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Analysis of transmission problems on Lipschitz boundaries in stronger norms

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Abstract — We concentrate on a model diffusion equation on a Lipschitz simply connected bounded domain with a small diffusion coef£cient in a Lipschitz simply connected subdomain located strictly inside of the original domain. We study asymptotic properties of the solution with respect to the small diffusion coef£cient vanishing. It is known that the solution asymptotically turns into a solution of a corresponding diffusion equation with Neumann boundary conditions on a part of the boundary. One typical proof technique of this fact utilizes a reduction of the problem to the interface of the subdomain, using a transmission condition. An analogous approach appears in studying domain decomposition methods without overlap, reducing the investigation to the surface that separates the subdomains and in theoretical foundation of a £ctitious domain, also called embedding, method, e.g., to prove a classical estimate that guaranties convergence of the solution of the £ctitious domain problem to the solution of the original Neumann boundary value problem.

On a continuous level, this analysis is usually performed in an $H^{1/2}$ norm for second order elliptic equations. This norm appears naturally for Poincaré-Steklov operators, which are convenient to employ to formulate the transmission condition. Using recent advances in regularity theory of Poincaré-Steklov operators for Lipschitz domains, we provide, in the present paper, a similar analysis in an $H^{1/2+\alpha}$ norm with $\alpha > 0$, for a simple model problem. This result leads to a convergence theory of the £ctitious domain method for a second order elliptic PDE in an $H^{1+\alpha}$ norm, while the classical result is in an H^1 norm. Here, $\alpha < 1/2$ for the case of Lipschitz domains we consider.

Keywords: Discontinuity, Lipschitz domains, embedding, extension theorems, regularity, diffusion equation, asymptotic properties

AMS(MOS) subject classi£cations. 65N55, 35R05, 35J25.

1. INTRODUCTION

Partial differential equations (PDEs) with highly discontinuous coef£cients between subdomains often appear in engineering applications to model processes in materials with contrast coef£cients, e.g., in typical composite materials. Such problems were historically studied in relation to so-called £ctitious domain, or embedding, method, which is a classical technique of approximating a solution of a boundary value problem for a PDE by a solution of a similar a PDE on

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an extended domain, see [1, 6, 18, 19, 29] and some modern treatment, e.g., [2, 5, 9, 10, 13, 14, 21, 22, 31]. The complement of the original domain embedded into the extended domain is usually called a £ctitious domain. Typically, the PDE coef£cients are kept the same in the original domain. In the £ctitious domain, the coef£cients are chosen to force the solution on the extended domain to satisfy approximately the boundary conditions on the original domain.

A simple example is a problem for the Laplacian with a cavity in the original domain, i.e. with homogeneous Neumann boundary conditions on the surface of the cavity. We £ll the cavity with the £ctitious domain and consider the diffusion equations with the diffusion coef£cient equal to one in the original domain and equal to a small positive constant value in the £ctitious domain. Informally speaking, a small diffusion coef£cient forces the ¤uxes to be small in the £ctitious domain and, in the limit, when the small diffusion coef£cient becomes zero, leads to the no-¤ux case, i.e. to the original homogeneous Neumann boundary condition on the surface of the cavity.

Let us highlight that the current state of the art in £ctitious domain methods does not require actually constructing explicitly the problem with discontinuous coef£cients, neither it needs the £ctitious coef£cient to be small, but nonzero. Thus, an analysis of behavior of the solution of PDEs with highly discontinuous coef£cients is no longer of importance in £ctitious domain methods. Nevertheless, it is convenient to utilize the historical terminology of the £ctitious domain method to study PDEs with highly discontinuous coef£cients between subdomains, and we follow it in the present paper, even though in a way it turns things upside down in terms of what the original problem and domains are.

Asymptotic properties of the solution with respect to the small diffusion coef£cient vanishing are described by a well-known estimate that guaranties convergence in an H^1 norm of the solution of the £ctitious domain problem to the solution of the original problem in the original domain; e.g., [3]. One typical proof technique of the estimate reduces the problem to the interface of the £ctitious domain extension, using a transmission condition. An analogous approach appears in studying domain decomposition methods without overlap, reducing the investigation to the surface that separates the subdomains; e.g., [4, 27].

