On the giant component of the sparse random graph

A. Puhalskii

Abstract

This paper provides large deviation asymptotics for the sizes of giant components and the number of components of sparse random graph $G(n, c/n)$.

1 Introduction

We consider random graph $G(n, p_n)$ that contains n vertices, every two of which are independently connected by an edge with probability $p_n = c/n$, $c > 0$. Let β^n denote the size of the largest connected component of the graph. It is known that if $c > 1$ then β^n/n tends in probability, as $n \to \infty$, to the solution of the equation $\beta + \exp(-\beta c) = 1$ that belongs to $(0, 1)$. This component is thus a giant component in that its size is of order n. Moreover, there exist no other giant components, i.e., given arbitrary $\delta > 0$ with probability tending to 1 any other component has size less than δn, actually, the sizes of the other components are of order $\log n$. Along with this law-of-large-numbers-type result, there is a central-limit-theorem-type result stating that the random variables $\sqrt{n}\bigl(\beta^n/n - \beta\bigr)$ converge in distribution, as $n \to \infty$, to a Gaussian random variable with mean 0 and variance $\sigma^2 = \bigl((1-\beta)\beta\bigr)/\bigl(1 - c(1-\beta)\bigr)^2$. For $c \leq 1$ with probability tending to 1 no giant component exists. The purpose of this paper is to evaluate the probabilities that there exist giant components of sizes other than βn in the supercritical case $c > 1$ and that there exist giant components for the subcritical case $c \leq 1$. In view of the above law of large numbers, this is a large-deviation problem. Therefore, it is not surprising that the probabilities in question are exponentially small, so we are concerned with finding the exponents. Thus, more specifically, we study the logarithmic asymptotics of the probabilities that there exist connected components of size between $n(u - \epsilon)$ and $n(u + \epsilon)$, where $u \in (0, 1]$ and $\epsilon > 0$ is small. The other issue with which we are concerned in this paper is the asymptotics of the total number of connected components.

To give an idea of the sort of results presented in the paper we state the following theorem. Let, given $\epsilon > 0$, $m \in \mathbb{N}$, and $u_k \in (0, 1]$, $k = 1, 2, \ldots, m$, such that $\sum_{k=1}^m u_k \leq 1$, $A_\epsilon^n(u_1, \ldots, u_m)$ denote the event that there exist m connected components, whose respective sizes are between $n(u_k - \epsilon)$ and $n(u_k + \epsilon)$ for $k = 1, 2, \ldots, m$, and $B_\epsilon^n(u_1, \ldots, u_m)$ be the intersection of $A_\epsilon^n(u_1, \ldots, u_m)$ and the event that there is no other connected component of size greater than $n\epsilon$. Let for $u \in [0, 1]$

$$K_c(u) = u \log u - u \log\bigl(1 - e^{-cu}\bigr) - \frac{cu^2}{2}, \tag{1.1}$$

$$L_c(u) = (1 - u)\log(1 - u) + cu - \frac{cu^2}{2}, \tag{1.2}$$

$$M_c(u) = u \log c - \log c + \frac{c}{2} - \frac{1}{2c}, \tag{1.3}$$

where we adopt the convention $0 \cdot \infty = 0$.

Theorem 1.1. *1. If $\sum_{k=1}^m u_k \ge 1 - 1/c$, then*

$$\lim_{\epsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\log P\big(A_\epsilon^n(u_1,\dots,u_m)\big) = \lim_{\epsilon\to 0}\liminf_{n\to\infty}\frac{1}{n}\log P\big(A_\epsilon^n(u_1,\dots,u_m)\big)$$
$$= \lim_{\epsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\log P\big(B_\epsilon^n(u_1,\dots,u_m)\big) = \lim_{\epsilon\to 0}\liminf_{n\to\infty}\frac{1}{n}\log P\big(B_\epsilon^n(u_1,\dots,u_m)\big)$$
$$= -\Big(\sum_{k=1}^m K_c(u_k) + L_c\big(\sum_{k=1}^m u_k\big)\Big).$$

2. If $\sum_{k=1}^m u_k < 1 - 1/c$, then

$$\lim_{\epsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\log P\big(A_\epsilon^n(u_1,\dots,u_m)\big) = \lim_{\epsilon\to 0}\liminf_{n\to\infty}\frac{1}{n}\log P\big(A_\epsilon^n(u_1,\dots,u_m)\big)$$
$$= -\Big(\sum_{k=1}^m K_c(u_k) + L_c\big(\sum_{k=1}^m u_k\big)\Big),$$

and

$$\lim_{\epsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\log P\big(B_\epsilon^n(u_1,\dots,u_m)\big) = \lim_{\epsilon\to 0}\liminf_{n\to\infty}\frac{1}{n}\log P\big(B_\epsilon^n(u_1,\dots,u_m)\big)$$
$$= -\Big(\sum_{k=1}^m K_c(u_k) + M_c\big(\sum_{k=1}^m u_k\big)\Big).$$

We note that part 2 constitutes a non-void statement only for the supercritical case $c > 1$. As an interesting consequence of the theorem, we mention the following 0-1 law: if $\sum_{k=1}^m u_k \ge 1 - 1/c$, then

$$\lim_{\epsilon\to 0}\liminf_{n\to\infty} P\big(B_\epsilon^n(u_1,\dots,u_m)|A_\epsilon^n(u_1,\dots,u_m)\big) = 1,$$

while if $\sum_{k=1}^m u_k < 1 - 1/c$, then

$$\lim_{\epsilon\to 0}\limsup_{n\to\infty} P\big(B_\epsilon^n(u_1,\dots,u_m)|A_\epsilon^n(u_1,\dots,u_m)\big) = 0.$$

In particular, provided there exists one component of size close to nu, with probability tending to 1 there exists no other giant component for $u \ge 1 - 1/c$ and there exists one more giant component for $u < 1 - 1/c$.

Another contribution of the paper is an introduction of a new approach in the problem, which reveals a surprising connection with queueing theory. While previous approaches heavily rely on combinatorial arguments, we make no use of them. The stated large deviation results are derived as consequences of the large deviation principle for a stochastic process that models the formation of the connected components and builds on an earlier construction of a similar sort, see, e.g., Janson, Luczak and Rucinski [7]. This stochastic process asymptotically has the same behaviour as the waiting time process in a certain queueing system with state-dependent arrivals, and the connected components correspond to the busy cycles in this queueing system. The connection allows us to capitalise in our analysis on the techniques developed earlier in the queueing theory framework, Puhalskii [9], as well as to make use of our intuition for large deviations of queues. Furthermore, the local-time-at-zero process corresponds to the number of connected components. Thus, our approach also recovers the results of Stepanov [12] on the logarithmic asymptotics of the moment generating function of the number of connected components in a random graph. We then translate the large deviation results for the process into the assertions on the large-deviation behaviour of the connected components. As one of the consequences of our results, we derive the large deviation principle for the largest component obtained earlier by O'Connell [8] by different methods.

The techniques used in the paper have broader implications. Though the described stochastic process unambiguously specifies the size of all the connected components of the random graph, the corresponding dependence is not expressed by a continuous (or almost continuous) mapping so that one cannot apply the contraction principle. Thus, we have to carry out more delicate analysis by establishing the asymptotics for suitable subsets and supsets of $A^n_\epsilon(u_1, \ldots, u_m)$ and $B^n_\epsilon(u_1, \ldots, u_m)$ first, and relying on them to derive the asserted limits. By using similar methods one should be able to analyse the large deviations of busy cycles in a queueing system for which the large-deviation analysis of the waiting-time (or queue-length) process has been carried out (e.g., as in Puhalskii [9]).

The crucial argument in proving the LDP for the basic process is provided by the large deviation theory of semimartingales, Puhalskii [10]. The LDP involves a degenerate cumulant, which is known to present certain complications in proving the LD lower bound and is usually tackled via a perturbation approach: one perturbs the stochastic process in question so that the perturbed process has a non-degenerate cumulant and then takes a limit in the lower bound for the perturbed process as the perturbation tends to zero. The perturbation typically depends on the process and has to be introduced on a case-by-case basis. Our approach, instead of studying the upper and lower bounds, requires that the limiting maxingale problem have a unique solution. When the cumulant is degenerate, no general uniqueness results are available. As in the standard approach to the LD lower bound, we prove that the maxingale problem associated with the random graph has a unique solution by using the idea of perturbation as well. The difference, however, is that we perturb the (limit) maxingale problem. Therefore, if different stochastic frameworks lead to the same maxingale problem (which is quite common), for which uniqueness has been proved, no case-specific perturbation is needed. In order to prove the uniqueness we invoke methods of idempotent probability theory: we represent solutions to the perturbed maxingale problems as the laws of idempotent processes, which solve certain equations. Any solution of the original maxingale problem is a weak solution of the limit equation obtained as the perturbation magnitude tends to zero. The solution of the latter equation being unique, signifies the solution to the maxingale problem is unique. Interestingly, we draw on Ethier and Kurtz [5, Chapter 6] for the particular technical implementation, so we use probabilistic ideas in an idempotent probability setting.

We now outline the structure of the paper. The next section contains the construction of and the large deviation limit theorems for the basic stochastic process. Section 3 is concerned with proving Theorem 1.1. Section4 contains some large deviation results on the size of the largest component. Section 5 studies large deviation asymptotics for the number of connected components. The appendix contains auxiliary results.

2 The model equation and large deviations of the underlying process

We model the growth of the giant component of a sparse random graph on n vertices with edge probability c/n by a stochastic process that extends a branching process construction described in Janson, Luczak and Rucinski [7]. The process, denoted by $V^n = (V^n_i,\, i = 0, 1, \ldots, n)$, evolves in discrete time and starts at $V^n_0 = 0$. At time 1 an arbitrary vertex of the graph is picked and is connected by an edge to the other vertices independently with probability c/n, after which the vertex is called "saturated" and the vertices, to which it has been connected, are called "non-saturated"; all thus produced vertices are called "generated". The value V_1^n is the number of vertices in the resulting connected component, i.e., the number of the generated vertices. At time 2 we pick one of the generated non-saturated vertices if any and "saturate" it as before by connecting it independently with probability c/n to the vertices that haven't been generated yet; if there are no generated non-saturated vertices, we pick an arbitrary non-generated vertex, declare it generated and saturate it. We denote as V_2^n the number of vertices in the resulting component, which is the total number of vertices generated at times 1 and 2. We proceed in this manner by saturating one vertex each time-step until "time" n. By construction, the sizes of the connected components of the configuration of edges at time n have the same distribution as the sizes of the connected components of the random graph $G(n, c/n)$. Also, the sizes of the connected components are recovered from the process V^n as the times elapsed between the moments when V_i^n becomes equal to i, because these are the moments when there are no non-saturated vertices left. Besides, the number of connected components equals the number of times when all the generated vertices have been saturated. Therefore, the vertex-process V^n captures the probabilistic properties of the connected components of $G(n, c/n)$.

Since at time i there are i saturated vertices and $V_i^n - i$ generated non-saturated vertices, the evolution of V^n is given by the following recursion

$$V_i^n = \big(V_{i-1}^n + \sum_{j=1}^{n-V_{i-1}^n} \xi_{ij}^n\big)\mathbf{1}(V_{i-1}^n > i-1) + \big(1 + \sum_{j=1}^{n-i} \xi_{ij}^n\big)\mathbf{1}(V_{i-1}^n = i-1),$$
$$i = 1, 2, \ldots, n, \; V_0^n = 0, \quad (2.1)$$

where the ξ_{ij}^n are independent Bernoulli random variables with $P(\xi_{ij}^n = 1) = c/n$ and $\mathbf{1}(A)$ is the indicator function of event A that equals 1 on A and 0 outside of A. Let W_i^n denote the number of non-saturated generated vertices at time i. Since

$$W_i^n = V_i^n - i, \quad (2.2)$$

(2.1) implies that

$$W_i^n = \big(W_{i-1}^n + \sum_{j=1}^{n-W_{i-1}^n-(i-1)} \xi_{ij}^n - 1\big)\mathbf{1}(W_{i-1}^n > 0) + \sum_{j=1}^{n-i} \xi_{ij}^n \,\mathbf{1}(W_{i-1}^n = 0),$$
$$i = 1, 2, \ldots, n, \; W_0^n = 0. \quad (2.3)$$

We use for the analysis of the solution of (2.3) the following observation. Let us introduce a related process $\tilde{W}^n = (\tilde{W}_i^n, \, i = 1, 2, \ldots, n)$ by

$$\tilde{W}_i^n = \big(\tilde{W}_{i-1}^n + \sum_{j=1}^{n-\tilde{W}_{i-1}^n-(i-1)} \xi_{ij}^n - 1\big)^+, \; i = 1, 2, \ldots, n, \; \tilde{W}_0^n = 0, \quad (2.4)$$

where $a^+ = \max\{a, 0\}$. A simple induction argument shows that $\tilde{W}_i^n \le W_i^n \le \tilde{W}_i^n + 1$, $i = 0, 1, \ldots, n$, so the asymptotic properties of W^n (and hence V^n) multiplied by a vanishing constant are the same as those of $\tilde{W}^n$. We note that $\tilde{W}_i^n$ is the waiting time of the i-th request, where $i = 0, 1, \ldots, n-1$, in a queueing system that starts empty, has $\sum_{j=1}^{n-\tilde{W}_i^n - i} \xi_{i+1,j}^n$ as the ith request's service time and 1 as the interarrival times. (Alternatively, $\tilde{W}_i^n$ can be considered as the queue length at time $i = 0, 1, \ldots, n$ for a discrete-time queueing system that serves one request per unit time, the number of arrivals in $[i,i+1]$ being equal to $\sum_{j=1}^{n-\tilde{W}_i^n-i}\xi_{i+1,j}^n$.) The connected components of the random graph correspond to the busy cycles of the queueing system.

