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A Method for Finding an Optimal Investment and Distribution Strategy for an Individual Retiree

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Abstract: We describe a method for finding an optimal investment and distribution strategy for an individual retiring with a pool of assets and both fixed and indexed annuities. The method finds a strategy that optimizes an expected utility function. The method uses a discrete time model and a technique for estimating the cumulative distribution function of a function of independent random variables that reduces multidimensional integration to function evaluations. This reduces the expectation calculation to one dimensional integration allowing for simple computations of derivative information for use in a standard optimization routine. We provide one simple example to illustrate the method.

Keywords: Probability Theory, Stochastic Programing, Risk Analysis, Financial Modeling

1 Introduction

Increasingly employers are passing the responsibility of preparing for retirement on to their employees. Fixed employer pensions are being reduced or eliminated in favor of defined contribution plans such as 401(k) plans. As a result, individuals often find themselves at retirement with a pool of assets as one of their major sources of income for their retirement years. This requires them to make two key decisions. First, how much of this pool should they withdraw annually to cover living expenses and second, how should they invest the balance. The primary risk that drives these decisions is the risk of running out of money which could leave the individual with limited means at an advanced age. Both the withdrawal and investment decisions are key to minimizing this risk. Withdrawing too much will deplete the assets prematurely, while withdrawing too little will reduce the individuals standard of living standard. Similarly, investing the assets too conservatively will increase the risk of running out of money since the assets will grow more slowly but too aggressive an investment strategy may do the same since it will increase the risk of a large asset loss. These decisions are complicated by the uncertainties of investment returns and mortality.

In this paper we describe a method for finding a solution to this problem in the context of expected utility theory. The solution relies on a deterministic method for estimating the cumulative distribution function of a function of several independent random variables [2]. This method estimates the c.d.f. without utilizing integration or discretizing and gives a very good estimate when the number of random variables involved is not too large (less than 20 or so). Since the method does not utilize integration, finding expected values is reduced to single dimensional integration and this allows us to easily develop partial derivative information. Thus we reduce the problem to one that can be solved using standard optimization routines for deterministic equations.

2 The Retirees' Asset Allocation Problem

Consider an individual who retires with an accumulation of assets (for example IRA's, 401(k) profit sharing plans, personal savings etc.) and who has a fixed annual annuity (for example an annuity contract or an employer pension plan) and an inflation indexed benefit (for example Social Security or a governmental pension). The individual must decide on an annual withdrawal and asset allocation strategy that will maximize the expected utility of his/her income stream.

In this paper we propose a formulation of this problem as a dynamic stochastic program using a discrete time model. Our objective is to maximize the expected value of the sum of the utility of the retirees annual income stream. The formulation as a dynamic program allows use to keep the number of random variables involved at each step to a reasonable size (twelve for our sample case). This allows us to use the technique discussed in the introduction and in detail below, that gives an accurate estimate of the cumulative distribution function of the utility without integration. Thus in turn gives us a good estimate of the expected value via one dimensional integration so that we can use standard optimization approaches to solving the problem.

2.1 The Model

In order to ease understanding, throughout this paper we will use capital letters without embellishment (e.g. W) to denote a random variable, capital letters of the form $\vec{R}$ to denote a random vector, small letters such as c for constants, small letters of the from $\vec{c}$, for vectors of constants, and (finally) capital letters $\bar{D}_i$ and $\bar{L}_i$ (to be defined below) are decision variables and vectors of decision variables respectively

The proposed model is a discrete time model. The basic variables for the model are as follows:

CPI_i - change in consumer price index from attained age i to attained age $i+1$

$\bar{D}_i$ - distribution from accumulated retirement assets for age $i \geq r$, a decision variable

$fage$ - age the fixed annual annuity becomes payable

$fbft$ - the fixed annual annuity, payable for the retiree's life

i - the retirees age from age at retirement until maximum possible survival age

$iage$ - age the inflation indexed benefit will begin

$ibft$ - annual inflation indexed benefit in dollars at retirement age r

INC_i - total annual income for the retiree during age i

$\bar{L}_i$ - an n-dimensional column vector whose j-th entry is the percent of retirement assets invested in asset class j for $1 \leq j \leq n-1$ at the beginning of age i (the n^{th} entry is always equal to zero since it corresponds to the entry in $\vec{R}_i$ which is used for inflation as discussed below). This is a vector of decision variables.

mx - maximum possible survival age

${}_ip_r$ - the probability that the retiree will survive from age r to age i.

r - age of retirement

$\vec{R}_i$ - an n-dimensional column vector whose j-th entry is the rate of return on asset class j for $1 \leq j \leq n-1$ from attained age i to attained age $i+1$, the n^{th} entry is inflation for the period.

$\vec{r}_{r-1}$ - actual return on each asset class for the year prior to retirement

W_i - accumulated retirement assets at age $i > r$, before distributions for the upcoming year

w_r - accumulated assets at retirement

Derived variables are:

$ACPI_i = \begin{cases} \Pi_{j=r}^{i-1} CPI_i & \text{if } i > r \\ 1.0 & \text{if } i = r \end{cases}$ the accumulated change in CPI from retirement age to age i

$b_i = \begin{cases} 0 & \text{if } i < fage \\ fbft & \text{if } i \geq fage \end{cases}$ the fixed annual annuity at age i

$S_i = \begin{cases} 0 & \text{if } i < iage \\ ibft * ACPI_i & \text{if } i \geq iage \end{cases}$ the inflation indexed annuity at age i

The amount available for consumption during age i is as follows:

$$INC_i = \bar{D}_i + S_i + b_i \tag{1}$$

This is the sum of the amount withdrawn from savings for the year plus income from the fixed annuity and the indexed annuity. Then the accumulated retirement assets at age $i + 1 > r$ is given by:

$$W_{i+1} = \max\left(0, W_i - \bar{D}_i\right) * \left(\vec{R}_i * \bar{L}_i\right) \tag{2}$$

The decision variables for the problem are the elements of $\bar{L}_i$, the percentages to be invested in each asset class at the beginning of age i and $\bar{D}_i$ the amount to withdraw from the fund each year. We assume the retiree will reallocate the investments annually at the beginning of each age i in the percentages specified by $\bar{L}_i$.

We assume the retiree wishes to maximize the expected value of the sum of the utilities of the income stream that is produced from his/her allocation and distribution strategy (this formulation is similar to that of J.R. Brown, see discussion section of [5]). Moreover, it is assumed that the retiree's utility function remains fixed for all ages (although it would not be difficult to adjust for a variable utility function). All future income levels are adjusted for the passage of time by discounting them to the age at retirement by the accumulated inflation and weight them by the probability that the retiree will survive to age i to receive the income. Thus our problem can be stated as follows.

Given w_r and $\vec{r}_{r-1}$ (the retiree's wealth at retirement and the return performance of each asset class and inflation in the prior year) solve the following stochastic program:

$$\max_{\bar{D}_i, \bar{L}_i \; i=r,\ldots,mx} E\left(\Sigma_{i=r}^{mx}\left({}_ip_r\right) U\left(\frac{INC_i}{ACPI_i}\right)\right) \tag{3}$$

Subject to

$$\begin{array}{ll} \bar{D}_i \leq W_i \; \forall i & \\ \Sigma_{j=1}^{n-1} \bar{L}_i\left(j\right) = 1 \; \forall i & \text{(recall } L_i\left(n\right) \text{ is inflation)} \\ 0 \leq \bar{L}_i\left(j\right) \; \forall j \leq n-1, \; \forall i & \end{array} \tag{4}$$

where E is the expectation operator and U is the retirees utility function at retirement. We assume asset returns and inflation are the only random and uncertain elements in the problem.