On a continuous level, this analysis is usually performed in an $H^{1/2}$ norm for second order elliptic equations. This norm appears naturally for Poincaré-Steklov operators, which are convenient to employ to formulate the transmission condition. A regularity theory of Poincaré-Steklov operators for Lipschitz domains, which is closely related to *regularity of transmission and diffraction problems* and properties of *layer potentials* on nonsmooth interfaces, have been extensively studied recently; see, e.g., papers [24, 30], the book [23], and references therein. In [17], these results have been used to develop a regularity theory of the diffusion equation with highly discontinuous coef£cients; see also [25, 26].

Taking advantage of the recent progress, we provide an analysis of the transmission problem in an $H^{1/2+\alpha}$ norm with some $\alpha > 0$. This result leads to a convergence theory of the £ctitious domain method for a second order elliptic PDE in an $H^{1+\alpha}$

norm, while the standard result is in an H^1 norm. Here, $\alpha < 1/2$ is required for the case of Lipschitz domains we consider. For smooth domains a similar result is given in [19] using a different technique.

2. A MODEL DIFFUSION EQUATION

Following [17], we consider a boundary value problem in two dimensions

$$\operatorname{div}(k \operatorname{grad} u - \varphi) = 0,\ u \in \overset{0}{H}{}^1(\Box),\ \varphi \in (L_2(\Box))^2, \tag{2.1}$$

where $\Box$ is a Lipschitz simply connected bounded domain.

Let $\mathscr{D} \subset \Box$ be a Lipschitz connected domain and let the open set $\mathscr{D}^\perp$ be de£ned by the conditions:

$$\mathscr{D} \cap \mathscr{D}^\perp = \emptyset, \quad \overline{\mathscr{D}} \cup \overline{\mathscr{D}^\perp} = \overline{\Box}.$$

We assume that $\mathscr{D}^\perp$ is a Lipschitz simply connected domain.

Let us also assume for simplicity that $\mathscr{D}^\perp$ is strictly inside $\Box$. This assumption, in particular, forces the intersection $\partial\mathscr{D} \cap \partial\Box = \partial\Box$ to have a positive Lebesgue measure on $\partial\Box$, which ensures that any function in $\overset{0}{H}{}^1(\Box)$, which is constant in $\mathscr{D}$, actually vanishes there. Informally speaking, these assumptions simply mean that the domain $\mathscr{D} \subset \Box$ has a cavity $\mathscr{D}^\perp$ in it.

We de£ne the boundary of the cavity as

$$\Gamma = \partial\mathscr{D} \cap \partial\mathscr{D}^\perp = \partial\mathscr{D}^\perp.$$

We assume that k is a piecewise constant function on $\Box$, and highly discontinuous :

$$k = \omega \text{ on } \mathscr{D}^\perp,\ k = 1 \text{ on } \mathscr{D},\ \text{where } 0 < \omega \leqslant 1. \tag{2.2}$$

The interface Γ separates subdomains $\mathscr{D}$ and $\mathscr{D}^\perp$, where the diffusion coef£cient takes different values.

For the "right-hand side" φ, we assume that

$$\varphi \in (H^\alpha(\Box))^2,\ 0 \leqslant \alpha < 1/2$$

and that its restriction on $\mathscr{D}^\perp$ satis£es:

$$\varphi|_{\mathscr{D}^\perp} = 0.$$

We need the later assumption as we want to £x φ and to have the solution u uniformly bounded as a function of $\omega \to 0$ at the same time.

3. PROPERTIES OF THE SOLUTION AS A FUNCTION OF $\omega \to 0$

The following several statements are already known, e.g., [3, 17, 20].

First, we examine the H^1 boundedness of the solution u, uniform in ω [3]:

Theorem 3.1. *Here and throughout, let C be a generic constant, independent of ω, i.e. $C \neq C(\omega)$. Under the assumptions of Section 2, we have*

$$\|u\|_{H^1(\Box)} \leqslant C\|\varphi\|_{(L_2(\Box))^2}. \tag{3.1}$$

Second, we present an estimate [3, 20] that guaranties convergence in an H^1 norm of the solution u of the £ctitious domain problem (2.1) to the solution of the "original" problem, de£ned below, in the domain $\mathscr{D}$ when $\omega \to 0$.