There is a large body of techniques developed for analysing equations of the form (2.4). For example, the solution of (2.4) is known to be given by $\tilde{W}^n=\mathcal{R}(\tilde{S}^n)$, where the process $\tilde{S}^n=(\tilde{S}_i^n,\, i=0,1,\ldots,n)$ with $\tilde{S}_0^n=0$ is defined for $i=1,2,\ldots,n$ by $\tilde{S}_i^n=\sum_{k=1}^{i}\sum_{j=1}^{n-\tilde{W}_{k-1}^n-(k-1)}\xi_{kj}^n-i$, and $\mathcal{R}$ is Skorohod's reflection operator: $\mathcal{R}(\mathbf{x})_t=\mathbf{x}_0-\inf_{s\in[0,t]}\mathbf{x}_s\wedge 0$. However, we find it more convenient to work directly with W^n.

A manipulation of (2.3) yields the following equality:

$$W_i^n=S_i^n+\epsilon_i^n+\Phi_i^n,\ i=1,2,\ldots,n,\ W_0^n=0, \tag{2.5}$$

where

$$S_i^n=\sum_{k=1}^{i}\Big(\sum_{j=1}^{n-W_{k-1}^n-(k-1)}\xi_{kj}^n-1\Big), \tag{2.6}$$

$$\epsilon_i^n=1-\mathbf{1}(W_i^n=0)-\sum_{k=1}^{i}\xi_{k,n-k+1}^n\,\mathbf{1}(W_{k-1}^n=0), \tag{2.7}$$

$$\Phi_i^n=\sum_{k=1}^{i}\mathbf{1}(W_k^n=0). \tag{2.8}$$

Hence,

$$W^n=\mathcal{R}(S^n+\epsilon^n), \tag{2.9}$$

which paves the way for the following theorem.

Let us introduce the continuous-time processes $\overline{W}^n=(W_{\lfloor nt\rfloor}^n/n,\ t\in[0,1])$, $\overline{\Phi}^n=(\Phi_{\lfloor nt\rfloor}^n/n,\ t\in[0,1])$, and $\overline{S}^n=(S_{\lfloor nt\rfloor}^n/n,\ t\in[0,1])$. These processes are considered as random elements of $\mathbb{D}_C([0,1])$ that is the space of right-continuous with left-hand limits real-valued functions equipped with uniform metric. We introduce more notation. We denote by $\dot{\mathbf{x}}_t$ the derivative of an absolutely continuous function $\mathbf{x}=(\mathbf{x}_t,\ t\in[0,1])\in\mathbb{D}_C([0,1])$ at t; "almost everywhere (a.e.)" in the below statement refers to the Lebesgue measure; we let $0/0=0$. We also denote by $\mathbb{D}(\mathbb{R}_+)$ the space of real-valued right-continuous with left-hand limits functions on $\mathbb{R}_+$, which is equipped with the Skorohod topology. For a sequence of random variables α^n we write $\alpha^n\overset{P^{1/n}}{\to}0$ if $\lim_{n\to\infty}\big(P(|\alpha^n|>\epsilon)\big)^{1/n}=0$ for arbitrary $\epsilon>0$. We let $\mathbf{e}=(t,\ t\in\mathbb{R}_+)$.

Theorem 2.1. *The processes $\overline{S}^n$ and $\overline{W}^n$ satisfy large deviation principles (LDPs) in $\mathbb{D}_C([0,1])$ with respective action functionals I^S and I^W given by*

$$I^S(\mathbf{x})=\int_0^1\Big((\dot{\mathbf{x}}_t+1)\log\frac{\dot{\mathbf{x}}_t+1}{c(1-t-\mathcal{R}(\mathbf{x})_t)}-(\dot{\mathbf{x}}_t+1)+c(1-t-\mathcal{R}(\mathbf{x})_t)\Big)\,dt$$

for absolutely continuous $\mathbf{x}=(\mathbf{x}_t,\ t\in[0,1])\in\mathbb{D}_C([0,1])$ with $\mathbf{x}_0=0$, $\mathcal{R}(\mathbf{x})_t\le 1-t$, and $\dot{\mathbf{x}}_t\ge-1$ a.e., $t\in[0,1]$, and $I^S(\mathbf{x})=\infty$ for other $\mathbf{x}$, and

$$I^W(\mathbf{x})=\int_0^1\Big((\dot{\mathbf{x}}_t+1)\log\frac{\dot{\mathbf{x}}_t+1}{c(1-t-\mathbf{x}_t)}-(\dot{\mathbf{x}}_t+1)+c(1-t-\mathbf{x}_t)\Big)\,\mathbf{1}(\mathbf{x}_t>0)\,dt$$

$$+ \int_0^{(1-1/c)^+} \big(-\log(c(1-t)) - 1 + c(1-t)\big)\,\mathbf{1}(\mathbf{x}_t = 0)\,dt$$

for absolutely continuous $\mathbf{x} = (\mathbf{x}_t,\, t \in [0,1]) \in \mathbb{D}_C([0,1])$ *with* $\mathbf{x}_0 = 0$, $\mathbf{x}_t \in [0, 1-t]$, *and* $\dot{\mathbf{x}}_t \geq -1$ *a.e.,* $t \in [0,1]$, *and* $I^W(\mathbf{x}) = \infty$ *for other* $\mathbf{x}$.

We preview the proof by reminding the reader a characterisation of reflection (see, e.g., Puhalskii [9], Puhalskii and Whitt [11]).

Lemma 2.1. *Let* $\mathbf{y}$ *be an absolutely continuous function with* $\dot{\mathbf{y}}_t \geq -1$ *a.e. Then* $\mathbf{x} = \mathcal{R}(\mathbf{y})$ *if and only if* $\mathbf{x}$ *is absolutely continuous and* $\dot{\mathbf{x}}_t = \dot{\mathbf{y}}_t + \gamma_t$ *a.e., where* $\gamma_t \in [0,1]$ *and* $\gamma_t = 0$ *for* $\mathbf{x}_t > 0$. *Also* $\dot{\mathbf{x}}_t = 0$ *a.e. on the set* $\{\mathbf{x}_t = 0\}$

Proof of Theorem 2.1. Let $A^n = (A^n_t,\, t \in [0,1])$ be defined by

$$A^n_t = \frac{1}{n} \sum_{k=1}^{\lfloor nt \rfloor} \sum_{j=1}^{n - W^n_{k-1} - (k-1)} \xi^n_{k,j}. \tag{2.10}$$

Our first step is to prove that the A^n as elements of $\mathbb{D}_C([0,1])$ satisfy an LDP with action functional

$$I^A(\mathbf{x}) = \int_0^1 \Big(\dot{\mathbf{x}}_t \log \frac{\dot{\mathbf{x}}_t}{c(1 - t - \mathcal{R}(\mathbf{x} - \mathbf{e})_t)} - \dot{\mathbf{x}}_t + c(1 - t - \mathcal{R}(\mathbf{x} - \mathbf{e})_t)\Big)\,dt \tag{2.11}$$

if $\mathbf{x}$ is absolutely continuous, $\mathbf{x}_0 = 0$, $\dot{\mathbf{x}}_t \geq 0$ a.e., and $\mathcal{R}(\mathbf{x} - \mathbf{e})_t \leq 1 - t$ for $t \in [0,1]$, and $I^A(\mathbf{x}) = \infty$ otherwise.

Let us extend the A^n to processes $\overline{A}^n$ defined on $\mathbb{R}_+$ by letting $\overline{A}^n_t = A^n_{t\wedge 1}$ and show that the $\overline{A}^n$ satisfy the hypotheses of Theorem 5.1.5 in Puhalskii [10]. We define σ-algebras $\mathcal{F}^n_t$, $t \in \mathbb{R}_+$, as the σ-algebras generated by the random variables ξ^n_{kj}, $k = 1, 2, \ldots, \lfloor n(t \wedge 1) \rfloor$, $j \in \mathbb{N}$, completed with sets of P-measure zero, and introduce the filtrations $\mathbf{F}^n = (\mathcal{F}^n_t,\, t \in \mathbb{R}_+)$. By (2.10) $\overline{A}^n$ is a totally discontinuous $\mathbf{F}^n$-adapted semimartingale with predictable measure of jumps $\big(\nu^n([0,t], \Gamma),\, t \in \mathbb{R}_+,\, \Gamma \in \mathcal{B}(\mathbb{R})\big)$ given by

$$\nu^n([0,t], \Gamma) = \sum_{k=0}^{\lfloor n(t\wedge 1) \rfloor - 1} F^n\big(1 - \overline{W}^n_{k/n} - k/n,\, n(\Gamma \setminus \{0\})\big),$$

where

$$F^n(s, \Gamma') = P\Big(\sum_{j=1}^{\lfloor ns \rfloor} \xi^n_{1j} \in \Gamma'\Big),\ \Gamma' \in \mathcal{B}(\mathbb{R}). \tag{2.12}$$

Since the jumps of $\overline{A}^n$ are bounded from above by 1, $\overline{A}^n$ satisfies the Cramér condition, so its stochastic (or Doléans-Dade) exponential is well defined. Noting that the first predictable characteristic of $\overline{A}^n$ has the form

$$B^n_t = \frac{c}{n} \sum_{k=0}^{\lfloor n(t\wedge 1) \rfloor - 1} \big(1 - \overline{W}^n_{k/n} - \frac{k}{n}\big),$$

and the continuous martingale part of $\overline{A}^n$ equals 0, we have that the stochastic exponential is given by

$$\mathcal{E}_t^n(\lambda) = \prod_{k=1}^{\lfloor n(t\wedge 1)\rfloor} \Bigl(1 + \int_{\mathbb{R}} (e^{\lambda x} - 1)\, \nu^n(\{\frac{k}{n}\}, dx)\Bigr) = \prod_{k=0}^{\lfloor n(t\wedge 1)\rfloor - 1} \int_{\mathbb{R}} e^{\lambda x}\, F^n\bigl(1 - \overline{W}^n_{k/n} - \frac{k}{n},\, n\, dx\bigr).$$

By (2.9), (2.6) and (2.10)

$$\overline{W}^n = \mathcal{R}(A^n - \mathbf{e}^n + \overline{\epsilon}^n), \tag{2.13}$$

where $\mathbf{e}^n = (\lfloor nt\rfloor/n,\, t \in \mathbb{R}_+)$, $\overline{\epsilon}^n = (\overline{\epsilon}_t^n,\, t \in \mathbb{R}_+)$, and $\overline{\epsilon}_t^n = \epsilon^n_{\lfloor n(t\wedge 1)\rfloor}/n$. Hence, recalling the fact that the ξ_i^n are Bernoulli and equal 1 with probability c/n, we have

$$\frac{1}{n} \log \mathcal{E}_t^n(n\lambda) = n \log\bigl(1 + (e^{\lambda} - 1)\frac{c}{n}\bigr) \int_0^{t\wedge 1} \bigl(1 - \mathcal{R}(\overline{A}^n - \mathbf{e}^n + \overline{\epsilon}^n)_s - \frac{\lfloor ns\rfloor}{n}\bigr)\, ds.$$

It is straightforward to see that

$$\sup_{t\in\mathbb{R}_+} |\overline{\epsilon}_t^n| \overset{P^{1/n}}{\to} 0 \text{ as } n \to \infty. \tag{2.14}$$

Thus, denoting

$$G_t(\lambda, \mathbf{x}) = c(e^{\lambda} - 1) \int_0^{t\wedge 1} (1 - \mathcal{R}(\mathbf{x} - \mathbf{e})_s - s)\, ds, \tag{2.15}$$

we have that for arbitrary $\delta > 0$ and $T > 0$

$$\sup_{t\in[0,T]} \Bigl|\frac{1}{n} \log \mathcal{E}_t^n(n\lambda) - G_t(\lambda, \overline{A}^n)\Bigr| \overset{P^{1/n}}{\to} 0 \text{ as } n \to \infty.$$

Since $G_t(\lambda, \mathbf{x})$ satisfies the uniform continuity and majoration conditions of Theorem 5.1.5 of [10], by the theorem the sequence of laws of the $\overline{A}^n$ on space $\mathbb{D}(\mathbb{R}_+)$ is $\mathbb{C}$-exponentially tight (of order n), and its every large deviation accumulation point solves the maxingale problem $(0, G)$ with $G = \bigl(G_t(\lambda, \mathbf{x}),\, t \in \mathbb{R}_+, \lambda \in \mathbb{R}, \mathbf{x} \in \mathbb{C}(\mathbb{R}_+)\bigr)$. Let deviability Π^A on $\mathbb{C}(\mathbb{R}_+)$ be a solution of $(0, G)$. We prove that $\Pi^A(\mathbf{x}) = \exp(-I^A(p_1\mathbf{x}))$, $\mathbf{x} \in \mathbb{D}(\mathbb{R}_+)$, where $(p_1\mathbf{x})_t = \mathbf{x}_{t\wedge 1}$. The idea is to show that the equation

$$A_t = N_{L_t(A)},\ t \in \mathbb{R}_+,\ \ \Pi\text{-a.e.}, \tag{2.16}$$

where N is idempotent Poisson and

$$L_t(\mathbf{x}) = c \int_0^{t\wedge 1} (1 - \mathcal{R}(\mathbf{x} - \mathbf{e})_s - s)\, ds,$$

has a (weak) solution A with idempotent distribution Π^A. In more detail, let $\Omega = \mathbb{C}(\mathbb{R}_+) \times \mathbb{C}(\mathbb{R}_+)$ with canonical component processes $A = (A_t(\mathbf{x}, \mathbf{x}'),\, t \in \mathbb{R}_+, (\mathbf{x}, \mathbf{x}') \in \Omega)$ and $N = (N_t(\mathbf{x}, \mathbf{x}'),\, t \in \mathbb{R}_+, (\mathbf{x}, \mathbf{x}') \in \Omega)$ defined by the respective equalities $A_t(\mathbf{x}, \mathbf{x}') = \mathbf{x}_t$ and $N_t(\mathbf{x}, \mathbf{x}') = \mathbf{x}'_t$. We will prove that there exists deviability Π on Ω such that A and N satisfy (2.16) Π-a.e., $\sup_{\mathbf{x}'\in\mathbb{C}(\mathbb{R}_+)} \Pi(\mathbf{x}, \mathbf{x}') = \Pi^A(\mathbf{x})$ and $\sup_{\mathbf{x}\in\mathbb{C}(\mathbb{R}_+)} \Pi(\mathbf{x}, \mathbf{x}') = \Pi^N(\mathbf{x}')$. After that we will draw on Ethier and Kurtz [5, Theorem 1.1, Chapter 6] to conclude that (2.16) has a unique solution. The reasoning used to establish (2.16) is also along the lines of the approaches developed in [5].