The problem can be reformulated as a stochastic dynamic programming problem. For constants w_i and $acpi_i$ and vector of constants $\vec{r}_{i-1}$, let

$$\begin{array}{l} EU_i^*\left(w_i, acpi_i, \vec{r}_{i-1}\right) = \max_{\bar{D}_j, \bar{L}_j \; j=i,\ldots,mx} E(\Sigma_{j=i}^{mx}\left({}_jp_r\right) U\left(\frac{INC_j}{ACPI_j}\right) \\ \mid W_i = w_i, ACPI_i = acpi_i, \vec{R}_{i-1} = \vec{r}_{i-1}) \end{array}$$

i.e., $EU_i^*\left(w_i, acpi_i, \vec{r}_{i-1}\right)$ is the function that gives the maximum expected utility of the income stream from age i to age mx given the retiree's wealth at age i is w_i and accumulated inflation up to age i has been $acpi_i$ and asset returns from age $i-1$ to age i were $\vec{r}_{i-1}$. Then we assume all wealth is distributed in the last year of life, i.e.,

$$EU_{mx}^*\left(w_{mx}, acpi_{mx}, \vec{r}_{mx-1}\right) =_{mx} p_r U\left(\frac{w_{mx} + ibft * acpi_{mx} + b_{mx}}{acpi_{mx}}\right)$$

If we have determined $EU_{i+1}^*\left(w_{i+1}, acpi_{i+1}, \vec{r}_{i+1}\right)$ then

$$EU_i^*\left(w_i, acpi_i, \vec{r}_{i-1}\right) =$$

$$\begin{array}{l} \max_{\bar{D}_j, \bar{L}_j \; j=i,\ldots,mx-1} E[\Sigma_{j=i}^{mx-1}\left({}_jp_r\right) U\left(\frac{INC_j}{ACP_j}\right) \\ \mid W_i = w_i, ACPI_i = acpi_i, \vec{R}_{i-1} = \vec{r}_{i-1}] = \end{array} \tag{5}$$

$$\begin{array}{l} \max_{\bar{D}_j, \bar{L}_j \; j=i,\ldots,mx-1} {}_i p_r [{}_jp_r U\left(\frac{inc_i}{acpi_i}\right) + E(\Sigma_{j=i+1}^{mx-1}\left({}_jp_r\right) U\left(\frac{INC_j}{ACPI_j}\right) \\ \mid W_i = w_i, ACPI_i = acpi_i, \vec{R}_{i-1} = \vec{r}_{i-1})] = \end{array} \tag{6}$$

$$\max_{\bar{D}_i,\bar{L}_i}[{}_ip_rU\left(\frac{inc_i}{acpi_i}\right)+\max_{\bar{D}_j,\bar{L}_j\ j=i-1,...,mx-1}E(\Sigma_{j=i+1}^{mx-1}\left({}_jp_r\right)U\left(\frac{INC_j}{ACPI_j}\right)$$
$$\mid W_i=w_i,ACPI_i=acpi_i,\vec{R}_{i-1}=\vec{r}_{i-1})]= \tag{7}$$

$$\max_{\bar{D}_i,\bar{L}_i}[{}_ip_rU\left(\frac{inc_i}{acpi_i}\right)+E(EU_{i+1}^{*}\left(W_{i+1},ACPI_{i+1},\vec{R}_i\right)$$
$$\mid W_i=w_i,ACPI_i=acpi_i,\vec{R}_{i-1}=\vec{r}_{i-1})] \tag{8}$$

Equation 7 follows from the principle of optimality from dynamic programming since the equation is monotone increasing in the second term of the sum. Equation 8 follows since

$$\max_{\bar{D}_j,\bar{L}_j\ j=i-1,...,mx-1}E(\Sigma_{j=i+1}^{mx-1}\left({}_jp_r\right)U\left(\frac{INC_j}{ACPI_j}\right)$$
$$\mid W_i=w_i,ACPI_i=acpi_i,\vec{R}_{i-1}=\vec{r}_{i-1})=$$
$$\max_{\bar{D}_j,\bar{L}_j\ j=i-1,...,mx-1}E[E(\Sigma_{j=i+1}^{mx}\left({}_jp_r\right)U\left(\frac{INC_j}{ACPI_j}\right)$$
$$\mid W_i=w_i,ACPI_i=acpi_i,\vec{R}_{i-1}=\vec{r}_{i-1},W_{i+1},ACPI_{i+1},\vec{R}_i)]$$

and $\forall\bar{D}_j,\bar{L}_j\ j=i-1,...,mx-1$ and each $W_{i+1},ACPI_{i+1},\vec{R}_i$

$$E(\Sigma_{j=i+1}^{mx}\left({}_jp_r\right)U\left(\frac{INC_j}{ACPI_j}\right)$$
$$\mid W_i=w_i,ACPI_i=acpi_i,\vec{R}_{i-1}=\vec{r}_{i-1},W_{i+1},ACPI_{i+1},\vec{R}_i)$$
$$\leq EU_{i+1}^{*}\left(W_{i+1},ACPI_{i+1},\vec{R}_i\right)$$

This implies

$$E(E[\Sigma_{j=i+1}^{mx}\left({}_jp_r\right)U\left(\frac{INC_j}{ACPI_j}\right)$$
$$\mid W_i=w,ACPI_i=acpi,\vec{R}_{i-1}=\vec{r},W_{i+1},ACPI_{i+1},\vec{R}_i])$$
$$\leq E\left(EU_{i+1}^{*}\left(W_{i+1},ACPI_{i+1},\vec{R}_i\right)\mid W_i=w_i,ACPI_i=acpi_i,\vec{R}_{i-1}=\vec{r}_{i-1}\right)$$

and the right hand being independent of $\bar{D}_j,\bar{L}_j\ j=i-1,...,mx-1$ gives us

$$\max_{\bar{D}_j,\bar{L}_j\ j=i-1,...,mx-1}E(\Sigma_{j=i+1}^{mx-1}\left({}_jp_r\right)U\left(\frac{INC_j}{ACPI_j}\right)$$
$$\mid W_i=w_i,ACPI_i=acpi_i,\vec{R}_{i-1}=\vec{r}_{i-1})=$$
$$E\left(EU_{i+1}^{*}\left(W_{i+1},ACPI_{i+1},\vec{R}_i\right)\mid W_i=w_i,ACPI_i=acpi_i,\vec{R}_{i-1}=\vec{r}_{i-1}\right)$$

The recursive steps are as follows

$$\text{Find } EU_r^* \left(w_r, 1.0, \vec{r}_{r-1} \right) \tag{9}$$

by solving recursively, for the function

$$\begin{array}{l} EU_i^* \left(w_i, acpi_i, \vec{r}_{i-1} \right) = \\ \max_{\bar{D}_i, \bar{L}_i} [\left({}_ip_r U \left(\frac{inc_i}{acpi_i} \right) \right) + E(EU_{i+1}^* \left(W_{i+1}, ACPI_{i+1}, \vec{R}_i \right) \\ \mid W_i = w_i, ACPI_i = acpi_i, \vec{R}_{i-1} = \vec{r}_{i-1})] \end{array} \tag{10}$$

subject to

$$\begin{array}{l} \bar{D}_i \leq w \\ \Sigma_{j=1}^{n-1} \bar{L}_i \left(j \right) = 1 \\ 0 \leq \bar{L}_i \left(j \right) \, \forall j \leq n-1 \end{array} \tag{11}$$

with

$$EU_{mx}^* \left(w_{mx}, acpi_{mx}, r_{mx-1} \right) =_{mx} p_r U \left(\frac{w_{mx} + ibft * acpi_{mx} + b_{mx}}{acpi_{mx}} \right) \tag{12}$$

At each step, for selected values of $w_i, acpi_i$ and $\vec{r}_{i-1}$ we will estimate the c.d.f. for $EU_{i+1}^* \left(W_{i+1}, ACPI_{i+1}, \vec{R}_i \right)$ given $W_i = w_i, ACPI_i = acpi_i, \vec{R}_{i-1} = \vec{r}_{i-1}$ using the technique described in [2] (discussed below) and use this estimate to calculate the optimal expected value. Using these values we will estimate $EU_{i+1}^* \left(w_{i+1}, acpi_{i+1}, \vec{r}_i \right)$ for all values of $w_{i+1}, acpi_{i+1}$ and $\vec{r}_i$ by interpolation. This function will then be used to solve for $EU_i^* \left(w_i, acpi_i, \vec{r}_{i-1} \right)$ and so on.

