$.

On the other hand,

$$E - F = \min(\mu_E(x,y), 1 - \mu_F(x,y)).$$

As $F \subseteq E$, $\forall(x,y) \in core(F)$ $\mu_F(x,y)=1 \wedge \mu_E(x,y)=1$. Then

$$\min\left(\mu_E(x,y), 1-\mu_F(x,y)\right)=0.$$

That is, $E-F$ does not contain the core of *F*. Then it can be stated that, $\forall(x,y)$

$$Area(E-F)_\alpha \leq Area(E_\alpha) - Area(F_1).$$

Therefore, as $e_\alpha^+ = Area(E_\alpha)$ and $f_\alpha^- = Area(F_1)$,

$$Area(E-F)_\alpha \leq e_\alpha^+ - f_\alpha^-.$$

Since

$$AF(E-F)_\alpha^+ = Area(E-F)_\alpha$$

then

$$AF(E-F)_\alpha^+ \leq \left[AF(E) - AF(F)\right]_\alpha^+ .\blacksquare$$

Proof of property 3.1.8:

If $AF(E)_\alpha = \left[e_\alpha^-, e_\alpha^+\right]$ and $AF(F)_\alpha = \left[f_\alpha^-, f_\alpha^+\right]$, then

$$AF(E)_\alpha - AF(F)_\alpha = \left[e_\alpha^- - f_\alpha^+, e_\alpha^+ - f_\alpha^-\right].$$

Considering $AF(E-F)_\alpha = \left[g_\alpha^-, g_\alpha^+\right]$, we need to prove that $g_\alpha^- = \min\left(e_\alpha^- - f_\alpha^+\right)$. As e_α^- is constant for every alpha and equal to $Area\left(core(E)\right) = AF(E)_1^-$, the minimum is attained when f_α^+ is maximum, and corresponds to the $Area\left(support(F)\right) = AF(F)_{0^+}^+$.

The value g_α^- is the area of the core of *E*-*F*, which is the set

$$\begin{aligned}
&\left\{(x,y) : \min\left(\mu_E(x,y), 1-\mu_F(x,y)\right)=1\right\} = \\
&\left\{(x,y) : \mu_E(x,y)=1 \wedge 1-\mu_F(x,y)=1\right\} = \\
&\left\{(x,y) : \mu_E(x,y)=1 \wedge \mu_F(x,y)=0\right\}
\end{aligned}$$

Therefore this is the set of points belonging to the core of E and not belonging to the support of F. Then, as $F \subseteq core(E)$,

$$\begin{aligned}
g_\alpha^- &= Area\left(core(E-F)\right) \\
&= Area\left(core(E)\right) - Area\left(support(F)\right) \\
&= \min\left(e_\alpha^- - f_\alpha^+\right).
\end{aligned}$$

and the validity of property 3.1.8 is proved.■