Let us first note that since $1 - \mathcal{R}(\overline{A}^n - \mathbf{e}^n - \overline{\epsilon}^n)_s - \lfloor ns \rfloor / n \geq 0$ for all s and (2.14) holds, we have that $1 - \mathcal{R}(A - \mathbf{e})_s - s \geq 0 \quad \Pi^A$-a.e., so

$$L_t(\mathbf{x}) = \tilde{L}_t(\mathbf{x}) \;\; \Pi^A\text{-a.e.,} \tag{2.17}$$

where

$$\tilde{L}_t(\mathbf{x}) = c \int_0^{t\wedge 1} (1 - \mathcal{R}(\mathbf{x} - \mathbf{e})_s - s)^+ \, ds.$$

Given $\epsilon > 0$, we define an idempotent process $A^\epsilon = (A^\epsilon_t(\mathbf{x}, \mathbf{x}'),\, t \in \mathbb{R}_+, (\mathbf{x}, \mathbf{x}') \in \Omega)$ on Ω by

$$A^\epsilon_t(\mathbf{x}, \mathbf{x}') = A_t(\mathbf{x}, \mathbf{x}') + \mathbf{x}'_{\epsilon t} \tag{2.18}$$

and let

$$G^\epsilon_t(\lambda; (\mathbf{x}, \mathbf{x}')) = G_t(\lambda; \mathbf{x}) + (e^\lambda - 1)\epsilon t.$$

Deviability Π^A being a solution of the maxingale problem $(0, G)$ signifies that the idempotent process $\big(\exp(\lambda \mathbf{x}_t - G_t(\lambda; \mathbf{x})),\, t \in \mathbb{R}_+, \mathbf{x} \in \mathbb{C}(\mathbb{R}_+)\big)$ is a local exponential maxingale on $(\mathbb{C}(\mathbb{R}_+), \Pi^A)$ adapted to the canonical τ-flow $\mathbf{A} = (\mathcal{A}_t,\, t \in \mathbb{R}_+)$. Also, $\big(\exp(\lambda \mathbf{x}'_{\epsilon t} - (e^\lambda - 1)\epsilon t),\, t \in \mathbb{R}_+, \mathbf{x}' \in \mathbb{C}(\mathbb{R}_+)\big)$ is a local exponential maxingale on $(\mathbb{C}(\mathbb{R}_+), \Pi^N)$ adapted to the τ-flow $\mathbf{A}^\epsilon = (\mathcal{A}_{\epsilon t},\, t \in \mathbb{R}_+)$ Therefore, under product deviability $\Pi^A \times \Pi^N$ the process $\big((\lambda A^\epsilon_t(\mathbf{x}, \mathbf{x}') - G^\epsilon_t(\lambda; (\mathbf{x}, \mathbf{x}')),\, t \in \mathbb{R}_+, (\mathbf{x}, \mathbf{x}') \in \Omega\big)$ is a local exponential maxingale on Ω relative to the canonical τ-flow $\mathbf{F} = (\mathcal{F}_t,\, t \in \mathbb{R}_+)$. Let

$$\sigma^\epsilon_t(\mathbf{x}, \mathbf{x}') = \inf\{s \in \mathbb{R}_+ : \; \tilde{L}_s(\mathbf{x}) + \epsilon s \geq t\}. \tag{2.19}$$

The idempotent variable σ^ϵ_t is a finite idempotent $\mathbf{F}$-stopping time and $G^\epsilon_{\sigma^\epsilon_t(\mathbf{x}, \mathbf{x}')}(\lambda; (\mathbf{x}, \mathbf{x}')) = (e^\lambda - 1)t$, so the process $\big(\exp(\lambda N^\epsilon_t(\mathbf{x}, \mathbf{x}') - (e^\lambda - 1)t),\, t \in \mathbb{R}_+, (\mathbf{x}, \mathbf{x}') \in \Omega\big)$, where $N^\epsilon_t(\mathbf{x}, \mathbf{x}') = A^\epsilon_{\sigma^\epsilon_t(\mathbf{x}, \mathbf{x}')}(\mathbf{x}, \mathbf{x}')$, is a local exponential maxingale relative to the flow $\mathbf{F}^\epsilon = (\mathcal{F}_{\sigma^\epsilon_t},\, t \in \mathbb{R}_+)$, which implies by Lemma 2.4.20 in [10] that $N^\epsilon = (N^\epsilon_t(\mathbf{x}, \mathbf{x}'),\, t \in \mathbb{R}_+, (\mathbf{x}, \mathbf{x}') \in \Omega)$ is an $\mathbf{F}^\epsilon$-Poisson idempotent process, so it is a Poisson idempotent process. In view of (2.18) and (2.19) we can thus write that $\Pi^A \times \Pi^N$-a.e.

$$A_t + \mathbf{x}'_{\epsilon t} = N^\epsilon_{L_t(A) + \epsilon t},\; t \in \mathbb{R}_+. \tag{2.20}$$

The pair (A, N^ϵ) specifies a mapping of Ω into itself. Let Π^ϵ denote the image of $\Pi^A \times \Pi^N$ under this mapping, i.e., $\Pi^\epsilon(\mathbf{x}, \mathbf{x}') = \sup_{\substack{(\mathbf{y}, \mathbf{y}') \in \Omega:\, A(\mathbf{y}, \mathbf{y}') = \mathbf{x},\\ N^\epsilon(\mathbf{y}, \mathbf{y}') = \mathbf{x}'}} \Pi^A(\mathbf{y}) \Pi^N(\mathbf{y}')$; briefly, Π^ϵ is the joint idempotent distribution of (A, N^ϵ) on Ω. The net Π^ϵ, $\epsilon > 0$, is tight as $\epsilon \to 0$. Let Π denote an accumulation point of the Π^ϵ. By (2.20) for $T > 0$ and $\eta > 0$, using the fact that $\mathbf{x}'$ and N^ϵ are Poisson under $\Pi^A \times \Pi^N$,

$$\begin{aligned}
\Pi^\epsilon\big(\sup_{t \in [0,T]} |\mathbf{x}_t - \mathbf{x}'_{L_t(\mathbf{x})}| \geq \eta\big) &= (\Pi^A \times \Pi^N)\big(\sup_{t \in [0,T]} |A_t - N^\epsilon_{L_t(A)}| \geq \eta\big) \\
&\leq (\Pi^A \times \Pi^N)(\sup_{t \in [0,T]} |\mathbf{x}'_{\epsilon t}| \geq \eta/2) \vee (\Pi^A \times \Pi^N)(\sup_{\substack{s,t \in [0, cT]:\\ |s-t| \leq \epsilon T}} |N^\epsilon_s - N^\epsilon_t| \geq \eta/2) = \Pi^N(\mathbf{x}'_{\epsilon T} \geq \eta/2).
\end{aligned}$$

By the Markov inequality $\Pi^N(\mathbf{x}'_{\epsilon T} \geq \eta/2) \leq (e-1)\epsilon T / e^{\eta/2}$, hence, $\lim_{\epsilon \to 0} \Pi^\epsilon\big(\sup_{t \in [0,T]} |\mathbf{x}_t - \mathbf{x}'_{L_t(\mathbf{x})}| \geq \eta\big) = 0$. Since $\sup_{t \in [0,T]} |\mathbf{x}_t - \mathbf{x}'_{L_t(\mathbf{x})}|$ is a continuous function of $\mathbf{x} \in \mathbb{C}(\mathbb{R}_+)$ and $\mathbf{x}' \in \mathbb{C}(\mathbb{R}_+)$, we thus conclude that $\Pi\big(\sup_{t \in [0,T]} |\mathbf{x}_t - \mathbf{x}'_{L_t(\mathbf{x})}| \geq \eta\big) = 0$, so $\Pi\big(\sup_{t \in [0,T]} |\mathbf{x}_t - \mathbf{x}'_{L_t(\mathbf{x})}| > 0\big) = \sup_{\eta > 0} \Pi\big(\sup_{t \in [0,T]} |\mathbf{x}_t - \mathbf{x}'_{L_t(\mathbf{x})}| \geq \eta\big) = 0$, which is equivalent to (2.16) by A and N being the respective first and second component processes on Ω.

The argument of the proof of Ethier and Kurtz [5, Theorem 1.1, Ch. 6] (the theorem itself does not apply unfortunately) allows us to conclude that (2.16) has a unique solution for A given by $A_t = N_{\sigma_t(N)}$, where $\sigma_t(\mathbf{x}') = \inf\bigl\{s \in \mathbb{R}_+ : \int_0^s \bigl((1 - \mathcal{R}(\mathbf{x}' - \mathbf{e})_r - r)^+\bigr)^{-1} dr \geq t\bigr\} \wedge 1$. Therefore, $\Pi(\mathbf{x}, \mathbf{x}') = 0$ if $(\mathbf{x}_t,\, t \in \mathbb{R}_+) \neq (\mathbf{x}'_{\sigma_t(\mathbf{x}')},\, t \in \mathbb{R}_+)$, so $\Pi(\mathbf{x}, \mathbf{x}') = \Pi^N(\mathbf{x}')$ if $(\mathbf{x}_t,\, t \in \mathbb{R}_+) = (\mathbf{x}'_{\sigma_t(\mathbf{x}')},\, t \in \mathbb{R}_+)$, and we can write

$$\Pi^A(\mathbf{x}) = \sup_{\mathbf{x}'} \Pi(\mathbf{x}, \mathbf{x}') = \sup_{\mathbf{x}' :\, \mathbf{x}_t = \mathbf{x}'_{\sigma_t(\mathbf{x}')}} \Pi^N(\mathbf{x}') = \sup_{\mathbf{x}' :\, \mathbf{x}_t = \mathbf{x}'_{L_t(\mathbf{x})}} \Pi^N(\mathbf{x}'). \tag{2.21}$$

Recalling that $\Pi^N(\mathbf{x}') = \exp(-I^N(\mathbf{x}'))$, where $I^N(\mathbf{x}') = \int_0^\infty (\dot{\mathbf{x}}'_t \log \dot{\mathbf{x}}'_t - \dot{\mathbf{x}}'_t + 1)\, dt$ if $\mathbf{x}'$ is absolutely continuous, $\mathbf{x}'_0 = 0$, and $\dot{\mathbf{x}}'_t \geq 0$ a.e., and $I^N(\mathbf{x}') = \infty$ otherwise, we derive by using a change of variables that the right-most side of (2.21) equals $\exp(-I^A(p_1\mathbf{x}))$, as required. Thus, the processes $\overline{A}^n$ satisfy an LDP in $\mathbb{D}(\mathbb{R}_+)$ with action functional $I^A \circ p_1$. By the contraction principle the A^n satisfy an LDP in $\mathbb{D}([0,1])$ with I^A. The A^n being random elements of $\mathbb{D}_C([0,1])$ implies the LDP for the uniform topology. The LDP for the A^n has been proved. The LDP for the $\overline{S}^n$ follows by the equality $\overline{S}^n = A^n - \mathbf{e}$ (see (2.13) and (2.10)) and the contraction principle.