2.2 The Asset Model

The asset model we use is similar to that described in [1]. The proposed model for asset returns on each asset class and inflation is the following:

Let R_t be an n-dimensional column vector whose n-th entry is the rate of inflation for year beginning at $t-1$ and whose i-th entry, for $i \neq n$, is the rate of return on asset class i for the same one-year period. We assume $\mu_{\vec{R}_i}$, $\Sigma_{\vec{R}_i}$ and the regression coefficients for R, ρ_R^2 are all known. For our analysis we will use the model

$$R_t = e^{I_t} \tag{13}$$

where I_t is an n-dimensional column vector defined as follows:

$$I_t = \vec{\phi}_0 + \vec{\phi}_1 I_{t-1} + \vec{X}A \tag{14}$$

with $\vec{X} \overset{i.i.d.}{\sim} N(0, Id_n)$ where Id_n is the n-dimensional identity matrix, $\vec{\phi}_0$ and $\vec{\phi}_1$ are n-dimensional column vectors and A is an $n \times n$ matrix of constants such that if $Y = \vec{X}A$ then $\vec{Y} \sim N(0, \Sigma_{\vec{Y}})$ with $\Sigma_{\vec{Y}}$ the $n \times n$ variance-covariance matrix of $\vec{Y}$. Thus our returns are lognormally distributed, correlated and have a one-year auto regression term.

The parameters $\vec{\phi}_0$, $\vec{\phi}_1$ and A are defined as follows (where $\vec{\phi}_0$ is like the mean, $\vec{\phi}_1$ the auto correlation and A correlation, see Appendix A for the derivation).

$$\vec{\phi}_0(i) = \left(1 - \vec{\phi}_1(i)\right) \log\left(\mu^2_{R(i)} \left(\frac{1}{\mu^2_{R(i)} + \sigma^2_{R(i)}}\right)^{\frac{1}{2}}\right) \tag{15}$$

$$\vec{\phi}_1(i) = \left(\log\left(1 + \frac{\sigma^2_{R(i)}}{\mu^2_{R(i)}}\right)\right)^{-1} \log\left(1 + \frac{\sigma^2_{R(i)}}{\mu^2_{R(i)}} \rho_{R_t(i)R_{t+1}(i)}\right) \tag{16}$$

and

$$A = (S)^{\frac{1}{2}} * V^T \tag{17}$$

where

$$\Sigma_{\vec{Y}} = U * S * V^T \tag{18}$$

for random vector $\vec{Y}$ such that

$$cov_{\vec{Y}(i)\vec{Y}(j)} = \left(1 - \vec{\phi}_1(i)\,\vec{\phi}_1(j)\right) \log\left(1 + \frac{\sigma_{R(i)}\sigma_{R(j)}}{\mu_{R(i)}\mu_{R(j)}} \rho_{R(i)R(j)}\right) \text{ when } i \neq j \tag{19}$$

and

$$\sigma^2_{\vec{Y}(i)} = \left(1 - \vec{\phi}_1^2(i)\right) \log\left(1 + \frac{\sigma^2_{R(i)}}{\mu^2_{R(i)}}\right) \text{ when } i = j \tag{20}$$

(i.e. we use the singular value decomposition of $\Sigma_{\vec{Y}}$).

2.3 Estimating the c.d.f.

As stated, to evaluate the expected value we estimate the c.d.f. using the technique described in [2]. Let $Y = f\left(\vec{X}\right)$ where $\vec{X} = (X_1, ..., X_n)$ is a vector of continuous independent random variables with c.d.f.'s $F_{X_i}(x)$ (i.e.$F_{X_i}(x) = prob\,(X_i < x)$ and $f : R^n \to R$ is continuous and monotone increasing in each x_i (it is a simple adjustment to consider functions that increase in some variables and decrease in others, see Appendix B for details). Assume that the range of variable X_i is $[b_i, c_i]$, that is, we have finite support.

The first step of the method is to divide the range of each variable into subintervals thus creating a partition of $[b_1, c_1] \times ... \times [b_n, c_n]$. If $\{A_j \mid j = 1, m\}$ is such a partition (i.e. A_j is also an n-dimensional box) and if $F_{Y|A_i}(y)$ is the c.d.f. of the variable $Y \mid X \in A_i$ then we can combine these c.d.f.s to reproduce the c.d.f. of interest by setting

$$F_Y(y) = \sum_{j=1}^{m} F_{Y|A_i}(y) \Pr ob\left(\vec{X} \in A_j\right)$$

The next step of the method is to construct an estimate $\tilde{F}_{Y|A_i}(y) \approx F_{Y|A_i}(y)$. For this we use the formula

$$\tilde{F}_{Y|A_i}\left(f\left(F^{-1}_{X_1|A_i}(\beta), ..., F^{-1}_{X_n|A_i}(\beta)\right)\right) = \beta \tag{21}$$

to obtain

$$F_Y(y) \approx \tilde{F}_Y(y) = \sum_{j=1}^{m} \tilde{F}_{Y|A_i}(y) \Pr ob\left(\vec{X} \in A_j\right) \tag{22}$$

This estimate will converge as the measures of the A_j's decrease (see [2]).

For our problem we need to calculate $E(Y)$ where $Y = EU^*\left(W_{i+1}, ACPI_{i+1}, \vec{R}_i\right)$. For this we use the formula $E(Y) = \int_{-\infty}^{+\infty} (1 - F_Y(y))\, dy$. Replacing $F_Y(y)$ with $\tilde{F}_Y(y)$ from above and noting that

$$\begin{aligned} 1 - \tilde{F}_Y(y) &= 1 - \textstyle\sum_{j=1}^{m} \tilde{F}_{Y|A_i}(y) \Pr ob\left(\vec{X} \in A_j\right) \\ &= \textstyle\sum_{j=1}^{m} \Pr ob\left(\vec{X} \in A_j\right) - \sum_{j=1}^{m} \tilde{F}_{Y|A_i}(y) \Pr ob\left(\vec{X} \in A_j\right) \\ &= \textstyle\sum_{j=1}^{m} \Pr ob\left(\vec{X} \in A_j\right)\left(1 - \tilde{F}_{Y|A_i}(y)\right) \end{aligned}$$

we obtain

$$E(Y) \simeq \sum_{j=1}^{m} \Pr ob\left(\vec{X} \in A_j\right) \int_{-\infty}^{+\infty} \left(1 - \tilde{F}_{Y|A_i}(y)\right) dy$$

However for a continuously increasing distribution function we can use $\int_{-\infty}^{+\infty} (1 - F_Y(y))\, dy = \int_0^1 F_Y^{-1}(\beta)\, d\beta$ and using the formula $\tilde{F}_{Y|A_i}^{-1}(\beta) = f\left(F_{X_1|A_i}^{-1}(\beta), ..., F_{X_n|A_i}^{-1}(\beta)\right)$ the problem can be restated as

$$E(Y) \simeq \sum_{j=1}^{m} \Pr ob\left(\vec{X} \in A_j\right) \int_0^1 f\left(F_{X_1|A_i}^{-1}(\beta), ..., F_{X_n|A_i}^{-1}(\beta)\right) d\beta \qquad (23)$$