Theorem 3.2. *Under the assumptions of Section 2, let u_0 be de£ned in domain $\mathscr{D}$ as satisfying the same differential equation as u, i.e.*

$$\operatorname{div}(\operatorname{grad} u_0 - \varphi) = 0 \ \text{ in } \mathscr{D}, \tag{3.2}$$

and taking homogeneous Dirichlet boundary conditions on the external boundary $\partial\mathscr{D} \cap \partial\Box = \partial\Box$ of domain $\mathscr{D}$ and homogeneous Neumann boundary conditions on its internal boundary Γ. Then

$$\|u|_{\mathscr{D}} - u_0\|_{H^1(\mathscr{D})} \leqslant C\omega\,\|\varphi|_{\mathscr{D}}\|_{(L_2(\mathscr{D}))^2}. \tag{3.3}$$

Now we turn our attention to regularity. It is well-known that under our assumptions the solution u is regular strictly inside of subdomains $\mathscr{D}$ and $\mathscr{D}^\perp$ for a suf£ciently smooth right-hand side φ, but may have singularities near the boundary $\partial\Box$ and near the interface Γ. Thus, regularity of the solution u is closely related to *regularity of transmission and diffraction problems*, properties of *layer potentials* and *Steklov-Poincaré operators* on nonsmooth interfaces, which have been extensively studied; see, e.g., recent papers [24, 30], the book [23], and references therein. Under our assumption that domain $\mathscr{D}^\perp$ is strictly inside (SI) of $\Box$, the question of regularity can be reduced to studying layer potentials on Γ without boundary conditions, because in the SI case the boundary Γ is a closed Lipschitz curve without sel£ntersection and it does not have any common points, also called *junction* points, with the boundary $\partial\Box$, where the homogeneous Dirichlet boundary condition is enforced.

For a £xed ω, a regularity result is known to hold, according to [8, 30]. A similar regularity of the solution u, but *uniform* in ω, under additional assumptions is established in [17]:

Theorem 3.3. *Under the assumptions of Section 2, $\|u|_{\mathscr{D}}\|_{H^{1+\alpha}(\mathscr{D})}$ is uniformly bounded in ω, with a £xed positive α up to a certain value $\alpha_{\max}$, in the following* *sense:*

$$\|u|_{\mathscr{D}}\|_{H^{1+\alpha}(\mathscr{D})} \leqslant C\|\varphi\|_{(H^{\alpha}(\Box))^2},\ \alpha \in (0, \alpha_{\max}), \tag{3.4}$$

where $\alpha_{\max} = 1/2$.

We note that in the limit $\omega = 0$, which corresponds to a Neumann boundary value problem on $\mathscr{D}$, according to Theorem 3.2, the regularity result also holds with $\alpha_{\max} = 1/2$; see, e.g., [16]:

Theorem 3.4. *Under the assumptions of Theorem 3.2, $u_0 \in H^{1+\alpha}(\mathscr{D})$ with a £xed positive α up to a certain value $\alpha_{\max}$, in the following sense:*

$$\|u_0\|_{H^{1+\alpha}(\mathscr{D})} \leqslant C\|\varphi\|_{(H^{\alpha}(\Box))^2},\ \alpha \in (0, \alpha_{\max}), \tag{3.5}$$

where $\alpha_{\max} = 1/2$.

The following statement is new.

Theorem 3.5. *Under the assumptions of Theorem 3.2, for a small enough non-negative ω, we have*

$$\|u|_{\mathscr{D}} - u_0\|_{H^{1+\alpha}(\mathscr{D})} \leqslant C\omega\,\|\varphi|_{\mathscr{D}}\|_{(H^{\alpha}(\mathscr{D}))^2},\ \alpha \in (0, \alpha_{\max}), \tag{3.6}$$

where $\alpha_{\max} = 1/2$.

We prove Theorem 3.5 in the next section. In the course of the proof, we will also establish Theorems 3.1, 3.2, and 3.3.