We now consider $\overline{W}^n$. By (2.13), (2.14) and the contraction principle the $\overline{W}^n$ satisfy an LDP with action functional $I^W(\mathbf{x}) = \inf_{\mathbf{y} :\, \mathbf{x} = \mathcal{R}(\mathbf{y})} I^S(\mathbf{y})$, so $I^W(\mathbf{x}) = \infty$ if either of the conditions $\mathbf{x}_0 = 0$ or $\mathbf{x}_t \in [0, 1-t]$ for $t \in [0,1]$ is not satisfied. Assume now that both these conditions hold. Since $\mathbf{y}$ in the latter inf has to be absolutely continuous in order for $I^S(\mathbf{y})$ to be finite, so does $\mathbf{x}$ in order for $I^W(\mathbf{x})$ to be finite. In particular, since $\dot{\mathbf{y}}_t \geq -1$ a.e., we have that $\dot{\mathbf{x}}_t \geq -1$ a.e. Thus, for $\mathbf{x}$ such that $\mathbf{x}_0 = 0$, $\mathbf{x}_t \in [0, 1-t]$ when $t \in [0,1]$, and $\dot{\mathbf{x}}_t \geq -1$ a.e. on $[0,1]$ we have by Lemma 2.1 and Puhalskii [9]

$$\begin{aligned}
I^W(\mathbf{x}) &= \inf_{\mathbf{y} :\, \mathbf{x} = \mathcal{R}(\mathbf{y})} I^S(\mathbf{y}) \\
&= \inf_{\substack{(\gamma_t) :\, \gamma_t \in [0,1], \\ \gamma_t \mathbf{x}_t = 0,\, t \in [0,1]}} \int_0^1 \Bigl((\dot{\mathbf{x}}_t - \gamma_t + 1) \log \frac{\dot{\mathbf{x}}_t - \gamma_t + 1}{c(1 - t - \mathbf{x}_t)} - (\dot{\mathbf{x}}_t - \gamma_t + 1) + c(1 - t - \mathbf{x}_t)\Bigr)\, dt \\
&= \int_0^1 \inf_{\substack{\gamma \in [0,1]: \\ \gamma \mathbf{x}_t = 0}} \Bigl((\dot{\mathbf{x}}_t - \gamma + 1) \log \frac{\dot{\mathbf{x}}_t - \gamma + 1}{c(1 - t - \mathbf{x}_t)} - (\dot{\mathbf{x}}_t - \gamma + 1) + c(1 - t - \mathbf{x}_t)\Bigr)\, dt \\
&= \int_0^1 \Bigl((\dot{\mathbf{x}}_t + 1) \log \frac{\dot{\mathbf{x}}_t + 1}{c(1 - t - \mathbf{x}_t)} - (\dot{\mathbf{x}}_t + 1) + c(1 - t - \mathbf{x}_t)\Bigr) \mathbf{1}(\mathbf{x}_t > 0)\, dt \\
&\quad + \int_0^1 \inf_{\gamma \in [0,1]} \Bigl((1 - \gamma) \log \frac{1 - \gamma}{c(1 - t)} - (1 - \gamma) + c(1 - t)\Bigr) \mathbf{1}(\mathbf{x}_t = 0)\, dt \\
&= \int_0^1 \Bigl((\dot{\mathbf{x}}_t + 1) \log \frac{\dot{\mathbf{x}}_t + 1}{c(1 - t - \mathbf{x}_t)} - (\dot{\mathbf{x}}_t + 1) + c(1 - t - \mathbf{x}_t)\Bigr) \mathbf{1}(\mathbf{x}_t > 0)\, dt \\
&\qquad + \int_0^{(1-1/c)^+} \Bigl(-\log(c(1-t)) - 1 + c(1-t)\Bigr) \mathbf{1}(\mathbf{x}_t = 0)\, dt.
\end{aligned}$$

□

Remark 2.1. We have thus proved that the $(\overline{W}^n, \overline{S}^n)$ large deviation converge at rate n as $n \to \infty$ on $\mathbb{D}([0,1], \mathbb{R}^2)$ to the idempotent processes (W, S) given by the equations

$$W = \mathcal{R}(S),\ S_t = N_{\int_0^t (1-s-W_s)\,ds} - t.$$

3 Asymptotics of the giant component

In this section we prove Theorem 1.1. We start with auxiliary results.

Lemma 3.1. *Let $c > 1$. If $u \in [1-1/c, 1]$, then the function $K_c(x) + L_c(u+x)$ is strictly increasing for $x \in [0, 1-u)$. If $u \in (0, 1-1/c)$, the function $K_c(x) + L_c(u+x)$ is strictly decreasing for $x \in [1-1/c, x^*]$ and strictly increasing for $x \in [x^*, 1-u)$, where x^* is the solution of the equation*

$$\frac{x}{1-e^{-cx}} = 1-u. \tag{3.1}$$

Also $K_c(x^*) + L_c(u+x^*) = L_c(u)$.

We defer the proof to the appendix.

Let $\tilde{A}$ denote the set of absolutely continuous $\mathbf{x}$ with $\dot{\mathbf{x}}_r \geq -1$ a.e. and $1 - r - \mathcal{R}(\mathbf{x})_r \geq 0$ on $[0,1]$, and let, given $0 \leq s \leq t \leq 1$ and $\mathbf{x} \in \tilde{A}$,

$$I^S_{s,t}(\mathbf{x}) = \int_s^t \Bigl((\dot{\mathbf{x}}_r + 1)\log\frac{\dot{\mathbf{x}}_r + 1}{c(1 - r - \mathcal{R}(\mathbf{x})_r)} - (\dot{\mathbf{x}}_r + 1) + c(1 - r - \mathcal{R}(\mathbf{x})_r)\Bigr)\,dr,$$

$$\tilde{I}_{s,t} = \inf_{\substack{\mathbf{x}\in\tilde{A}:\,\mathcal{R}(\mathbf{x})_t = \mathcal{R}(\mathbf{x})_s = 0,\\ \mathbf{x}_r \geq \mathbf{x}_s,\, r\in[s,t]}} I^S_{s,t}(\mathbf{x}). \tag{3.2}$$

Lemma 3.2. *The infimum in* (3.2) *is attained at*

$$\hat{\mathbf{x}}^{s,t}(r) = \mathbf{x}_s + s - r + \frac{t-s}{1-e^{-c(t-s)}}\bigl(1 - e^{-c(r-s)}\bigr),\ r \in [s,t] \tag{3.3}$$

and is given by

$$\tilde{I}_{s,t} = K_c(t-s) + L_c(t) - L_c(s). \tag{3.4}$$

Proof. Noting that in (3.2) $\mathcal{R}(\mathbf{x})_r = \mathbf{x}_r - \mathbf{x}_s$, the claim follows by the Lagrange multipliers method applied to the convex function $\int_s^t \bigl(\mathbf{z}_r \log(\mathbf{z}_r/\mathbf{y}_r) - \mathbf{z}_r + \mathbf{y}_r\bigr)\,dr$ with constraints $\mathbf{y}_s = 0$, $\mathbf{y}_t = t-s$, $\mathbf{y}_r \geq 0$, $\mathbf{z}_r \geq 0$, and $\mathbf{y}_r = c(1 - \int_0^s \mathbf{z}_p\,dp)$, $r \in [s,t]$. ☐

Proof of Theorem 1.1. Let us begin by establishing the upper bounds. For fixed $\epsilon > 0$, $m \in \mathbb{N}$, and $u_k \in (0,1]$, $k = 1, 2, \ldots, m$, such that $\sum_{k=1}^m u_k \leq 1$, let us denote by $\hat{A}_\epsilon(u_1, \ldots, u_m)$ the set of functions $\mathbf{x} \in \mathbb{D}_C([0,1])$ starting at zero for which there exist points $0 = t_0 \leq t_1 \leq t_2 \leq \ldots \leq t_{2m} \leq 1 = t_{2m+1}$ with $|t_{2i} - t_{2i-1} - u_i| \leq \epsilon$, $i = 1, 2\ldots, m$ such that $\mathcal{R}(\mathbf{x})_{t_{2i-1}} = \mathcal{R}(\mathbf{x})_{t_{2i}} = 0$ and $\mathbf{x}_t \geq \mathbf{x}_{t_{2i-1}}$ for $t \in [t_{2i-1}, t_{2i}]$. By the construction of W^n, if there exists a connected component of size l of the random graph, then there exist natural numbers σ^n and τ^n ranging in $\{0, 1, 2, \ldots, n\}$ such that $\tau^n - \sigma^n = l$, $W^n_{\sigma^n} = 0$, $W^n_{\tau^n} = 0$, and $W^n_i \geq 1$ for $i = \sigma^n + 1, \ldots, \tau^n - 1$. Since in view of (2.3), (2.6), and (2.7) $W^n_i = (S^n_i + \epsilon^n_i) - (S^n_{\sigma^n} + \epsilon^n_{\sigma^n})$ for $i \in [\sigma^n, \tau^n]$, we have that $S^n_i + \epsilon^n_i \geq S^n_{\sigma^n} + \epsilon^n_{\sigma^n}$ on $[\sigma^n, \tau^n]$. Therefore, recalling (2.9), we have that $A^n_\epsilon(u_1, \ldots, u_m) \subset \{\overline{S}^n + \overline{\epsilon}^n \subset \hat{A}_\epsilon(u_1, \ldots, u_m)\}$. The set $\hat{A}_\epsilon(u_1, \ldots, u_m)$ being closed in $\mathbb{D}_C([0,1])$ implies by Theorem 2.1 and (2.14) that

$$\limsup_{n\to\infty}\frac{1}{n}\log P(A^n_\epsilon(u_1,\ldots,u_m))\le\limsup_{n\to\infty}\frac{1}{n}\log P(\overline{S}^n+\overline{\epsilon}^n\subset\hat{A}_\epsilon(u_1,\ldots,u_m))$$
$$\le-\inf_{\mathbf{x}\in\hat{A}_\epsilon(u_1,\ldots,u_m)} I^S(\mathbf{x}).$$

Let $\hat{A}(u_1,\ldots,u_m)$ denote the set $\hat{A}_\epsilon(u_1,\ldots,u_m)$ corresponding to $\epsilon=0$. Thus, $\hat{A}(u_1,\ldots,u_m)$ consists of functions $\mathbf{x}\in\mathbb{D}_C([0,1])$ starting at zero for which there exist points $0=t_0\le t_1\le t_2\le\ldots\le t_{2m}\le 1=t_{2m+1}$ with $t_{2i}-t_{2i-1}=u_i$, $i=1,2\ldots,m$ such that $\mathcal{R}(\mathbf{x})_{t_{2i-1}}=\mathcal{R}(\mathbf{x})_{t_{2i}}=0$ and $\mathbf{x}_t\ge\mathbf{x}_{t_{2i-1}}$ for $t\in[t_{2i-1},t_{2i}]$. Since $\cap_{\epsilon>0}\hat{A}_\epsilon(u_1,\ldots,u_m)=\hat{A}(u_1,\ldots,u_m)$, we have that

$$\limsup_{\epsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\log P(A^n_\epsilon(u_1,\ldots,u_m))\le-\inf_{\mathbf{x}\in\hat{A}(u_1,\ldots,u_m)} I^S(\mathbf{x}). \tag{3.5}$$

We now evaluate the infimum on the right of (3.5). Letting

$$\hat{I}_{s,t}=\inf_{\mathbf{x}\in\tilde{A}:\,\mathcal{R}(\mathbf{x})_t=\mathcal{R}(\mathbf{x})_s=0} I^S_{s,t}(\mathbf{x}), \tag{3.6}$$

we have in view of (3.2) that it can be written as

$$\inf_{\mathbf{x}\in\hat{A}(u_1,\ldots,u_m)} I^S(\mathbf{x})=\inf_{(t_i)}\Big(\sum_{i=1}^m\tilde{I}_{t_{2i-1},t_{2i}}+\sum_{i=0}^m\hat{I}_{t_{2i},t_{2i+1}}\Big). \tag{3.7}$$

As it follows by the proof of Theorem 2.1,

$$\hat{I}_{s,t}=\inf_{\mathbf{x}\in\hat{A}:\,\mathbf{x}_t=\mathbf{x}_s=0} I^W_{s,t}(\mathbf{x}), \tag{3.8}$$

where $\hat{A}$ is the set of trajectories $\mathbf{x}$ that are absolutely continuous, nonnegative and $\dot{\mathbf{x}}_t\ge -1$ a.e., and

$$I^W_{s,t}(\mathbf{x})=\int_s^t\Big((\dot{\mathbf{x}}_r+1)\log\frac{\dot{\mathbf{x}}_r+1}{c(1-r-\mathbf{x}_r)}-(\dot{\mathbf{x}}_r+1)+c(1-r-\mathbf{x}_r)\Big)\mathbf{1}(\mathbf{x}_r>0)\,dr$$
$$+\int_{s\wedge(1-1/c)^+}^{t\wedge(1-1/c)^+}\big(-\log(c(1-r))-1+c(1-r)\big)\mathbf{1}(\mathbf{x}_r=0)\,dr. \tag{3.9}$$