For our problem $\vec{X}$ is a vector of i.i.d. standard normals. Thus, if $A_j = \Pi_{k=1}^{n-1}\left[a_k^l, a_k^u\right]$ and Φ is the c.d.f. of the standard normal

$$F_{X_i|A_j}^{-1}(\beta) = \sqrt{2}\,\mathrm{erf}^{-1}\left(2\left(\beta\left(\Phi(a_i^u) - \Phi\left(a_i^l\right)\right) + \Phi\left(a_i^l\right)\right) - 1\right)$$

and the function of evaluation is

$$f\left(\vec{X}\right) = EU^*\left(W_{i+1}, ACPI_{i+1}, \vec{R}_i\right)$$

where

$$ACPI_{i+1} = \vec{R}_i(12) * acpi_i$$

$$W_{i+1} = \max\left(0, w_i - \bar{D}_i\right) * \left(\vec{R}_i * \bar{L}_i\right)$$

$$\vec{R}_i = e^{I_i}$$

with

$$I_i = \vec{\phi}_0 + \vec{\phi}_1 I_{i-1} + \vec{X}A$$

and

$$I_{i=1} = \log(\vec{r}_{i-1})$$

We also need to evaluate partials with respect to each variable $\bar{L}_i(X_k)$ and $\bar{D}_i$ (for optimization) but this is straight forward as the partial derivatives pass through the integral.

3 Example

We will solve the problem for the case of an individual who retires at age 107. This will require finding the optimal expected utility (and distribution and allocation strategy) at age 108 for various possible scenarios of age 108 wealth, asset returns and inflation from age 107 to age 108, and using this data to construct the function $EU_{108}^*(w, acpi, \vec{r})$ (we can start at age 108 since from our assumptions we have set $mx = 109$). Then we will use this function to solve for the optimal age 107 distribution and allocation strategy. For realistic problems, with much younger retirement age, we would then continue backward in this manner to the age at retirement. The purpose of this example is simply to describe the method.

3.1 Data for Individual Retiring at age 107

$r = 107$

$mx = 109$

$W_r = \$130,000$

$fbft = \$5,000$

$fage = 107$

$ibft = \$4,000$

$iage = 107$

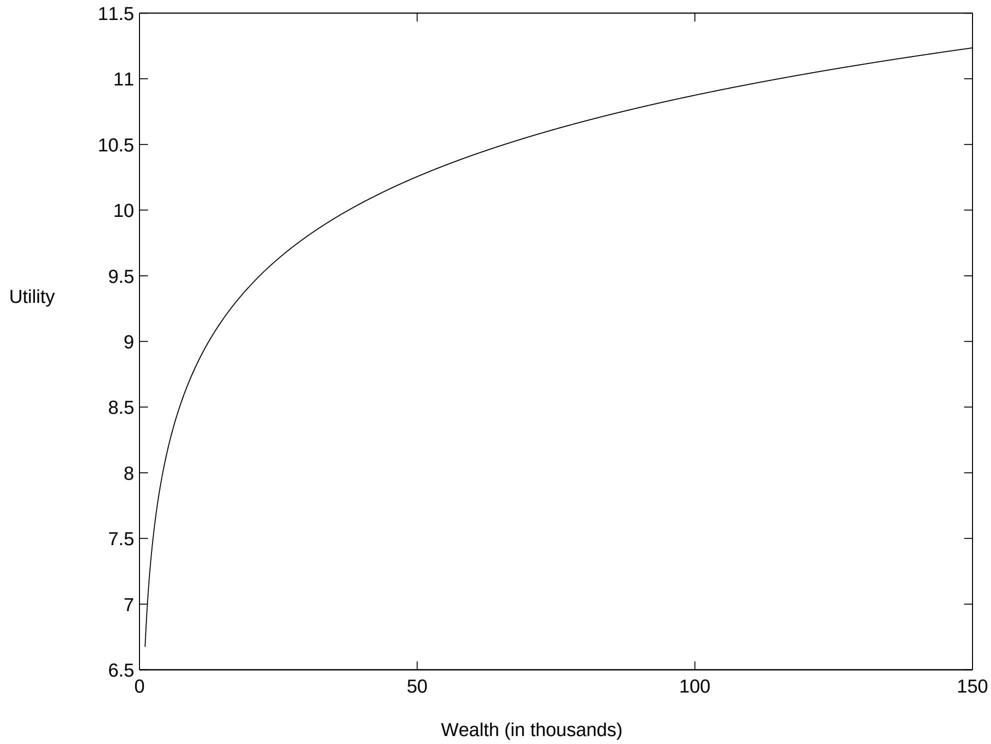
${}_{109}p_{108} = .525$

${}_{108}p_{109} = .526$

$U(x) = \frac{x^{-.01}-1}{-.01}$

$\vec{r}_{106} = [\ .7\ .7\ .8\ .8\ 1.04\ 1.03\ 1.01\ 1.06\ 1.02\ 1.02\ 1.03];$

For this utility function, the relative risk aversion is $R(x) = \frac{1.01}{x}$ which decreases as wealth increases.



We will assume the retiree can invest in the following asset classes and will experience inflation with the following characteristics (means, std.s and correlation coefficients are from the Watson Wyatt 2003 Asset Return Assumptions, serial correlations are estimated with some reference to Ibbotson historical values):

	Class	Mean	Std.	Serial Corr.
Large/Mid Cap Stocks	1	9.8%	17.2%	0
Small Cap Stocks	2	10.5	22.3	0
Non-US Developed Stocks (unhedged)	3	10.3	19.6	0
Emerging Market Stocks	4	12.7	32.6	0
Real Estate REITs	5	8.6	15.0	0
Fixed Income	6	4.7	7.2	0
Long-term Govt. Bonds	7	6.0	10.8	0
High Yield Bonds	8	8.0	14.0	0
TIPS Inflation Linked Bonds	9	4.5	6.1	.3
Non-US Bonds (unhedged)	10	5.0	11.0	0
Cash/Treasury Bills	11	3.5	2.1	.75
Inflation	12	2.4	3.1	.75

Corr.Coef.

asset class	1	2	3	4	5	6	7	8	9	10	11	inflation
1	1.00											
2	.76	1.00										
3	.68	.52	1.00									
4	.43	.32	.46	1.00								
5	.49	.56	.39	-.06	1.00							
6	.33	.26	.11	-.06	.22	1.00						
7	.41	.33	.14	.02	.25	.97	1.00					
8	.56	.34	.40	.15	.05	.46	.47	1.00				
9	-.11	-.15	-.26	-.04	-.08	.28	.21	-.08	1.00			
10	.08	.06	.54	.18	.08	.22	.22	.13	-.22	1.00		
11	.03	.02	.12	.04	-.04	-.05	-.13	-.08	.26	-.10	1.00	
inflation	-.24	-.22	-.19	.00	-.17	-.11	-.18	-.26	.65	-.24	.39	1.00

3.2 Results

In experimentation we found that, for our example, the age 108 maximum expected utility was insensitive to the prior year asset returns since the allocation of assets did not include assets with a non-zero auto regression term. Thus the age 108 maximum expected utility is only a function of wealth at age 108 and inflation from age 107 to age 108. Using simulation we estimated that for the bulk of the probability, wealth at age 108 will fall in the range zero to $338,000 and inflation from age 107 to age 108 in the range of -6% to 10%. Using these values we solved for the optimal expected utility at age 108 ($EU^*_{108}(w, acpi)$) for various values of inflation and wealth. The results (weighted by the factor $(1 +_{109} p_{108}) /_{108}p_{107}$ for scaling purposes) are shown in the following table.