4. PROOF OF THEOREM 3.5

Let function $\hat{u} \in \overset{0}{H}{}^1(\Box)$ satis£es the same equation as function u, i.e.

$$\int_{\mathscr{D}} (\operatorname{grad}\hat{u} - \varphi) \cdot \operatorname{grad} v \, d\mathscr{D} = 0,\ \forall v \in \overset{0}{H}{}^1(\mathscr{D}), \tag{4.1}$$

in $\mathscr{D}$ and vanishes in $\mathscr{D}^{\perp}$, so that $\hat{u}|_{\mathscr{D}} \in \overset{0}{H}{}^1(\mathscr{D})$.

We note that equation (4.1) is a uniformly elliptic homogeneous Dirichlet boundary value problem on a Lipschitz domain and is independent of ω. Therefore,

$$\|\hat{u}|_{\mathscr{D}}\|_{H^{1+\alpha}(\mathscr{D})} \leqslant C\|\varphi|_{\mathscr{D}}\|_{(H^{\alpha}(\mathscr{D}))^2},\ \alpha \in (0, \alpha_{\max}) \tag{4.2}$$

by well-known regularity results; see, e.g., [11, 12].

We now simplify equation (2.1) by introducing a new function

$$w = u - \hat{u} \in \overset{0}{H^1}(\Box).$$

By construction, the function w is piece-wise harmonic, i.e. the restriction $w|_{\mathscr{D}}$ is harmonic in $\mathscr{D}$ and the restriction $w|_{\mathscr{D}^\perp}$ is harmonic in $\mathscr{D}^\perp$. On the interface of the jump in the coef£cients, Γ, the functions $w|_{\mathscr{D}}$ and $w|_{\mathscr{D}^\perp}$ have the same trace $w|_\Gamma$ from $\mathscr{D}$ and $\mathscr{D}^\perp$ and satisfy the following standard transmission condition on Γ between $\mathscr{D}$ and $\mathscr{D}^\perp$ for the normal components:

$$\mathbf{n} \cdot \left(\operatorname{grad}\left(w|_{\mathscr{D}} + \hat{u}|_{\mathscr{D}}\right) - \varphi|_{\mathscr{D}}\right)\big|_\Gamma = \mathbf{n} \cdot \left(\omega \operatorname{grad}\left(w|_{\mathscr{D}^\perp}\right)\right)\big|_\Gamma, \tag{4.3}$$

where the left side corresponds to $\mathscr{D}$ and the right side to $\mathscr{D}^\perp$, and $\mathbf{n}$ is the normal direction on Γ, oriented outward $\mathscr{D}$.

Let us rewrite equation (4.3) by collecting all terms with w in the left-hand side:

$$\begin{aligned} \frac{\partial}{\partial \mathbf{n}} w|_{\mathscr{D}}\Big|_\Gamma - \omega \frac{\partial}{\partial \mathbf{n}} w|_{\mathscr{D}^\perp}\Big|_\Gamma \\ = \mathbf{n} \cdot \left(\varphi|_{\mathscr{D}} - \operatorname{grad} \hat{u}|_{\mathscr{D}}\right)\big|_\Gamma . \end{aligned} \tag{4.4}$$

Let us now de£ne a Steklov-Poincaré operator $S_{\mathscr{D}}$ that, for a given harmonic function $w|_{\mathscr{D}}$, where $w \in \overset{0}{H^1}(\Box)$, maps its trace on Γ into its Neumann datum on Γ:

$$S_{\mathscr{D}} : w|_{\mathscr{D}}\big|_\Gamma \mapsto \frac{\partial}{\partial \mathbf{n}} w|_{\mathscr{D}}\Big|_\Gamma .$$

Let us similarly de£ne the Steklov-Poincaré operator $S_{\mathscr{D}^\perp}$ for the other domain $\mathscr{D}^\perp$ using the normal direction on Γ, oriented outwards from $\mathscr{D}^\perp$. Steklov-Poincaré operators act between the corresponding space of trace functions

$$\Lambda = \{w|_\Gamma, \forall w \in \overset{0}{H^1}(\Box)\}.$$

and its dual Λ'. These operators are symmetric and positive semi-de£nite on Λ and bounded as mappings $\Lambda \to \Lambda'$; see, e.g., [28]. Moreover, the operator $S_{\mathscr{D}}$ is positive de£nite, because of the homogeneous Dirichlet boundary conditions on $\partial\Box$. As the domain $\mathscr{D}^\perp$ is strictly inside, i.e., Γ is a closed curve, the space Λ is simply $H^{1/2}(\Gamma)$ and Λ' is $H^{-1/2}(\Gamma)$.