Thus, $\hat{I}_{s,t}=\tilde{I}_{s,t}$ if $t\le(1-1/c)^+$ and $\hat{I}_{s,t}=0$ if $s\ge(1-1/c)^+$, and by (3.7)

$$\inf_{\mathbf{x}\in\hat{A}(u_1,\ldots,u_m)} I^S(\mathbf{x})=\inf_{(t_i)}\Big(\sum_{i=1}^m\tilde{I}_{t_{2i-1},t_{2i}}+\sum_{i=0}^{m^*-1}\tilde{I}_{t_{2i},t_{2i+1}}+\hat{I}_{t_{2m^*},t_{2m^*+1}}\Big), \tag{3.10}$$

where m^* is minimal i such that $t_{2i+1}\ge(1-1/c)^+$. Let us consider $\hat{I}_{t_{2m^*},t_{2m^*+1}}$. If $t_{2m^*}\ge(1-1/c)^+$, then $\hat{I}_{t_{2m^*},t_{2m^*+1}}=0$. Let us assume that $t_{2m^*}<(1-1/c)^+$. Let $\mathbf{x}$ be a nonnegative trajectory with $\mathbf{x}_{t_{2m^*}}=\mathbf{x}_{t_{2m^*+1}}=0$ that delivers the value of $\hat{I}_{t_{2m^*},t_{2m^*+1}}$. Then $\mathbf{x}$ can be assumed to be positive on $(t_{2m^*},(1-1/c)^+)$ since otherwise we can consider the trajectory that is optimal on $[t_{2m^*},(1-1/c)^+]$ for values at the end points equal to 0 and extend it by 0 to $[(1-1/c)^+,t_{2m^*+1}]$, the cost associated with this trajectory being not greater than that for $\mathbf{x}$. Now if $\mathbf{x}$ equals 0 at some $x^*\in[(1-1/c)^+,t_{2m^*+1}]$, then it is optimal to assume that it equals 0 on $[x^*,t_{2m^*+1}]$ since the cost of such an extension is zero. Therefore, $\hat{I}_{t_{2m^*},t_{2m^*+1}} = \tilde{I}_{t_{2m^*},x^*}$ for some $x^* \in [(1-1/c)^+, t_{2m^*+1}]$. As it follows by (3.4) the derivative of $\tilde{I}_{s,t}$ with respect to s for $t-s$ fixed equals $-c(t-s)-\log(1-t)+\log(1-s)$, so it is positive for $s \geq (1-1/c)^+$. Therefore, the first sum on the right-hand side of (3.10) does not increase if we move all the intervals $[t_{2i}, t_{2i+1}]$ for $i = m^*+1, \ldots, m$ to the left so they are arranged with no gaps to the right of x^* beginning at x^*. By Lemma 3.2 for this arrangement

$$\sum_{i=1}^{m} \tilde{I}_{t_{2i-1},t_{2i}} + \sum_{i=0}^{m^*-1} \tilde{I}_{t_{2i},t_{2i+1}} + \hat{I}_{t_{2m^*},t_{2m^*+1}} = \sum_{i=1}^{m} K_c(u_i) + \sum_{i=0}^{m^*} K_c(v_i) + L_c\Big(\sum_{i=1}^{m} u_i + \sum_{i=0}^{m^*} v_i\Big), \quad (3.11)$$

where $v_i = t_{2i+1} - t_{2i}$, $i = 0, 1, \ldots, m^*-1$ and $v_{m^*} = x^* - t_{2m^*}$. The function K_c being subadditive implies that $\sum_{i=0}^{m^*} K_c(v_i) \geq K_c\big(\sum_{i=0}^{m^*} v_i\big)$, so by (3.10) and (3.11)

$$\inf_{\mathbf{x} \in \hat{A}(u_1,\ldots,u_m)} I^S(\mathbf{x}) = \sum_{i=1}^{m} K_c(u_i) + \inf_{v \geq (1-1/c-\sum_{i=1}^m u_i)^+} \Big(K_c(v) + L_c\big(\sum_{i=1}^{m} u_i + v\big)\Big). \quad (3.12)$$

If $\sum_{i=1}^m u_i \geq 1 - 1/c$, then by Lemma 3.1 the above infimum is attained at $v = 0$ and by (3.5)

$$\limsup_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log P(A^n_\epsilon(u_1, \ldots, u_m)) \leq -\Big(\sum_{k=1}^{m} K_c(u_k) + L_c\big(\sum_{k=1}^{m} u_k\big)\Big). \quad (3.13)$$

The inclusion $B^n_\epsilon(u_1, \ldots, u_m) \subset A^n_\epsilon(u_1, \ldots, u_m)$ also yields

$$\limsup_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log P(B^n_\epsilon(u_1, \ldots, u_m)) \leq -\Big(\sum_{k=1}^{m} K_c(u_k) + L_c\big(\sum_{k=1}^{m} u_k\big)\Big). \quad (3.14)$$

If $\sum_{i=1}^m u_i < 1 - 1/c$ (so $c > 1$), according to (3.12) we have to minimise $K_c(x) + L_c(\sum_{k=1}^m u_k + x)$ over $x \geq 1 - 1/c - \sum_{i=1}^m u_i$, the minimum equals $L_c(\sum_{k=1}^m u_k)$ by Lemma 3.1. Thus, (3.13) holds for this case too.

We now turn to establishing a counterpart of (3.14) when $\sum_{i=1}^m u_i < 1 - 1/c$. Let $\hat{B}_\epsilon(u_1, \ldots, u_m)$ be the subset of $\hat{A}_\epsilon(u_1, \ldots, u_m)$ of those $\mathbf{x}$ for which $\mathcal{R}(\mathbf{x})_t = 0$ for some t on every interval of length ϵ that does not intersect with the intervals $[t_{2i-1}, t_{2i}]$. Let $\hat{B}(u_1, \ldots, u_m)$ be the set of those functions $\mathbf{x} \in \hat{A}(u_1, \ldots, u_m)$ for which $\mathcal{R}(\mathbf{x})_t = 0$ for t outside of the associated intervals $[t_{2i-1}, t_{2i}]$. By (2.9), $B^n_\epsilon(u_1, \ldots, u_m) \subset \{\overline{S}^n + \overline{\epsilon}^n \subset \hat{B}_\epsilon(u_1, \ldots, u_m)\}$, so since the sets $\hat{B}_\epsilon(u_1, \ldots, u_m)$ are closed and $\cap_{\epsilon>0} \hat{B}_\epsilon(u_1, \ldots, u_m) = \hat{B}(u_1, \ldots, u_m)$, we conclude in analogy with (3.5) that

$$\limsup_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log P(B^n_\epsilon(u_1, \ldots, u_m)) \leq -\inf_{\mathbf{x} \in \hat{B}(u_1,\ldots,u_m)} I^S(\mathbf{x}). \quad (3.15)$$

Let

$$J_{s,t} = \inf_{\mathbf{x} \in \tilde{A}:\, \mathcal{R}(\mathbf{x})_r = 0,\, r \in [s,t]} I^S_{s,t}(\mathbf{x}).$$

Then

$$J_{s,t} = \int_{s\wedge(1-1/c)}^{t\wedge(1-1/c)} \big(-\log(c(1-r)) - 1 + c(1-r)\big)\, dr$$
$$= -(t-s)\log c - \frac{c(t-s)^2}{2} + c(t-s)(1-s) + (1-t)\log(1-t) - (1-s)\log(1-s). \quad (3.16)$$

In analogy with (3.7) and (3.10) we can write

$$\inf_{\mathbf{x}\in\hat{B}(u_1,\ldots,u_m)} I^S(\mathbf{x}) = \inf_{(t_i)}\Big(\sum_{i=1}^{m}\tilde{I}_{t_{2i-1},t_{2i}} + \sum_{i=0}^{m^*} J_{t_{2i},t_{2i+1}}\Big), \tag{3.17}$$

where, as above, m^* is minimal i such that $t_{2i+1}\geq 1-1/c$. In effect, since $J_{s,t}=0$ for s to the right of $1-1/c$, we can assume that $t_{2m^*+1}=1-1/c$. As in the case of the event $\hat{A}$, the total cost can only decrease if all the intervals $[t_{2i-1},t_{2i}]$, $i=m^*+1,\ldots,m$ are arranged next to each other starting at $1-1/c$. Next, a calculation using (3.4) and (3.16) shows that for such an arrangement

$$\sum_{i=1}^{m}\tilde{I}_{t_{2i-1},t_{2i}} + \sum_{i=0}^{m^*} J_{t_{2i},t_{2i+1}} = \sum_{i=1}^{m} K_c(u_i) + L_c\Big(\sum_{i=1}^{m} u_i + \sum_{i=0}^{m^*} v_i\Big) - \sum_{i=0}^{m^*} v_i \log c, \tag{3.18}$$

where $v_i = t_{2i+1}-t_{2i}$. The right-hand side of (3.18) is a monotonically increasing function of $\sum_{i=0}^{m^*} v_i$, so the minimum is attained at $1-1/c-\sum_{i=1}^{m} u_i$, and we obtain by (3.15) and (3.17) that

$$\limsup_{\epsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\log P(B_\epsilon^n(u_1,\ldots,u_m)) \leq -\Big(\sum_{i=1}^{m} K_c(u_i) + M_c\big(\sum_{i=1}^{m} u_i\big)\Big).$$

The proofs of the upper bounds are over.

Let us consider the lower bounds and begin again with the case $\sum_{i=1}^{m} u_i \geq 1-1/c$. Let $s_0=0$ and $s_k=\sum_{i=1}^{k} u_i$, $k=1,\ldots,m$. We define motivated by Lemma 3.2 for $\epsilon\in(0,\min_{k=1,\ldots,m} u_k)$

$$\breve{\mathbf{x}}_0^\epsilon = 0, \quad \breve{\mathbf{x}}_t^\epsilon = \breve{\mathbf{x}}_{s_{k-1}}^\epsilon + s_{k-1} - t + \frac{u_k-\epsilon/3}{1-e^{-c(u_k-\epsilon/3)}}\big(1-e^{-c(t-s_{k-1})}\big), \quad t\in[s_{k-1},s_k],\ \ k=1,\ldots,m,$$
$$\breve{\mathbf{x}}_t^\epsilon = \breve{\mathbf{x}}_{s_m}^\epsilon + (c-1)(t-s_m) - \frac{c}{2}(t^2-s_m^2), \quad t\in[s_m,1]. \tag{3.19}$$

The function $\mathcal{R}(\breve{\mathbf{x}})$ is positive on the intervals $(s_{k-1},s_k-\epsilon/3)$, $k=1,2,\ldots,m$ and equals zero elsewhere. Given $\delta>0$, let $\breve{B}_{\epsilon,\delta}(u_1,\ldots,u_m)$ denote the δ-neighbourhood of $\breve{\mathbf{x}}^\epsilon$ for the uniform metric on $\mathbb{D}_C([0,1])$. It follows from the definition of the reflection mapping and $\breve{\mathbf{x}}^\epsilon$ that if $\mathbf{x}\in\breve{B}_{\epsilon,\delta}(u_1,\ldots,u_m)$, then $\mathcal{R}(\mathbf{x})$ is positive on intervals $(\tilde{s}_{k-1},\tilde{s}_k)$, $k=1,2,\ldots,m$, where $|\tilde{s}_k-s_k|<\epsilon/3$, and the rest of the intervals where $\mathcal{R}(\mathbf{x})$ is positive are of length less than ϵ. Since $W^n=\mathcal{R}(S^n+\epsilon^n)$, we conclude that $\{\overline{S}^n+\overline{\epsilon}^n\subset\breve{B}_{\epsilon,\delta}(u_1,\ldots,u_m)\}\subset B_\epsilon^n(u_1,\ldots,u_m)$. The set $\breve{B}_{\epsilon,\delta}(u_1,\ldots,u_m)$ being $\mathbb{C}([0,1])$-open in $\mathbb{D}_C([0,1])$, we have, recalling (2.14),

$$\liminf_{n\to\infty}\frac{1}{n}\log P(B_\epsilon^n(u_1,\ldots,u_m)) \geq \liminf_{n\to\infty}\frac{1}{n}\log P(\overline{S}^n+\overline{\epsilon}^n\subset\breve{B}_{\epsilon,\delta}(u_1,\ldots,u_m))$$
$$\geq -\inf_{\mathbf{x}\in\breve{B}_{\epsilon,\delta}(u_1,\ldots,u_m)} I^S(\mathbf{x}) \geq -I^S(\breve{\mathbf{x}}^\epsilon). \tag{3.20}$$

By (3.19) $I^S(\breve{\mathbf{x}}^\epsilon)\to\sum_{k=1}^{m} K_c(u_k)+L_c\big(\sum_{k=1}^{m} u_k\big)$ as $\epsilon\to 0$, so

$$\liminf_{\epsilon\to 0}\liminf_{n\to\infty}\frac{1}{n}\log P(B_\epsilon^n(u_1,\ldots,u_m)) \geq -\Big(\sum_{k=1}^{m} K_c(u_k) + L_c\big(\sum_{k=1}^{m} u_k\big)\Big). \tag{3.21}$$

Since $B_\epsilon^n(u_1,\ldots,u_m)\subset A_\epsilon^n(u_1,\ldots,u_m)$, we also have that

$$\liminf_{\epsilon\to 0}\liminf_{n\to\infty}\frac{1}{n}\log P(A_\epsilon^n(u_1,\ldots,u_m)) \geq -\Big(\sum_{k=1}^{m} K_c(u_k) + L_c\big(\sum_{k=1}^{m} u_k\big)\Big).$$

Let us now assume that $\sum_{k=1}^m u_k < 1-1/c$ and let u^* be such that $u^*/(1-\exp(-cu^*)) = 1-\sum_{k=1}^m u_k$. Then $\sum_{k=1}^m u_k + u^* \geq 1-1/c$, so (3.21) and Lemma 3.1 yield the bound

$$\liminf_{\epsilon\to 0}\liminf_{n\to\infty}\frac{1}{n}\log P(B_\epsilon^n(u_1,\dots,u_m,u^*)) \geq -\Big(\sum_{k=1}^m K_c(u_k) + L_c\big(\sum_{k=1}^m u_k\big)\Big),$$

hence, by the inclusion $B_\epsilon^n(u_1,\dots,u_m,u^*) \subset A_\epsilon^n(u_1,\dots,u_m)$

$$\liminf_{\epsilon\to 0}\liminf_{n\to\infty}\frac{1}{n}\log P(A_\epsilon^n(u_1,\dots,u_m)) \geq -\Big(\sum_{k=1}^m K_c(u_k) + L_c\big(\sum_{k=1}^m u_k\big)\Big).$$