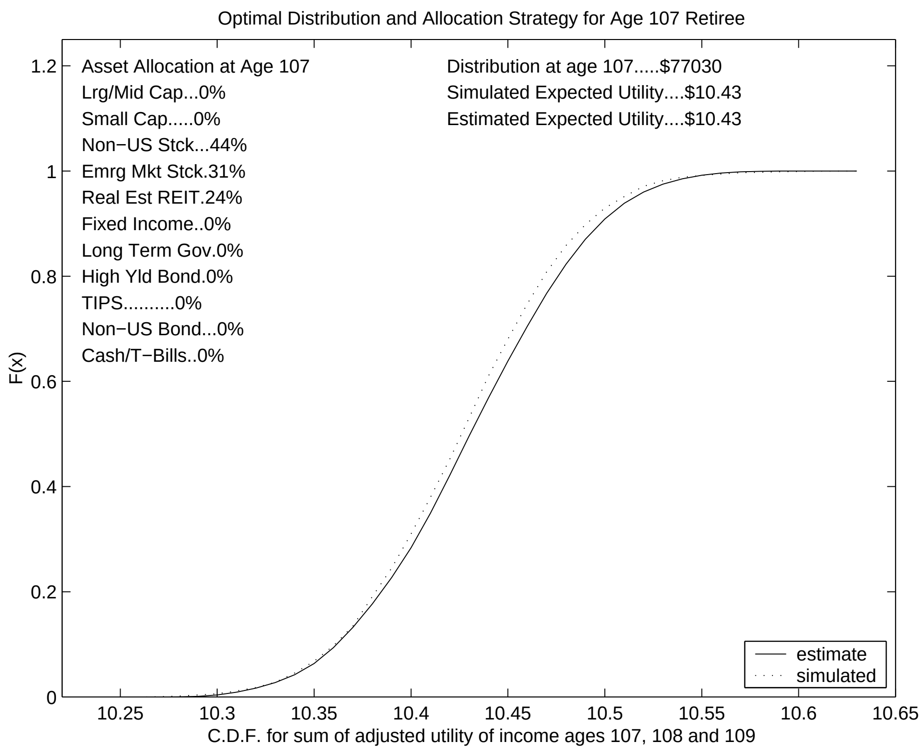
		Inflation				
		-6%	-2%	2%	6%	10%
	$500	8.76	8.74	8.72	8.70	8.68
	38,000	9.84	9.80	9.77	9.74	9.72
	75,500	10.31	10.27	10.24	10.21	10.18
	113,000	10.61	10.57	10.54	10.51	10.48
Wealth	150,500	10.84	10.80	10.77	10.73	10.70
	188,000	11.02	10.98	10.95	10.91	10.88
	225,500	11.17	11.13	11.10	11.06	11.03
	263,000	11.29	11.26	11.22	11.19	11.16
	300,500	11.41	11.37	11.33	11.30	11.27
	338,000	11.50	11.47	11.43	11.40	11.37

The optimal distribution for age 108 for each outcome is as follows:

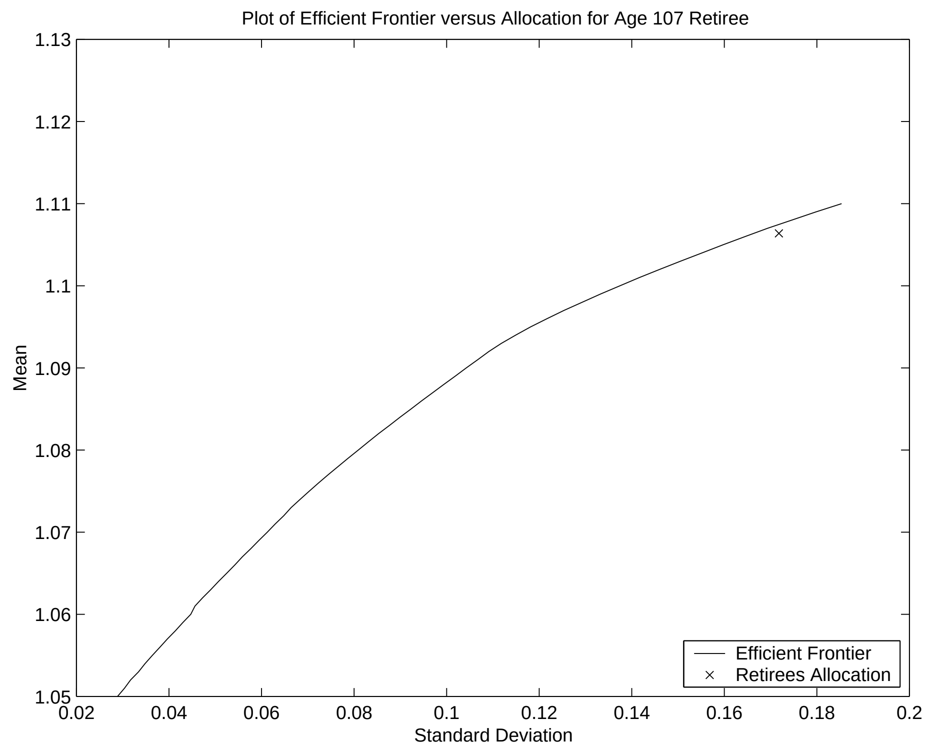
		Inflation				
		-6%	-2%	2%	6%	10%
	$500	500	500	500	500	500
	38,000	27,183	27,249	27,254	27,312	27,318
	75,500	51,757	51,805	51,832	51,888	51,919
	113,000	76,300	76,355	76,434	76,437	76,476
Wealth	150,500	100,859	100,908	100,931	100,989	100,996
	188,000	125,373	125,429	125,492	125,555	125,594
	225,500	149,954	149,989	150,003	150,054	150,092
	263,000	174,463	174,563	174,589	174,629	174,699
	300,500	199,036	199,059	199,111	199,161	199,182
	338,000	223,577	223,637	223,645	223,672	223,737

We used these values to estimate the function $EU^*_{108}(w, acpi)$ using spline interpolation. Then this function was used to solve for $EU^*_{107}(130000, 1, \vec{r}_{106})$

for the age 107 retiree. The optimal distribution and asset allocation for the age 107 retiree (derived by admittedly non-optimal matlab coding) is shown in the following graph. The graph also shows the estimated optimal expected utility and c.d.f. of age 107, 108 plus 109 utility (scaled by the factor $(1 +_{107} p_{108} +_{107} p_{109})^{-1}$) along with the same values derived from the empirical c.d.f. from a 5000 simulation run.



This allocation has an expected return of 10.64% and standard deviation of 17.18 and is close to being efficient under the Markowitz model as shown in the following figure. The standard deviation of an efficient portfolio with the same mean is 16.63 and for the fixed distribution at age 107 of $77,030 both produce the same expected utility of 10.43 suggesting that both are considered optimal under the tolerances of the coding used for our example. It is also interesting to note that the allocation is very aggressive indicating that the retiree is very nearly risk neutral at this wealth level, adopting a strategy that almost maximizes expected return.



4 Conclusion

We have provided an outline of a method for finding an optimal distribution and investment strategy for an individual retiree and solved the problem for an individual retiring at age 107 which demonstrates that, in principle, the problem can be solved for the more interesting case of retirees with much younger ages. Areas of future research include studying what utility functions might be more appropriate for a given retiree, allowing the utility function to vary over time, including joint life models, testing the sensitivity of the model to asset model parameters, and allowing for uncertainty in asset model parameters and incorporating this uncertainty into the problem.