Using the Steklov-Poincaré operators just de£ned, we rewrite equation (4.4) as

$$\left(S_{\mathscr{D}} + \omega S_{\mathscr{D}^\perp}\right)\lambda = \chi, \tag{4.5}$$

where we introduce new notations: λ for the common trace $w|_\Gamma$ of $w|_{\mathscr{D}}$ and $w|_{\mathscr{D}^\perp}$ on Γ and χ for the right-hand side of (4.4). The operator $S_{\mathscr{D}} + \omega S_{\mathscr{D}^\perp} \geqslant S_{\mathscr{D}}$ is symmetric, positive de£nite and bounded, *uniformly* in ω, $0 \leqslant \omega \leqslant 1$, e.g., [4]. The right-hand side χ does not depend on ω and is bounded, i.e.

$$\|\chi\|_{H^{-1/2}(\Gamma)} \leqslant C\|\varphi\|_{(L_2(\Box))^2} \tag{4.6}$$

as follows from its de£nition; see, e.g., [23]. Therefore, the solution λ is bounded uniformly in $H^{1/2}(\Gamma)$:

$$\|\lambda\|_{H^{1/2}(\Gamma)} \leqslant C\|\varphi\|_{(L_2(\Box))^2}, \tag{4.7}$$

which leads to uniform boundedness of w and, therefore, of u in $\overset{0}{H}{}^1(\Box)$, which proves Theorems 3.1.

Let us now consider similar arguments applied to u_0 instead of u. We replace w with w_0 de£ned only on $\mathscr{D}$ as

$$w_0 = u_0 - \hat{u}|_{\mathscr{D}} \in \overset{0}{H}{}^1(\mathscr{D}).$$

As a substitute for (4.3), we derive

$$\mathbf{n} \cdot \left(\operatorname{grad}(w_0 + \hat{u}|_{\mathscr{D}}) - \varphi|_{\mathscr{D}}\right)\Big|_{\Gamma} = 0, \tag{4.8}$$

where $\mathbf{n}$ is the normal direction on Γ, oriented outward $\mathscr{D}$. Calling λ_0 to be the trace $w_0\big|_{\Gamma}$ of w_0 on Γ and using the same χ as for the right-hand side of (4.4), we obtain

$$S_{\mathscr{D}}\lambda_0 = \chi. \tag{4.9}$$

We observe exactly as we expect that equation (4.5) turns into equation (4.9) when $\omega \to 0$.

To compare λ with λ_0, we use the equation

$$\left(S_{\mathscr{D}} + \omega S_{\mathscr{D}^\perp}\right)(\lambda - \lambda_0) = -\omega S_{\mathscr{D}^\perp}\lambda_0 = -\omega S_{\mathscr{D}^\perp} S_{\mathscr{D}}^{-1}\chi, \tag{4.10}$$

derived directly from (4.5) and (4.9). Using again (4.6) and the same arguments as those leading to (4.7), we obtain

$$\|\lambda - \lambda_0\|_{H^{1/2}(\Gamma)} \leqslant \omega C\|\varphi\|_{(L_2(\Box))^2}, \tag{4.11}$$

that leads to a similar bound for $w - w_0$ and, thus for $u|_{\mathscr{D}} - u_0$, in H^1 norm on domain $\mathscr{D}$, which is the statement of Theorem 3.2.

At this point, we are prepared to study the uniform regularity of λ, which shall demonstrate uniform regularity of u in $\mathscr{D}$.