We finally establish the lower bound for $B_\epsilon^n(u_1,\dots,u_m)$. In analogy with the argument used for the lower bound for the case $\sum_{k=1}^m u_k \geq 1-1/c$, we define for $\epsilon \in (0, \min_{k=1,\dots,m} u_k)$

$$\tilde{\mathbf{x}}_0^\epsilon = 0, \quad \tilde{\mathbf{x}}_t^\epsilon = \tilde{\mathbf{x}}_{s_{k-1}}^\epsilon + s_{k-1} - t + \frac{u_k-\epsilon/3}{1-e^{-c(u_k-\epsilon/3)}}\big(1-e^{-c(t-s_{k-1})}\big), \quad t\in[s_{k-1},s_k], \quad k=1,\dots,m,$$
$$\tilde{\mathbf{x}}_t^\epsilon = \tilde{\mathbf{x}}_{s_m}^\epsilon - \epsilon(t-s_m), \quad t\in[s_m, 1-1/c],$$
$$\tilde{\mathbf{x}}_t^\epsilon = \tilde{\mathbf{x}}_{1-1/c}^\epsilon + (c-1)\big(t-(1-1/c)\big) - \frac{c}{2}\big(t^2-(1-1/c)^2\big), \quad t\in[1-1/c,1]. \quad (3.22)$$

Denoting by $\tilde{B}_{\epsilon,\delta}(u_1,\dots,u_m)$ the δ-neighbourhood of $\tilde{\mathbf{x}}^\epsilon$ for the uniform metric, we have for small enough $\delta>0$ by an argument similar to the one used above that $\{\overline{S}^n + \overline{\epsilon}^n \subset \tilde{B}_{\epsilon,\delta}(u_1,\dots,u_m)\} \subset B_\epsilon^n(u_1,\dots,u_m)$, so analogously to (3.20)

$$\liminf_{n\to\infty}\frac{1}{n}\log P(B_\epsilon^n(u_1,\dots,u_m)) \geq -I^S(\tilde{\mathbf{x}}^\epsilon).$$

By the definition of I^S and (3.22) $I^S(\tilde{\mathbf{x}}^\epsilon) \to \sum_{k=1}^m K_c(u_k) + M_c\big(\sum_{k=1}^m u_k\big)$ as $\epsilon\to 0$, which results in the required inequality

$$\liminf_{\epsilon\to 0}\liminf_{n\to\infty}\frac{1}{n}\log P(B_\epsilon^n(u_1,\dots,u_m)) \geq -\Big(\sum_{k=1}^m K_c(u_k) + M_c\big(\sum_{k=1}^m u_k\big)\Big).$$

□

4 Other asymptotics

We present the LDP for the process of the sizes of the connected components of the random graph. Let $S^{\mathbb{N}}$ denote the set of decreasing sequences $\mathbf{u} = (u_1, u_2, \dots)$ of real numbers in $[0,1]$ with $\sum_{i=1}^\infty u_i \leq 1$; we equip $S^{\mathbb{N}}$ with the topology induced by the product topology on $\mathbb{R}^{\mathbb{N}}$. Let $(U_1^n, U_2^n, \dots,)$ be the sequence of the sizes of the connected components of the nth random graph arranged in descending order complemented with zeros, and $\overline{U}^n = (U_1^n/n, U_2^n/n, \dots)$ be the random element of $S^{\mathbb{N}}$ of the normalized sizes of the connected components.

Theorem 4.1. *The sequence $\overline{U}^n$, $n=1,2,\dots$, satisfies the LDP in $S^{\mathbb{N}}$ with action functional I^U defined for $\mathbf{u} = (u_1,u_2,\dots) \in S^{\mathbb{N}}$ by*

$$I^U(\mathbf{u}) = \begin{cases} \sum_{i=1}^\infty K_c(u_i) + L_c\big(\sum_{i=1}^\infty u_i\big) & \text{if } \sum_{i=1}^\infty u_i \geq 1-1/c, \\ \sum_{i=1}^\infty K_c(u_i) + M_c\big(\sum_{i=1}^\infty u_i\big) & \text{if } \sum_{i=1}^\infty u_i < 1-1/c. \end{cases}$$

Proof. Space $S^{\mathbb{N}}$ being compact, the sequence $\overline{U}^n$ is exponentially tight, so it suffices to check that

$$\lim_{\epsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\log P\big(d(\overline{U}^n,\mathbf{u})\le\epsilon\big)=\lim_{\epsilon\to 0}\liminf_{n\to\infty}\frac{1}{n}\log P\big(d(\overline{U}^n,\mathbf{u})\le\epsilon\big)=-I(\mathbf{u}),\tag{4.1}$$

where d is a metric on $\mathbb{R}^{\mathbb{N}}$. Let $\pi_m\mathbf{u}=(u_1,\dots,u_m)$, $\mathbf{u}=(u_1,u_2,\dots)$, $m\in\mathbb{N}$. Then by continuity of π_m, given $\delta>0$ and $m\in\mathbb{N}$, there exists $\epsilon>0$ such that if $d(\overline{U}^n,\mathbf{u})\le\epsilon$, then $d_m(\pi_m\overline{U}^n,\mathbf{u})\le\delta$, where d_m is the uniform metric on $\mathbb{R}^m$, so

$$\begin{aligned}\limsup_{n\to\infty}\frac{1}{n}\log P\big(d(\overline{U}^n,\mathbf{u})\le\epsilon\big)&\le\limsup_{n\to\infty}\frac{1}{n}\log P\big(\max_{i=1,\dots,m}|U_i^n/n-u_i|\le\delta\big)\\&\le\limsup_{n\to\infty}\frac{1}{n}\log P\big(A_\delta^n(u_1,\dots,u_m)\big).\end{aligned}\tag{4.2}$$

Theorem 1.1 then implies that

$$\limsup_{\epsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\log P\big(d(\overline{U}^n,\mathbf{u})\le\epsilon\big)\le-\Big(\sum_{i=1}^\infty K_c(u_i)+L_c\big(\sum_{i=1}^\infty u_i\big)\Big),\tag{4.3}$$

proving the required upper bound for $\sum_{i=1}^\infty u_i\ge 1-1/c$. If $\sum_{i=1}^\infty u_i<1-1/c$, then the second inequality in (4.2) is too crude. Given arbitrary $\gamma>0$, let m be such that $\sum_{i=m+1}^\infty u_i<\gamma/2$. Then we have that for suitable $\epsilon>0$,

$$\{d(\overline{U}^n,\mathbf{u})\le\epsilon\}\subset B_{\gamma,\delta}^n(u_1,\dots,u_m),\tag{4.4}$$

where $B_{\gamma,\delta}^n(u_1,\dots,u_m)$ is the event that there exist m connected components of sizes in $(n(u_i-\delta),n(u_i+\delta))$ for $i=1,2,\dots,m$ and there are no other connected components of size greater than $n\gamma$. Then for the set $\hat{B}_\gamma(u_1,\dots,u_m)$ defined as above we have similarly to (3.15) that

$$\limsup_{\delta\to 0}\limsup_{n\to\infty}\frac{1}{n}\log P(B_{\gamma,\delta}^n(u_1,\dots,u_m))\le-\inf_{\mathbf{x}\in\hat{B}_\gamma(u_1,\dots,u_m)}I^S(\mathbf{x}).\tag{4.5}$$

The argument used for deriving (3.10) and (3.11) shows that we can write the latter inf as

$$\inf_{\mathbf{x}\in\hat{B}_\gamma(u_1,\dots,u_m)}I^S(\mathbf{x})=\sum_{i=1}^m K_c(u_i)+\sum_{i=0}^{m^*}K_c(v_i)+L_c\big(\sum_{i=1}^m u_i+\sum_{i=0}^{m^*}v_i\big),\tag{4.6}$$

where the v_i i such that $v_i\le\gamma$ and $\sum_{i=1}^m u_i+\sum_{i=0}^{m^*}v_i\ge 1-1/c$. Since $|dK_c(u)/du+\log c|\to 0$ as $u\to 0$, we can write that $\sum_{i=0}^{m^*}K_c(v_i)=-\log c\sum_{i=0}^{m^*}v_i+\eta$, where η is small if γ is. Therefore,

$$\sum_{i=0}^{m^*}K_c(v_i)+L_c\big(\sum_{i=1}^m u_i+\sum_{i=0}^{m^*}v_i\big)=\eta-\log c\sum_{i=0}^{m^*}v_i+L_c\big(\sum_{i=1}^m u_i+\sum_{i=0}^{m^*}v_i\big).$$

As in (3.18), the minimum of the right-hand side over v_i equals $\eta+M_c\big(\sum_{i=1}^m u_i\big)$. Since $\eta\to 0$ and $\sum_{i=0}^m u_i\to\sum_{i=1}^\infty u_i$ as $\gamma\to 0$, we conclude by (4.4), (4.5), and (4.6) that

$$\begin{aligned}\limsup_{\epsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\log P\big(d(\overline{U}^n,\mathbf{u})\le\epsilon\big)&\le\limsup_{\gamma\to 0}\limsup_{\delta\to 0}\limsup_{n\to\infty}\frac{1}{n}\log P(B_{\gamma,\delta}^n(u_1,\dots,u_m))\\&\le-\Big(\sum_{i=1}^\infty K_c(u_i)+M_c\big(\sum_{i=1}^\infty u_i\big)\Big).\end{aligned}$$

The proof of the upper bound is over.

The argument for the lower bound is similar. Given $\epsilon > 0$ and $\mathbf{u} \in S^{\mathbb{N}}$, let $m \in \mathbb{N}$ be such that $\{d_m(\pi_m\overline{U}^n, \pi_m\mathbf{u}) \le 2\epsilon\} \subset \{d(\overline{U}^n, \mathbf{u}) \le \epsilon\}$. The required then follows by the inclusion $B^n_{2\epsilon}(u_1, \ldots, u_m) \subset \{d_m(\pi_m\overline{U}^n, \pi_m\mathbf{u}) \le 2\epsilon\}$ and Theorem 1.1. □

We next recover the result of O'Connell [8] on the LDP for the largest connected component. Let $x_0 = 1$ and x_k, $k \in \mathbb{N}$, for $c > 1$ be the solutions of the equations

$$\frac{x_k}{1 - e^{-cx_k}} = 1 - kx_k. \tag{4.7}$$

We note that $x_k \downarrow 0$ as $k \to \infty$. Let ξ^n denote the size of the largest connected component of the random graph on n vertices.

Corollary 4.1. *The sequence ξ^n/n satisfies the LDP in $[0, 1]$ with action functional defined by*

$$I_c(u) = \begin{cases} kK_c(u) + L_c(ku) & \text{if } u \in [x_k, x_{k-1}],\ k \in \mathbb{N} \\ M_c(0) & \text{if } u = 0 \end{cases}$$

when $c > 1$ and $I_c(u) = K_c(u) + L_c(u)$ when $c \le 1$.

Proof. By Theorem 4.1 and the contraction principle

$$I_c(u) = \inf_{\mathbf{u} \in O(u)} I^U(\mathbf{u}), \tag{4.8}$$

where $O(u) = \{\mathbf{u} \in S^{\mathbb{N}} : u_1 = u\}$. Hence, $I_c(0) = M_c(0)$. Let us assume now that $u > 0$. The infimum in (4.8) can be actually taken over $\mathbf{u}$ with $\sum_{i=1}^\infty u_i > 1 - 1/c$. Indeed, if $\sum_{i=1}^\infty u_i \le 1 - 1/c$ (so, $c > 1$), then $L_c\big(\sum_{i=1}^\infty u_i\big) < M_c\big(\sum_{i=1}^\infty u_i\big)$, hence, by Lemma 3.1

$$\sum_{i=1}^\infty K_c(u_i) + M_c\big(\sum_{i=1}^\infty u_i\big) > \sum_{i=1}^\infty K_c(u_i) + L_c\big(\sum_{i=1}^\infty u_i\big) = \sum_{i=1}^\infty K_c(u_i) + K_c(u^*) + L_c\big(\sum_{i=1}^\infty u_i + u^*\big),$$

where $u^*/(1 - \exp(-cu^*)) = 1 - \sum_{i=1}^\infty u_i$. The claim follows since $\sum_{i=1}^\infty u_i + u^* > 1 - 1/c$. We can thus assume that in (4.8) $I^U(\mathbf{u}) = \sum_{i=1}^\infty K_c(u_i) + L_c\big(\sum_{i=1}^\infty u_i\big)$ and $\sum_{i=1}^\infty u_i > 1 - 1/c$. Let $m^* = \min\{m : \sum_{i=1}^m u_i \ge 1 - 1/c\}$ and $l = \sum_{i=1}^{m^*} u_i$. By subadditivity of K_c, $\sum_{i=m^*}^\infty K_c(u_i) \ge K_c\big(\sum_{i=m^*}^\infty u_i\big)$; also since $l > 1 - 1/c$, $K_c\big(\sum_{i=m^*+1}^\infty u_i\big) + L_c\big(\sum_{i=1}^\infty u_i\big) \ge L_c(l)$. We thus conclude that it is optimal to assume that $u_i = 0$, $i = m^* + 1, m^* + 2, \ldots$, which yields the required result for $c \le 1$. We further assume that $c > 1$. The function K_c being concave and the fact that $u_i \le u$ imply that $\sum_{i=1}^{m^*} K_c(u_i) \ge \lfloor (1 - 1/c)/u \rfloor K_c(u) + K_c(l - \lfloor (1 - 1/c)/u \rfloor u)$, where $\lfloor a \rfloor$ denotes the integer part of a. We thus have to minimise $\lfloor (1-1/c)/u \rfloor K_c(u) + K_c(l - \lfloor (1-1/c)/u \rfloor u) + L_c(l)$ over $l \in [1 - 1/c, \lfloor (1-1/c)/u \rfloor u + u]$. By Lemma 3.1 the minimum is attained at $\lfloor (1-1/c)/u \rfloor u + l^* \wedge u$, where l^* satisfies

$$\frac{l^*}{1 - e^{-cl^*}} = 1 - \lfloor (1 - 1/c)/u \rfloor u. \tag{4.9}$$

Thus,

$$I_c(u) = \lfloor (1 - 1/c)/u \rfloor K_c(u) + K_c(l^* \wedge u) + L_c(\lfloor (1 - 1/c)/u \rfloor u + l^* \wedge u). \tag{4.10}$$

We show that this equality is consistent with the representaion of $I_c(u)$ in the statement of the theorem. We note that (4.7) implies that $kx_k < 1 - 1/c < (k+1)x_k$ so that $(1-1/c)/k \in (x_k, x_{k-1})$.