5 Bibliography

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6 Appendix A - Derivation of Asset Model Parameters

In this section we show the derivation of the model parameters $\vec{\phi}_0$, $\vec{\phi}_1$ and A. Recall that R_t is an n-dimensional column vector whose n-th entry is the rate of inflation for year beginning at $t-1$ and whose i-th entry, for $i \neq n$, is the rate of return on asset class i for the same one-year period and that

$$R_t = e^{I_t}$$

where I_t is the n-dimensional column vector

$$I_t = \vec{\phi}_0 + \vec{\phi}_1 I_{t-1} + XA$$

with $X \overset{i.i.d.}{\sim} N(0, Id_n)$ (where Id_n is the n-dimensional identity matrix).

Assume we know $\mu_{\vec{R}_i}$, $\Sigma_{\vec{R}_i}$ and the regression coefficients for R, ρ^2_R. Then we seek $\vec{\phi}_0$, $\vec{\phi}_1$ and A in terms of these knowns.

For this model we have ([1])

$$I_t \sim N\left(\mu_{I_t}, \Sigma_{I_t}\right)$$

where

$$\mu_I(i) = \frac{\vec{\phi}_0(i)}{\left(1 - \vec{\phi}_1(i)\right)}$$

$$\Sigma_I(i,j) = \frac{1}{\left(1 - \vec{\phi}_1(i)\,\vec{\phi}_1(j)\right)} \Sigma_{\vec{Y}}(i,j)$$

Consider the case $n = 1$, then since I_{-1} is a known constant,

$$\begin{array}{l} I_0 = \vec{\phi}_0 + \vec{\phi}_1 I_{-1} + X_0 A \\ I_1 = \vec{\phi}_0 + \vec{\phi}_1 \left(\vec{\phi}_0 + \vec{\phi}_1 I_{-1} + X_0 A\right) + X_1 A \\ \quad = \vec{\phi}_0 + \vec{\phi}_1 \vec{\phi}_0 + \vec{\phi}_1^2 I_{-1} + \vec{\phi}_1 X_0 A + X_1 A \\ \ldots \\ I_t = \vec{\phi}_0 \Sigma_{j=0}^t \vec{\phi}_1^j + \vec{\phi}_1^{t+1} I_{-1} + \Sigma_{j=0}^t \vec{\phi}_1^{t-j} X_j A \end{array}$$

and since $cov(X_i A, X_j A) = \begin{cases} 0 & \text{if } i \neq j \\ \Sigma_{\vec{Y}} = \sigma^2_{\vec{Y}} & \text{if } i = j \end{cases}$ ($X_i A$ and $X_j A$ are *i.i.d.*),

$$\begin{array}{l} cov(I_t, I_{t+1}) = \\ cov\left(\vec{\phi}_0 \Sigma_{j=0}^t \vec{\phi}_1^j + \vec{\phi}_1^{t+1} I_{-1} + \Sigma_{j=0}^t \vec{\phi}_1^{t-j} X_j A, \vec{\phi}_0 \Sigma_{j=0}^{t+1} \vec{\phi}_1^j + \vec{\phi}_1^{t+2} I_{-1} + \Sigma_{j=0}^{t+1} \vec{\phi}_1^{t+1-j} X_j A\right) \\ \qquad = \Sigma_{j=0}^t \vec{\phi}_1^{t-j} \vec{\phi}_1^{t+1-j} \sigma^2_{\vec{Y}} \\ \qquad = \frac{\vec{\phi}_1}{\left(1 - \vec{\phi}_1^2\right)} \sigma^2_{\vec{Y}} \end{array}$$

Thus $Z = (I_t(i), I_{t+1}(i))^T \sim N(\mu_Z, \Sigma_Z)$ where

$$\mu_Z = \left(\mu_{I_t(i)}, \mu_{I_{t+1}(i)}\right)^T$$

and

$$\Sigma_Z = \begin{pmatrix} \Sigma_{I_t}(i,i) & \frac{\vec{\phi}_1(i)}{\left(1-\vec{\phi}_1^2(i)\right)}\Sigma_{I_t}(i,i) \\ \frac{\vec{\phi}_1(i)}{\left(1-\vec{\phi}_1^2(i)\right)}\Sigma_{I_t}(i,i) & \Sigma_{I_t}(i,i) \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{\left(1-\vec{\phi}_1^2(i)\right)}\rho^2_{\vec{Y}(i)} & \frac{\phi_1(i)}{\left(1-\vec{\phi}_1^2(i)\right)^2}\rho^2_{\vec{Y}(i)} \\ \frac{\phi_1(i)}{\left(1-\vec{\phi}_1^2(i)\right)^2}\rho^2_{\vec{Y}(i)} & \frac{1}{\left(1-\vec{\phi}_1^2(i)\right)}\rho^2_{\vec{Y}(i)} \end{pmatrix}$$

Now we know that for $\vec{Y} = e^X$ where $X \sim N(\mu_X, \Sigma_X)$ then ([4])

$$\mu_{\vec{Y}(i)} = \exp\left(\mu_{X(i)} + \frac{1}{2}\sigma^2_{X(i)}\right)$$

$$\sigma_{\vec{Y}(i)} = \exp\left(\mu_{X(i)}\right)\left\langle \exp\left(\sigma^2_{X(i)}\right)\left[\exp\left(\sigma^2_{X(i)} - 1\right)\right]\right\rangle^{\frac{1}{2}}$$

and

$$\rho_{\vec{Y}(i)\vec{Y}(j)} = \frac{\exp\left[\frac{1}{2}\left(\sigma^2_{X(i)} + \sigma^2_{X(j)}\right)\right]\exp\left(\rho_{X(i)X(j)}\sigma_{X(i)}\sigma_{X(j)} - 1\right)}{\left[\exp\left(\sigma^2_{X(i)}\right)\left(\exp\left(\sigma^2_{X(i)}\right) - 1\right)\exp\left(\sigma^2_{X(j)}\right)\left(\exp\left(\sigma^2_{X(j)}\right) - 1\right)\right]^{\frac{1}{2}}}$$

But by assumption $\mu_{\vec{Y}}$, $\frac{1}{2}\sigma_{\vec{Y}}$ and $\rho_{\vec{Y}(i)\vec{Y}(j)}$ are the known constants. Solving for μ_X, $\frac{1}{2}\sigma^2_X$ and then $\rho_{X(i)X(j)}$ yields

$$\mu_{X(i)} = \log\left(\mu^2_{\vec{Y}(i)}\left(\frac{1}{\mu^2_{\vec{Y}(i)} + \sigma^2_{\vec{Y}(i)}}\right)^{\frac{1}{2}}\right)$$

$$cov_{X(i)X(j)} = \begin{cases} \log\left(1 + \frac{\sigma^2_{\vec{Y}(i)}}{\mu^2_{\vec{Y}(i)}}\right) & \text{if } i = j \\ \log\left(1 + \frac{\sigma_{\vec{Y}(i)}\sigma_{\vec{Y}(j)}}{\mu_{\vec{Y}(i)}\mu_{\vec{Y}(j)}}\rho_{\vec{Y}(i)\vec{Y}(j)}\right) & \text{otherwise} \end{cases}$$

This allows as to solve for $\vec{\phi}_0$, ϕ_1 and A in term of $\mu_{\vec{R}_i}$, $\Sigma_{\vec{R}_i}$ and the regression coefficients for $\vec{R}_i$ using the following relationships.