We £rst notice that $\chi \in H^{-1/2+\alpha}(\Gamma)$ because of extra smoothness of our functions φ and $\hat{u}$ that determine χ according to (4.2) and (4.4):

$$\|\chi\|_{H^{-1/2+\alpha}(\Gamma)} \leqslant C\|\varphi\|_{(H^\alpha(\Box))^2}.$$

In recent works [7, 15, 23], a regularity theory of Steklov-Poincaré operators is established for Lipschitz domains. Using these results for our situation, we have that for all $\alpha \in [0,1/2)$ our both Steklov-Poincaré operators are bounded:

$$S_{\mathscr{D}} : H^{1/2+\alpha}(\Gamma) \mapsto H^{-1/2+\alpha}(\Gamma),\ S_{\mathscr{D}^\perp} : H^{1/2+\alpha}(\Gamma) \mapsto H^{-1/2+\alpha}(\Gamma),$$

and the Steklov-Poincaré operator $S_{\mathscr{D}}$ is coercive:

$$\|S_{\mathscr{D}}^{-1}\psi\|_{H^{1/2+\alpha}(\Gamma)} \leqslant C\left(\|\psi\|_{H^{-1/2+\alpha}(\Gamma)} + \|S_{\mathscr{D}}^{-1}\psi\|_{H^{1/2}(\Gamma)}\right). \tag{4.12}$$

The coerciveness (modulo constants) of $S_{\mathscr{D}^\perp}$ is not important for our further arguments.

Let us rewrite equation (4.5) as

$$\left(I + \omega S_{\mathscr{D}}^{-1} S_{\mathscr{D}^\perp}\right)\lambda = S_{\mathscr{D}}^{-1}\chi. \tag{4.13}$$

and equation (4.10) as

$$\left(I + \omega S_{\mathscr{D}}^{-1} S_{\mathscr{D}^\perp}\right)(\lambda - \lambda_0) = -\omega S_{\mathscr{D}}^{-1} S_{\mathscr{D}^\perp} S_{\mathscr{D}}^{-1}\chi. \tag{4.14}$$

The operator $S_{\mathscr{D}}^{-1} S_{\mathscr{D}^\perp}$ is bounded in $H^{1/2+\alpha}(\Gamma)$, thus, the operator of equations (4.13) and (4.14) is a small perturbation of the identity and, therefore, it has a bounded inverse in $H^{1/2+\alpha}(\Gamma)$, for nonnegative ω small enough, e.g., for

$$0 \leqslant \omega \leqslant \frac{1}{2}\|S_{\mathscr{D}}^{-1} S_{\mathscr{D}^\perp}\|^{-1}_{H^{1/2+\alpha}(\Gamma)}. \tag{4.15}$$

Thus, the regularity estimate holds

$$\|\lambda\|_{H^{1/2+\alpha}(\Gamma)} \leqslant C\|\varphi\|_{(H^\alpha(\Box))^2},$$

uniformly in ω satisfying (4.15). As the function $w|_{\mathscr{D}}$ is a harmonic extension of λ to $\mathscr{D}$, it is uniformly bounded with respect to the $H^{1+\alpha}(\mathscr{D})$ norm:

$$\|w_{\mathscr{D}}\|_{H^{1/2+\alpha}(\mathscr{D})} \leqslant C\|\varphi\|_{(H^\alpha(\Box))^2}.$$

Indeed, $w|_{\mathscr{D}}$ solves the Laplace equation in the Lipschitz domain $\mathscr{D}$ with homogeneous the Dirichlet boundary conditions on $\partial\Box$ and nonhomogeneous the Dirichlet boundary conditions, given by the function $\lambda \in H^{1/2+\alpha}(\Gamma)$, on Γ. Thus, known regularity results (e.g., [16, 23, 30]) can be applied. Using $u = w + \hat{u}$, we combine the previous inequality with (4.2) to get (3.4) uniformly in ω, satisfying (4.15), which proves Theorem 3.3.

Similarly, from (4.14) we obtain

$$\|\lambda - \lambda_0\|_{H^{1/2+\alpha}(\Gamma)} \leqslant C\omega\,\|\varphi\|_{(H^\alpha(\Box))^2}.$$

The function w_0 is a harmonic extension of λ_0 to $\mathscr{D}$, therefore, function $w|_{\mathscr{D}} - w_0$ is a harmonic extension of $\lambda - \lambda_0$ to $\mathscr{D}$ and, as such, satis£es

$$\|w|_{\mathscr{D}} - w_0\|_{H^{1/2+\alpha}(\mathscr{D})} \leqslant C\omega\,\|\varphi\|_{(H^\alpha(\Box))^2}.$$

Finally, $w|_{\mathscr{D}} - w_0 = u|_{\mathscr{D}} - u_0$, by de£nition of w and w_0, which completes the proof of Theorem 3.5.

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