Hence, if k is such that $u \in [x_k, x_{k-1}]$, then either $u \in ((1-1/c)/k, x_{k-1}]$ or $u \in [x_k, (1-1/c)/k]$. In the former case $\lfloor (1-1/c)/u \rfloor = k-1$, and since

$$\frac{u}{1-e^{-cu}} \le \frac{x_{k-1}}{1-e^{-cx_{k-1}}} = 1-(k-1)x_{k-1} \le 1-(k-1)u,$$

it follows that $l^* \ge u$ and by (4.10) $I_c(u) = kK_c(u) + L_c(ku)$ as required. If $u \in [x_k, (1-1/c)/k]$, then $\lfloor (1-1/c)/u \rfloor = k$, and since

$$\frac{u}{1-e^{-cu}} \ge \frac{x_k}{1-e^{-cx_k}} = 1-kx_k \ge 1-ku,$$

it follows that $l^* \le u$ and by (4.10) $I_c(u) = kK_c(u) + K_c(l^*) + L_c(ku + l^*)$. The latter expression equals $kK_c(u) + L_c(ku)$ by (1.1), (1.2), and (4.9). □

Remark 4.1. Lemma 3.2 and (4.10) imply that for $c > 1$ the most probable configuration with the largest component of asymptotic size nu has $\lfloor (1-1/c)/u \rfloor$ components of asymptotic size nu and one component of asymptotic size $n(l^* \wedge u)$. A similar observation has been made by O'Connell [8].

Remark 4.2. The expression for I_c in Theorem 3.1 of O'Connell [8] contains a typo.

5 Large deviations for the number of connected components

In this section we prove a large deviation principle for the normalised number of connected components. As it follows by (2.8) the number of connected components equals Φ^n_n. We prove the large deviation principle for Φ^n_n/n as a consequence of the LDP for the process $\overline{\Phi}^n = (\Phi^n_{\lfloor nt \rfloor}/n,\, t \in [0,1])$.

Theorem 5.1. *The processes* $\overline{\Phi}^n$ *satisfy the LDP in* $\mathbb{D}_C([0,1])$ *with action functional*

$$I^\Phi(\phi) = \int_0^1 \Big[(1-\dot\phi_t) \log \frac{1-\dot\phi_t}{c(1-t)} - (1-\dot\phi_t) + c(1-t) \Big]\, dt + \sum (K_c(l_i) + l_i \log c) \tag{5.1}$$

if $\phi = (\phi_t,\, t \in [0,1])$ *is absolutely continuous and nondecreasing,* $\phi_0 = 0$ *and* $\dot\phi_t \le 1$ *a.e., where* l_i *are the lengths of the maximal intervals where* ϕ *is constant and the summation is performed over all such intervals, and* $I^\Phi(\phi) = \infty$ *otherwise.*

Proof. According to (2.5) and (2.9) we have that

$$\overline{\Phi}^n = \mathcal{R}(S^n + \epsilon^n) - S^n - \epsilon^n, \tag{5.2}$$

so Theorem 2.1, (2.14), and continuity of $\mathcal{R}$ yield by the contraction principle that $\overline{\Phi}^n$ satisfies an LDP in $\mathbb{D}_C([0,1])$ with action functional

$$I^\Phi(\phi) = \inf_{\mathbf{x}:\, \phi = \mathcal{R}(\mathbf{x}) - \mathbf{x}} I^S(\mathbf{x}). \tag{5.3}$$

We evaluate the right-hand side. By Lemma 2.1 we have that $I^\Phi(\phi) = \infty$ if any of the conditions that $\phi_0 = 0$, ϕ is nondecreasing and absolutely continuous with $\dot\phi_t \le 1$ a.e. is not met. Let all these conditions be met. By Lemma 2.1, on the set $\{t \in [0,1] : \dot\phi_t > 0\}$ we have that $\mathcal{R}(\mathbf{x})_t = 0$ and $d\mathcal{R}(\mathbf{x})_t/dt = 0$ a.e. Therefore, by the form of I^S in Theorem 2.1,

$$I^{\Phi}(\phi) = \inf_{\mathbf{x}\colon \phi=\mathcal{R}(\mathbf{x})-\mathbf{x}} \int_0^1 \Big((\dot{\mathbf{x}}_t+1)\log\frac{\dot{\mathbf{x}}_t+1}{c(1-t-\mathcal{R}(\mathbf{x})_t)} - (\dot{\mathbf{x}}_t+1) + c(1-t-\mathcal{R}(\mathbf{x})_t)\Big)\,dt$$

$$= \int_0^1 \mathbf{1}(\dot{\phi}_t>0)\Big((-\dot{\phi}_t+1)\log\frac{-\dot{\phi}_t+1}{c(1-t)} - (-\dot{\phi}_t+1) + c(1-t)\Big)\,dt$$

$$+ \inf_{\substack{\mathbf{y}\colon \mathbf{y}_0=0,\,0\le\mathbf{y}_t\le 1-t,\\ \mathbf{y}_t\dot{\phi}_t=0 \text{ a.e.}}} \int_0^1 \mathbf{1}(\dot{\phi}_t=0)\Big((\dot{\mathbf{y}}_t+1)\log\frac{\dot{\mathbf{y}}_t+1}{c(1-t-\mathbf{y}_t)} - (\dot{\mathbf{y}}_t+1) + c(1-t-\mathbf{y}_t)\Big)\,dt. \quad (5.4)$$

Let us consider the latter infimum and denote by A the union of maximal open intervals where ϕ is constant. The function ϕ increasing at almost every point of the set $B = \{t\in[0,1]:\ \dot{\phi}_t = 0\}\setminus A$ implies that an arbitrary neighbourhood of each such point contains a set of nonzero Lebesgue measure where $\dot{\phi}_t>0$. Since $\mathbf{y}_t=0$ at almost every t with $\dot{\phi}_t>0$, it follows that almost every point of B is an accumulation point of those t where $\mathbf{y}_t=0$, so by continuity of $\mathbf{y}$ we have that $\mathbf{y}=0$ almost everywhere on B and by Lemma 2.1 $\dot{\mathbf{y}}_t=0$ almost everywhere on B. Thus,

$$\inf_{\substack{\mathbf{y}\colon \mathbf{y}_0=0,\,0\le\mathbf{y}_t\le 1-t,\\ \mathbf{y}_t\dot{\phi}_t=0 \text{ a.e.}}} \int_0^1 \mathbf{1}(\dot{\phi}_t=0)\Big((\dot{\mathbf{y}}_t+1)\log\frac{\dot{\mathbf{y}}_t+1}{c(1-t-\mathbf{y}_t)} - (\dot{\mathbf{y}}_t+1) + c(1-t-\mathbf{y}_t)\Big)\,dt$$

$$= \int_B \Big(-\log\big(c(1-t)\big) - 1 + c(1-t)\Big)\,dt$$

$$+ \inf_{\substack{\mathbf{y}\colon \mathbf{y}_0=0,\,0\le\mathbf{y}_t\le 1-t,\\ \mathbf{y}_t\dot{\phi}_t=0 \text{ a.e.}}} \int_A \Big((\dot{\mathbf{y}}_t+1)\log\frac{\dot{\mathbf{y}}_t+1}{c(1-t-\mathbf{y}_t)} - (\dot{\mathbf{y}}_t+1) + c(1-t-\mathbf{y}_t)\Big)\,dt \quad (5.5)$$

If (s,t) is a maximal interval of constancy of ϕ, then since the $\mathbf{y}$ over which the infimum of the second integral on the right of (5.5) is taken equal zero at both ends of the interval the contribution of this interval is equal to $\tilde{I}_{s,t}$ as defined by (3.2). By Lemma 3.2 $\tilde{I}_{s,t} = K_c(t-s) + L_c(t) - L_c(s)$, so the right-hand side of (5.5) equals

$$\int_B \Big(-\log\big(c(1-t)\big) - 1 + c(1-t)\Big)\,dt + \sum K_c(l_i) + \int_A L_c'(t)\,dt,$$

where L_c' denotes the derivative of L_c. The assertion of the theorem follows by (5.4) via an algebraic manipulation. □

Let α^n denote the number of connected components of the random graph $G(n, c/n)$.

Theorem 5.2. *The sequence α^n/n satisfies the LDP in $[0,1]$ with action functional*

$$I^{\alpha}(a) = \inf_{\tau\in[0,1-a]} \Big((1-\tau-a)\big(\log(1-\tau-a) + \log 2 - \log c - 1\big) + (2a-1+\tau)\log(1-\tau) + \frac{c}{2}(1-\tau^2) + \tau\log\tau - \tau\log(1-e^{-c\tau})\Big).$$

Remark 5.1. In fact, one can assume that $\tau \in [(1-2a)^+, 1-a]$.

Remark 5.2. The derivative with respect to τ of the function in the infimum above equals

$$2\Big(1-\frac{a}{1-\tau}\Big)-\log\Big[2\Big(1-\frac{a}{1-\tau}\Big)\Big]-\Big(\frac{c\tau}{1-e^{-c\tau}}-\log\frac{c\tau}{1-e^{-c\tau}}\Big).$$

As a consequence, if $a\le 1/2$, then

$$I^\alpha(a)=(1-a)\log(1-a-\tau^*)-(1-a)\Big(1+\log\big(\frac{c}{2}\big)\Big)+\frac{c}{2}(1+\tau^{*2})+\tau^* \\ +(2a-1)\log(1-\tau^*), \quad (5.6)$$

where $\tau^*\in[0,1]$ is the unique solution (for τ) of the equation

$$\frac{c\tau}{e^{c\tau}-1}=2\Big(1-\frac{a}{1-\tau}\Big). \quad (5.7)$$

If $a>1/2$, then $I^\alpha(a)$ equals the minimum of $(1-a)\log(1-a)-(1-a)\big(1+\log(c/2)\big)+c/2$ and the right-hand side of (5.6) for τ^* being the largest solution of (5.7) in $[0,1]$.

Proof of Theorem 5.2. By Theorem 5.1 and the contraction principle the sequence α^n/n satisfies the LDP in $[0,1]$ with action functional

$$I^\alpha(a)=\inf_{\phi\colon\phi_1=a}I^\Phi(\phi). \quad (5.8)$$

We first show that the infimum is attained at a function ϕ that equals 0 on an interval $[0,\tau]$ and then strictly increases to a on $[\tau,1]$. The expression for I^Φ given by Theorem 5.1 can be written for a suitable function G as

$$I^\Phi(\phi)=\int_0^1 G(\dot\phi_t)\,dt+\int_0^1\big(c(1-t)-\log(c(1-t))\big)\,dt+\sum(K_c(l_i)+l_i\log c)+\int_0^1\dot\phi_t\log(1-t)\,dt.$$

Since the function $\log(1-t)$ is decreasing, the last integral on the right-hand side does not increase if we replace $\dot\phi$ with its increasing rearrangement, DeVore and Lorentz [3, Theorem 2.1], Bennett and Sharpley [1]. Also, if we group together all the intervals where $\dot\phi$ is constant, then $\sum K_c(l_i)$ will not increase either by subadditivity of K_c. The first integral on the right is not affected by the increasing rearrangement of $\dot\phi$. Therefore, the function $\dot\phi$ can be assumed to be nondecreasing, which proves the claim. Denoting by $P(\tau)$, where $\tau\in[0,1-a]$, the set of absolutely continuous nondecreasing functions ϕ such that $\phi_t=0$ for $t\in[0,\tau]$, $\dot\phi_t>0$ and $\dot\phi_t\le 1$ a.e. for $t\in(\tau,1]$, and $\phi(1)=a$, we obtain by Theorem 5.1 and (5.8) that

$$I^\alpha(a)=\inf_{\tau\in[0,1-a]}\Bigg(\inf_{\phi\in P(\tau)}\int_\tau^1 f\Big(\frac{1-\dot\phi_t}{c(1-t)}\Big)\,c(1-t)\,dt+K_c(\tau)+L_c(\tau)\Bigg), \quad (5.9)$$

where $f(x)=x\log x-x+1$. Let $I(\phi,\tau)$ denote the integral on the right. Convexity considerations and the condition $\phi_1=a$ provide us with the lower bound

$$I(\phi,\tau)\ge\frac{c(1-\tau)^2}{2}\,f\Bigg(\frac{2(1-\tau-a)}{c(1-\tau)^2}\Bigg), \quad (5.10)$$