When $i = j$

$$\sigma^2_{I(i)} = \frac{1}{\left(1 - \vec{\phi}^2_1(i)\right)} \sigma^2_{\vec{Y}(i)} = \log\left(1 + \frac{\sigma^2_{R(i)}}{\mu^2_{R(i)}}\right)$$

and when $i \neq j$

$$cov_{I(i)I(j)} = \frac{1}{\left(1 - \vec{\phi}_1(i)\,\vec{\phi}_1(j)\right)} cov_{\vec{Y}(i)\vec{Y}(j)} = \log\left(1 + \frac{\sigma_{R(i)}\sigma_{R(j)}}{\mu_{R(i)}\mu_{R(j)}}\rho_{R(i)R(j)}\right)$$

$$\mu_I(i) = \frac{\vec{\phi}_0(i)}{\left(1 - \vec{\phi}_1(i)\right)} = \log\left(\mu^2_{R(i)}\left(\frac{1}{\mu^2_{R(i)} + \sigma^2_{R(i)}}\right)^{\frac{1}{2}}\right)$$

and

$$cov_{I_t(i)I_{t+1}(i)} = \frac{\vec{\phi}_1(i)}{\left(1 - \vec{\phi}^2_1(i)\right)} \sigma^2_{\vec{Y}(i)} = \log\left(1 + \frac{\sigma^2_{R(i)}}{\mu^2_{R(i)}}\rho_{R_t(i)R_{t+1}(i)}\right)$$

The results are as follows:

$$\sigma^2_{\vec{Y}(i)} = \left(1 - \vec{\phi}^2_1(i)\right)\log\left(1 + \frac{\sigma^2_{R(i)}}{\mu^2_{R(i)}}\right)$$

gives (when substituted into the last equation)

$$\vec{\phi}_1(i) = \left(\log\left(1 + \frac{\sigma^2_{R(i)}}{\mu^2_{R(i)}}\right)\right)^{-1} \log\left(1 + \frac{\sigma^2_{R(i)}}{\mu^2_{R(i)}}\rho_{R_t(i)R_{t+1}(i)}\right)$$

which gives

$$\vec{\phi}_0(i) = \left(1 - \vec{\phi}_1(i)\right)\log\left(\mu^2_{R(i)}\left(\frac{1}{\mu^2_{R(i)} + \sigma^2_{R(i)}}\right)^{\frac{1}{2}}\right)$$

and when $i \neq j$

$$cov_{\vec{Y}(i)\vec{Y}(j)} = \left(1 - \vec{\phi}_1(i)\,\phi_1(j)\right)\log\left(1 + \frac{\sigma_{R(i)}\sigma_{R(j)}}{\mu_{R(i)}\mu_{R(j)}}\rho_{R(i)R(j)}\right)$$

when $i = j$

$$\sigma^2_{\vec{Y}(i)} = \left(1 - \vec{\phi}_1^2(i)\right) \log\left(1 + \frac{\sigma^2_{R(i)}}{\mu^2_{R(i)}}\right)$$

The only remaining task is to determine the matrix A. But we now have $\Sigma_{\vec{Y}}$ and since $\vec{Y} = XA$ where $X \sim N(0, Id_n)$, $\vec{Y}$ has a joint normal distribution with covariance matrix $\Sigma_{\vec{Y}} = A^T A$. Using the singular value decomposition of

$$\Sigma_{\vec{Y}} = U * S * V^T$$

where U and V are unitary and S is a diagonal matrix with non-negative entries and $U = V$ (since $\Sigma_{\vec{Y}}$ is symmetric). This gives us

$$A = (S)^{\frac{1}{2}} * V^T$$

where the square root is taken component wise.

6.1 Appendix B - Calculations for the Example

In this section we give the details of the equations used to solve the problem for the age 107 retiree.

6.1.1 Calculations for Age 108 Retiree - Finding EU$_{108}$

We give the calculation details for the formation of the problem for an individual who retirees at age 108 with wealth w_{108}, a fixed pension of $fbft$ and an indexed pension of $ibft$. Assume returns on the various asset classes for the prior year were $\vec{r}_{107}$.We assume the fixed and inflation adjusted benefits are in payment status.

At age 109

$$EU^*_{109}(w, acpi, \vec{r}) = U\left(\frac{w + ibft * acpi + fbft}{acpi}\right)$$

since distributing all assets for consumption during the last year of life is the utility maximizing strategy. Therefore our optimization problem for the age 108 retiree is

$$\max_{\bar{D}_{108}, \bar{L}_{108}} \left(U\left(\bar{D}_{108} + ibft + fbft\right) +_{109} p_{108} * E\left(U\left(\frac{W_{109} + ibft * ACPI_{109} + fbft}{ACPI_{109}}\right)\right)\right)$$

where

$$W_{109} = \left(w_{108} - \bar{D}_{108}\right) * \left(\vec{R}_{108} * \bar{L}_{108}\right)$$

$$ACPI_{109} = \vec{R}_{108}\,(12)$$

$$\vec{R}_{108} = e^{I_{108}}$$

$$I_{108} = \vec{\phi}_0 + \vec{\phi}_1 \vec{i}_{107} + XA \;(\vec{\phi}_1 \vec{i}_{107} \text{ is component wise multiplication})$$

$$\vec{i}_{107} = \log\left(\vec{r}_{107}\right)$$

Subject to

$$\begin{array}{l} \bar{D}_{108} \leq w_{108} \\ \Sigma_{j=1}^{11} \bar{L}_{108}\,(j) = 1 \\ 0 \leq \bar{L}_{108}\,(j)\ \forall j \leq 11 \end{array}$$

For our calculations we truncated the underlying normal variables at ± 3 stds. Let $\{A_j \mid j = 1, m\}$ be a partition of $[-3, 3]^{12}$ where each $A_j = \Pi_{i=1}^{12}\left[a_{j,i}^l, a_{j,i}^u\right]$ (so each A_j is a product of intervals). Let

$$f\left(\vec{X}\right) = \left(U\left(\frac{W_{109} + ibft * ACPI_{109} + fbft}{ACPI_{109}}\right)\right)$$

Then, letting $\Phi(x) = \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}} e^{-t^2/2\sigma^2} dt = \frac{1}{2}\,\mathrm{erf}\left(\frac{x}{\sqrt{2}}\right) + \frac{1}{2}$, the c.d.f. for the unit normal where erf is the error function, when f is an increasing function of X_i

$$F_{X_i|A_j}^{-1}(\beta) = \sqrt{2}\,\mathrm{erf}^{-1}\left(2\left(\beta\left(\Phi\left(a_{j,i}^u\right) - \Phi\left(a_{j,i}^l\right)\right) + \Phi\left(a_{j,i}^l\right)\right) - 1\right)$$

and when it is a decreasing function of X_i

$$F_{X_i|A_j}^{-1}(\beta) = \sqrt{2}\,\mathrm{erf}^{-1}\left(2\left((1-\beta)\left(\Phi\left(a_{j,i}^u\right) - \Phi\left(a_{j,i}^l\right)\right) + \Phi\left(a_{j,i}^l\right)\right) - 1\right)$$

and

$$prob\left(\vec{X} \in A_j\right) = \Pi_{i=1}^{12}\left(\Phi\left(a_{j,i}^u\right) - \Phi\left(a_{j,i}^l\right)\right) / \left(\Phi(3) - \Phi(-3)\right)^{12}$$

This gives us all of the pieces needed to estimate,

$$E\left(f\left(\vec{X}\right)\right) \simeq \sum_{j=1}^{m} \Pr ob\left(\vec{X} \in A_j\right) \int_0^1 f\left(F_{X_1|A_i}^{-1}(\beta), ..., F_{X_n|A_i}^{-1}(\beta)\right) d\beta$$

In summary, our objective function becomes

$$\max_{\bar{D}_i,\bar{L}_i}[U\left(\bar{D}_{108}+ibft+fbft\right)+_{109}p_{108}*\sum_{j=1}^{m}\Pr ob\left(\vec{X}\in A_j\right)$$
$$\int_0^1 U\left(\frac{\left(w_{108}-\bar{D}_{108}\right)*\left(\vec{R}_{108}*\bar{L}_{108}\right)+ibft*ACPI_{109}+fbft}{ACPI_{109}}\right)d\beta]$$

For the gradient we have

$$\frac{\vartheta}{\vartheta\bar{D}_{108}}\left(U\left(\bar{D}_{108}+ibft+fbft\right)+_{109}p_{108}E\left(U\left(\frac{W_{109}+ibft*ACPI_{109}+fbft}{ACPI_{109}}\right)\right)\right)\simeq$$
$$\left(\bar{D}_{108}+ibft+fbft\right)^{-1.01}+$$
$$_{109}p_{108}\sum_{j=1}^{m}\Pr ob\left(\vec{X}\in A_j\right)*$$
$$\int_0^1\frac{-\vec{R}_{108}*\bar{L}_{108}}{ACPI_{109}}\left(\frac{\left(w_{108}-\bar{D}_{108}\right)*\left(\vec{R}_{108}*\bar{L}_{108}\right)+ibft*ACPI_{109}+fbft}{ACPI_{109}}\right)^{-1.01}d\beta$$

and

$$\frac{\vartheta}{\vartheta\bar{L}_{108}(i)}\left(U\left(\bar{D}_{108}+ibft+fbft\right)+_{109}p_{108}*E\left(U\left(\frac{W_{109}+ibft*ACPI_{109}+fbft}{ACPI_{109}}\right)\right)\right)\simeq$$
$$_{109}p_{108}*\sum_{j=1}^{m}\Pr ob\left(\vec{X}\in A_j\right)*$$
$$\int_0^1\frac{\left(w_{108}-\bar{D}_{108}\right)*\vec{R}_{108}(i)}{ACPI_{109}}\left(\frac{\left(w_{108}-\bar{D}_{108}\right)*\left(\vec{R}_{108}*\bar{L}_{108}\right)+ibft*ACPI_{109}+fbft}{ACPI_{109}}\right)^{-1.01}d\beta$$

and second partials are

$$\frac{\vartheta^2}{\vartheta^2\bar{D}_{108}}\left(U\left(\bar{D}_{108}+ibft+fbft\right)+_{109}p_{108}E\left(U\left(\frac{W_{109}+ibft*ACPI_{109}+fbft}{ACPI_{109}}\right)\right)\right)\simeq$$
$$-1.01\left(\bar{D}_{108}+ibft+fbft\right)^{-2.01}+$$
$$_{109}p_{108}\sum_{j=1}^{m}\Pr ob\left(\vec{X}\in A_j\right)*$$
$$\int_0^1-1.01\left(\frac{\vec{R}_{108}*\bar{L}_{108}}{ACPI_{109}}\right)^2\left(\frac{\left(w_{108}-\bar{D}_{108}\right)*\left(\vec{R}_{108}*\bar{L}_{108}\right)+ibft*ACPI_{109}+fbft}{ACPI_{109}}\right)^{-2.01}d\beta$$

$$\frac{\vartheta}{\vartheta\bar{L}_{108}(j)\vartheta\bar{D}_{108}}\left(U\left(\bar{D}_{108}+ibft+fbft\right)+_{109}p_{108}E\left(U\left(\frac{W_{109}+ibft*ACPI_{109}+fbft}{ACPI_{109}}\right)\right)\right)\simeq$$
$$_{109}p_{108}\sum_{j=1}^{m}\Pr ob\left(\vec{X}\in A_j\right)*$$
$$\int_0^1[\frac{-\vec{R}_{108}(i)}{ACPI_{109}}\left(\frac{\left(w_{108}-\bar{D}_{108}\right)*\left(\vec{R}_{108}*\bar{L}_{108}\right)+ibft*ACPI_{109}+fbft}{ACPI_{109}}\right)^{-1.01}+$$
$$\frac{1.01*\vec{R}_{108}*\bar{L}_{108}*\left(w_{108}-\bar{D}_{108}\right)*\vec{R}_{108}(i)}{ACPI_{109}^2}\left(\frac{\left(w_{108}-\bar{D}_{108}\right)*\left(\vec{R}_{108}*\bar{L}_{108}\right)+ibft*ACPI_{109}+fbft}{ACPI_{109}}\right)^{-2.01}d\beta$$

$$\frac{\vartheta^2}{\vartheta\bar{L}_{108}(i)\vartheta\bar{L}_{108}(j)}\left(U\left(\bar{D}_{108}+ibft+fbft\right)+_{109}p_{108}*E\left(U\left(\frac{W_{109}+ibft*ACPI_{109}+fbft}{ACPI_{109}}\right)\right)\right)\simeq$$
$$_{109}p_{108}*\sum_{j=1}^{m}\Pr ob\left(\vec{X}\in A_j\right)*\int_0^1-1.01\left(\frac{\left(w_{108}-\bar{D}_{108}\right)*\vec{R}_{108}(i)}{ACPI_{109}}\right)$$
$$\left(\frac{\left(w_{108}-\bar{D}_{108}\right)*\vec{R}_{108}(i)}{ACPI_{109}}\right)\left(\frac{\left(w_{108}-\bar{D}_{108}\right)*\left(\vec{R}_{108}*\bar{L}_{108}\right)+ibft*ACPI_{109}+fbft}{ACPI_{109}}\right)^{-2.01}d\beta$$

6.1.2 Calculations for age 107 Retiree

In the analysis we found that the function $EU_{108}(w, acpi, r)$ was insensitive to the prior year return components r, therefore we can treat this as the two dimensional function $EU_{108}(w, acpi)$. This function is derived by interpolating over a matrix of values determined from the age 108 results. Then the optimization objective function is

$$\max_{\bar{D}_{107},\bar{L}_{107}} \left(U\left(\bar{D}_{107} + ibft + fbft\right) +_{108} p_{107} * E\left(EU\left(W_{108}, ACPI_{108}\right)\right)\right)$$

where

$$W_{108} = \left(w_{107} - \bar{D}_{107}\right) * \left(\vec{R}_{107} * \bar{L}_{107}\right)$$

$$ACPI_{108} = \vec{R}_{107}(12)$$

$$\vec{R}_{107} = e^{I_{107}}$$

$$I_{107} = \vec{\phi}_0 + \vec{\phi}_1 \vec{i}_{106} + XA \ (\vec{\phi}_1 \vec{i}_{106} \text{ is component wise multiplication})$$

$$\vec{i}_{106} = \log\left(\vec{r}_{106}\right)$$

Subject to

$$\begin{array}{l} \bar{D}_{107} \leq w_{107} \\ \Sigma_{j=1}^{11} \bar{L}_{107}(j) = 1 \\ 0 \leq \bar{L}_{107}(j) \ \forall j \leq 11 \end{array}$$

The partial derivatives are as follows:

$$\frac{\vartheta}{\vartheta \bar{D}_{107}} \left(U\left(\bar{D}_{107} + ibft + fbft\right) +_{108} p_{107} * E\left(EU\left(W_{108}, ACPI_{108}\right)\right)\right) = \left(\bar{D}_{107} + ibft + fbft\right)^{-1.01} + {}_{108}p_{107} * \int_0^1 \left(\frac{\left(-\vec{R}_{107} * \bar{L}_{107}\right)}{ACPI_{108}} * \frac{\vartheta}{\vartheta W_{108}} EU\left(W_{108}, ACPI_{108}\right)\right) d\beta$$

and

$$\frac{\vartheta}{\vartheta \bar{L}_{107}(i)} \left(U\left(\bar{D}_{107} + ibft + fbft\right) +_{108} p_{107} * E\left(EU\left(\frac{W_{108}}{ACPI_{108}}\right)\right)\right) = {}_{108}p_{107} * \int_0^1 \left(\frac{\left(w_{107} - \bar{D}_{107}\right) * \left(\vec{R}_{107}(i)\right)}{ACPI_{108}} * \frac{\vartheta}{\vartheta W_{108}} EU\left(W_{108}, ACPI_{108}\right)\right) d\beta$$

since only W_{108} is a function of $\bar{D}_{107}$ and $\bar{L}_{107}$.