which is attained at

$$\dot{\hat\phi}_t=1-\frac{2(1-\tau-a)}{(1-\tau)^2}\,(1-t). \quad (5.11)$$

If $\tau \geq 1-2a$, this function belongs to $P(\tau)$ and delivers the infimum to the integral on the right of (5.9). However, if $\tau < 1-2a$ (hence, $2a < 1$), then $\tilde{\phi}_t$ is negative for $t < 2-(1-\tau)^2/(1-\tau-a)-\tau$. We prove that for those τ the infimum of $I(\phi,\tau)$ over $\phi \in P(\tau)$ is not less than $I(\hat{\phi},\tau)$, where $\dot{\hat{\phi}}_t = 0$ when $t \in [\tau, 1-2a]$ and $\dot{\hat{\phi}}_t = 1-(1-t)/2a$, when $t \in [1-2a, 1]$. We consider $\dot{\phi} = (\dot{\phi}_t,\, t \in [0,1])$ for $\phi \in P(\tau)$ as an element of space L_1 of functions $h = (h_t,\, t \in [\tau, 1])$, with norm $\int_\tau^1 |h_t|\, c(1-t)\, dt$ and $I(\phi,\tau)$ as a functional on L_1. The latter is Gateau differentiable with Gateau derivative at $\dot{\hat{\phi}}$ given by $\langle F'(\dot{\hat{\phi}}), h\rangle = -\int_\tau^1 \log\big((1-\dot{\hat{\phi}}_t)/(c(1-t))\big)h(t)\, dt$. Therefore, for $\dot{\phi} \in L_1$ with $\dot{\phi}_t \in [0,1]$ and $\int_\tau^1 \dot{\phi}_t\, dt = a$,

$$\langle F'(\dot{\hat{\phi}}), \dot{\phi} - \dot{\hat{\phi}}\rangle = \int_\tau^{1-2a} \log(c(1-t))\dot{\phi}_t\, dt + \log(2ac) \int_{1-2a}^{1} (\dot{\phi}_t - \dot{\hat{\phi}}_t)\, dt$$

$$\geq \log(2ac) \int_{1-2a}^{1} \dot{\phi}_t\, dt + \log(2ac) \int_{1-2a}^{1} (\dot{\phi}_t - \dot{\hat{\phi}}_t)\, dt = 0,$$

implying (see, e.g., Ekeland and Temam [4]) that $I(\hat{\phi},\tau) \leq I(\phi,\tau)$ for $\phi \in P(\tau)$. A calculation shows that

$$I(\hat{\phi},\tau) = L_c(1-2a) - L_c(\tau) - (1-2a-\tau)\log c + \int_{1-2a}^{1} f\Big(\frac{1-\dot{\hat{\phi}}_t}{c(1-t)}\Big)\, c(1-t)\, dt,$$

so since $K_c(\tau) \geq K_c(1-2a) + (1-2a-\tau)\log c$, we have that for $\tau < 1-2a$

$$\inf_{\phi \in P(\tau)} \int_\tau^1 f\Big(\frac{1-\dot{\phi}_t}{c(1-t)}\Big)\, c(1-t)\, dt + K_c(\tau) + L_c(\tau) \geq \int_{1-2a}^{1} f\Big(\frac{1-\dot{\hat{\phi}}_t}{c(1-t)}\Big)\, c(1-t)\, dt + K_c(1-2a) + L_c(1-2a).$$

On noting that $\hat{\phi}$ restricted to $[1-2a, 1]$ is an element of $P(1-2a)$, we conclude that the infimum over τ in (5.9) for $2a < 1$ can be taken over $\tau \geq 1-2a$. Thus,

$$I^\alpha(a) = \inf_{\tau \in [(1-2a)^+, 1-a]} \Big(\int_\tau^1 f\Big(\frac{1-\dot{\tilde{\phi}}_t}{c(1-t)}\Big)\, c(1-t)\, dt + K_c(\tau) + L_c(\tau)\Big)$$

$$= \inf_{\tau \in [(1-2a)^+, 1-a]} \Big((1-\tau-a)\log\frac{2(1-\tau-a)}{c(1-\tau)^2} - (1-\tau-a) + \frac{c(1-\tau^2)}{2} + \tau\log\tau + (1-\tau)\log(1-\tau) - \tau\log(1-e^{-c\tau})\Big).$$

□

Remark 5.3. Theorem 5.1 implies in view of Varadhan's lemma, Dembo and Zeitouni [2], that for $\lambda \in \mathbb{R}$

$$\lim_{n\to\infty} \frac{1}{n} \log E e^{\lambda \alpha^n} = G(\lambda, c), \tag{5.12}$$

where

$$G(\lambda,c)=\sup_{\tau\in[(1-e^{\lambda}/c)^+,1]}\Big(\lambda(1-\tau)+\frac{c}{2}(1-\tau)^2e^{-\lambda}-(1-\tau)\log(1-\tau)$$
$$-\frac{c}{2}(1-\tau^2)-\tau\log\tau+\tau\log(1-e^{-c\tau})\Big).$$

This asymptotics has been obtained earlier by Stepanov [12], who carries out a detailed analysis of function $G(\lambda,c)$. In particular, it is continuous in λ for all c, differentiable for $c\le 2$, and piecewise differentiable for $c>2$, with the derivative having a jump at a point $\lambda(c)>\log c$. For the latter reason, one cannot apply Gärtner's theorem, Gärtner [6], to derive the LDP of Theorem 5.1 from limit (5.12) for $c>2$ (while for $c\le 2$ this is possible). Moreover, the function I^{α} is not convex for $c>2$, so it is not a Legendre-Fenchel transform.

Remark 5.4. The proof shows that an optimal configuration for the number of connected components of the order of na has a giant component of the size of the order of $n\tau$ for some $\tau\in[(1-2a)^+,1-a]$, the rest of the components being of size not greater than $1/\dot{\phi}_\tau=(1-\tau)/(2a-1+\tau)$.

A Appendix

Proof of Lemma 3.1. Let $f(x)=x-\log x-1$ and $\phi(x)=x/(1-e^{-x})$. Then the derivative of $K_c(x)+L_c(u+x)$ with respect to x equals $g(y)=f(z-y)-f(\phi(y))$, where $z=c(1-u)$ and $y=cx$. We have

$$\frac{dg}{dy}=\frac{1}{z-y}-1-\frac{1}{1-e^{-y}}+\frac{ye^{-y}}{1-e^{-y}}+\frac{1}{y}-\frac{e^{-y}}{1-e^{-y}}. \tag{A.1}$$

We note that

$$\frac{1}{1-e^{-y}}-\frac{ye^{-y}}{1-e^{-y}}-\frac{1}{y}+\frac{e^{-y}}{1-e^{-y}}=\Big(\frac{1}{y}-\frac{1}{e^{y}-1}\Big)\Big(\frac{y}{1-e^{-y}}-1\Big), \tag{A.2}$$

where $0<1/y-1/(e^y-1)<1$ and $0<y/(1-e^{-y})-1<y$ for $y>0$. Thus, if $u\ge 1-1/c$ so that $z\le 1$, we have that $dg/dy>y/(y-1)-y>0$, proving the assertion of the lemma for that case since $g(0)\ge 0$.

In the rest of the proof we assume that $u<1-1/c$, so $z>1$. We note that $y\in[z-1,z]$. The function $g(y)$ is negative for $y=z-1$ and tends to ∞ as $y\uparrow z$. We show that it first decreases and then increases on $[z-1,z)$. Since the product in (A.2) is positive, dg/dy is negative at $z-1$, so g decreases at $z-1$, also $dg/dy\to\infty$ as $y\uparrow z$, so g eventually increases. It thus suffices to prove that if $dg/dy\ge 0$, then $d^2g/dy^2>0$ for that would mean that once g has begun to increase it will continue increasing. Calculations show that

$$\frac{d^2g}{dy^2}=\frac{1}{(z-y)^2}+\frac{2e^{-y}}{(1-e^{-y})^2}-\frac{ye^{-y}}{(1-e^{-y})^2}-\frac{2ye^{-2y}}{(1-e^{-y})^3}-\frac{1}{y^2}+\frac{e^{-y}}{1-e^{-y}}+\frac{e^{-2y}}{(1-e^{-y})^2}. \tag{A.3}$$

By (A.1) and (A.2) the condition $dg/dy\ge 0$ implies that

$$\frac{1}{z-y}\ge 1+\frac{1}{1-e^{-y}}-\frac{ye^{-y}}{1-e^{-y}}-\frac{1}{y}+\frac{e^{-y}}{1-e^{-y}},$$

so by (A.3) $d^2g/dy^2>0$ provided

$$\Big(1+\frac{1}{1-e^{-y}}-\frac{ye^{-y}}{1-e^{-y}}-\frac{1}{y}+\frac{e^{-y}}{1-e^{-y}}\Big)^2$$

$$+\frac{2e^{-y}}{(1-e^{-y})^2}-\frac{ye^{-y}}{(1-e^{-y})^2}-\frac{2ye^{-2y}}{(1-e^{-y})^3}-\frac{1}{y^2}-\frac{e^{-y}}{1-e^{-y}}+\frac{e^{-2y}}{(1-e^{-y})^2}>0.$$

Simplifying shows that the latter inequality is equivalent to the inequality

$$y(1-e^{-y})(4y-5y^2e^{-y}+ye^{-y}-4+8e^{-y}+3y^2e^{-2y}-5ye^{-2y}-4e^{-2y})+y^3e^{-2y}>0.$$

We actually prove that

$$4y-5y^2e^{-y}+ye^{-y}-4+8e^{-y}+3y^2e^{-2y}-5ye^{-2y}-4e^{-2y}>0 \text{ for } y>0. \tag{A.4}$$

We write

$$\begin{aligned}&4y-5y^2e^{-y}+ye^{-y}-4+8e^{-y}+3y^2e^{-2y}-5ye^{-2y}-4e^{-2y}\\ &\quad=-4(1-e^{-y})^2+y(1-e^{-y})(4+5e^{-y})+y^2e^{-y}(3e^{-y}-5)=(1-e^{-y})R_1(y)+R_2(y),\end{aligned}$$

where

$$\begin{aligned}R_1(y)&=-4(1-e^{-y})+y(1.51+5e^{-y}),\\ R_2(y)&=2.49y(1-e^{-y})+y^2e^{-y}(3e^{-y}-5).\end{aligned}$$

Thus, the proof is complete provided

$$R_i(y)>0 \text{ for } y>0,\ i=1,2. \tag{A.5}$$

Since $R_1(0)=0$, to prove (A.5) for $i=1$ it suffices to check that $dR_1/dy>0$ for $y>0$, which is equivalent to

$$e^y>\frac{5y-1}{1.51}. \tag{A.6}$$

The minimum of $e^y-(5y-1)/1.51$ is attained at $y=\log(5/1.51)$ and equals $6/1.51-(5/1.51)\log(5/1.51)>0$ (in fact, $\log(5/1.51)\approx 1.1973<6/5$). Thus, (A.6) holds, and (A.5) for $i=1$ has been proved.

We now consider R_2. Its derivative equals $R_3(y)=(-2.51+5y+3e^{-y}-6ye^{-y})e^{-y}$. The expression in parentheses is positive for $y=0$ and $y=1$, negative for $y=0.5$, and is seen first to decrease and then to increase. Therefore, dR_2/dy is first positive, then negative and then positive again, changing signs on $[0,0.5)$ and $(0.5,1)$. Hence, the minimum of $R_2(y)$ is attained at $y^*\in(0.5,1)$. Since $R_3(y)\geq R_3(0.5)$ on $[0.5,1]$ and $R_3(0.5)=-0.01e^{-0.5}$, we have that

$$R_2(y^*)\geq R_2(0.5)+0.5R_3(0.5)=2.49-4.995e^{-0.5}+1.5e^{-1}>0.$$

(As a matter of fact, the latter sum approximately equals 0.0122.) Thus, (A.5) and hence (A.4) have been proved.

We have thus proved that $K_c(x)+L_c(u+x)$ decreases on $[1-1/c-u,\tilde{x}]$ and increases on $[\tilde{x},1-u]$, where $\tilde{x}$ is the unique solution of the equation

$$\log\frac{x}{1-e^{-cx}}-\frac{cx}{1-e^{-cx}}=\log\bigl(1-u-x\bigr)-c\bigl(1-u-x\bigr) \tag{A.7}$$

that is not less than $1-1/c-u$. Now algebraic manipulations show that the solution x^* of (3.1) is greater than $1-1/c-u$, satisfies (A.7), so $x^*=\tilde{x}$, and $K_c(x^*)+L_c(u+x^*)=L_c(u)$. □

Remark A.1. In fact, $x^*\in[1-1/(2c)-u,\beta-u]$.

References

[1] C. Bennett and R. Sharpley. *Interpolation of Operators.* Academic Press, 1988.

[2] A. Dembo and O. Zeitouni. *Large Deviations Techniques and Applications.* Springer, second edition, 1998.

[3] R.A. DeVore and G.G. Lorentz. *Constructive Approximation.* Springer, 1993.

[4] I. Ekeland and R. Temam. *Convex Analysis and Variational Problems.* North Holland, 1976.

[5] S.N. Ethier and T.G. Kurtz. *Markov Processes. Characterization and Convergence.* Wiley, 1986.

[6] J. Gärtner. On large deviations from the invariant measure. *Th. Prob. Appl.*, 22(1):24–39, 1977.

[7] S. Janson, T. Łuczak, and A. Ruciński. *Random Graphs.* Wiley, 2000.

[8] N. O'Connell. Some large deviation results for sparse random graphs. *Prob. Th. Rel. Fields*, 110:277–285, 1998.

[9] A. Puhalskii. Large deviation analysis of the single server queue. *Queueing Systems*, 21:5–66, 1995.

[10] A. Puhalskii. *Large Deviations and Idempotent Probability.* Chapman & Hall/CRC, 2001.

[11] A. Puhalskii and W. Whitt. Functional large deviation principles for waiting and departure processes. *Prob. Engin. Inform. Sciences*, 12:479–507, 1998.

[12] V.E. Stepanov. Phase transitions in random graphs. *Th. Prob. Appl.*, 15(2):200–216, 1970. (in Russian).