Solving Large-Scale Fuzzy and Possibilistic Optimization Problems: Theory, Algorithms and Applications

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Abstract: Fuzzy and possibilistic optimization are studied in the context of three methods for solving large fuzzy and possibilistic optimization problems. In particular, an optimization problem in radiation therapy with various orders of complexity from $1,000$ to $55,500$ *constraints for fuzzy and possibilistic linear and nonlinear programming implementations possessing (i) fuzzy inequalities, (ii) fuzzy right-hand side values, and (iii) possibilistic right-hand side values is used to test the performance of the three fuzzy and possibilistic optimization methods. We focus on the uncertainty in the right side which arises, in the context of the radiation therapy problem, from the fact that minimal and maximal radiation tolerances are target values rather than fixed real numbers. The results indicate that fuzzy/possibilistic optimization is a natural way to model various types of optimization under uncertainty problems and that large fuzzy and possibilistic optimization problems can be solved efficiently.*
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1 Introduction

Many of the hardest optimization problems are those that contain uncertainty because the meanings of inequalities and optima must be defined in the context of the problem in question. Moreover, the complexity of uncertain optimization is formidable. Our research focuses on three approaches to fuzzy and possibilistic uncertainty optimization - (1) **fuzzy optimization** of Tanaka, Okuda and Asai, [15], and Zimmermann, [18]; (2) the **fuzzy optimization** based on the surprise functions of Neumaier, [14] and [13]; and (3) the **possibilistic optimization** of Jamison and Lodwick, [6]. We assume that the reader is familiar with fuzzy set, possibilistic theory, and linear and nonlinear programming.

There are three aspects to our research that we present. Firstly, we solve very large fuzzy/possibilistic optimization problems, perhaps the largest reported application to date. Secondly, we make a case for separating fuzzy and possibilistic optimization both in terms of semantics as well as in computational methods. Lastly, we present new and updated algorithms for fuzzy and possibilistic optimization whose performance on large problems is excellent.

The purpose of this research is to demonstrate that the use of fuzzy and possibilistic optimization to solve large-scale optimization is not only tractable but the most direct way to model problems with embedded fuzzy and possibilistic uncertainty. To this end we present the three approaches, mentioned above, to solve a large-scale optimization problem where the uncertainty lies in right-hand side values. Moreover, we test a novel way to solve optimization under uncertainty (see [14]) and extend what has been done in [6] and [13]. The differences between fuzzy and possibilistic optimization are illustrated in our application.

This paper is organized as follows. This first introductory section contains the discussion of the general problem of optimization under uncertainty and the application that we consider. The second and third sections deal specifically with fuzzy and possibilistic optimization, respectively, and associated algorithms that will be used to solve the radiation therapy problem. The fourth section contains the exposition of the numerical experiments and their results. Conclusions are found in section five.

1.1 Optimization in the Presence of Uncertainty

The optimization problem that we consider is linear programming. Optimization under uncertainty is used here to mean optimization when at least one element of the input data is a real-valued interval, a real-valued random variable, a real-valued fuzzy number or a real-valued number described by a possibility/necessity distribution. We focus on large-scale fuzzy and possibilistic optimization. Large-scale stochastic optimization problems are well-studied. Interval numbers may be considered as a special case of fuzzy numbers so we do not study interval optimization separately.

There is often confusion about what is fuzzy and what is possibilistic optimization. These two types of optimization are *not* the same. Fuzzy and possibilistic entities have different meanings, semantics. Fuzzy and possibilistic optimization model different entities and the associated solution methods are different as we shall see. Fuzzy entities, as is well known, are sets with non-sharp boundaries in which there is a transition between elements that belong and elements that don't belong to the set. Possibilistic entities are sets that exist but the evidence associated with whether a particular element belongs to the set or not is incomplete or hard to obtain. Decision-making in the presence of fuzzy and possibilistic entities takes the following generic form (see [11] for the development). For notation, we use tilde to denote a fuzzy set, the Greek letter mu (subscripted by the fuzzy set from which it is derived) to denote a membership function and a circumflex to denote a possibility distribution.

1. *Decision Making Under Fuzzy Uncertainty:* Given the set of decisions $X = \{x \in \tilde{F} \text{ and } \tilde{G}\}$, find the optimal decision in the set X. That is,

$$\sup_{x \in X} \{\mu_F(x), \mu_G(x)\}. \tag{1}$$

   Note that the decision space X is a **crisp set** where $\tilde{F}$ and $\tilde{G}$ are **fuzzy sets.**

2. *Decision Making Under Possibilistic Uncertainty:* Given the set of decisions $\hat{Y} = \{\hat{y} \in F \text{ and } \hat{y} \in G\}$, find the optimal decision in the set $\hat{Y}$. That is, for $\hat{y}_F = \hat{y} \in F$ and $\hat{y}_G = \hat{y} \in G$

$$\sup_{\hat{y} \in \hat{Y}} U(\hat{y}_F, \hat{y}_G), \tag{2}$$

   where $U(\hat{y}_F, \hat{y}_G)$ represents a "utility" of the outcomes $\hat{y} \in F$ and $\hat{y} \in G$. Note that the decision space $\hat{Y}$ is a **set of possibility distributions** where F and G are **crisp sets.**

Very simply, fuzzy decision making selects from a set of crisp elements while possibility selects from a set of distributions. It should be noted that there are many ways to interpret U applied to the distributions associated with F and G. In this paper "and" is interpreted as solution to simultaneous inequalities (the intersection of individual sets as determined by each constraint) where the utility is measured by minimizing the penalized "expected value" of inequality violations. One form of optimization over distributions is to minimize the expected value of violations. There are, of course, many other approaches. An equivalent of "expected value" in the setting of possibility theory is the expected average, denoted EA below, (see [6]). Thus, for the expected average, the following form is obtained where $\hat{y}^{+}(\alpha)$ and $\hat{y}^{-}(\alpha)$ are the respectively right and left sides of the $\alpha - level$ of the possibility distribution $\hat{y}$. Let $\hat{y} = \{\hat{y}_F \text{ and } \hat{y}_G\} = \{\hat{y} \in F \text{ and } \hat{y} \in G\}$.

$$\begin{gathered}\sup_{\hat{y}\in\hat{Y}} U(\hat{y}_F, \hat{y}_G) = \\ \sup_{\hat{y}\in\hat{Y}} EA\{\hat{y}\} = \\ \sup_{\hat{y}\in\hat{Y}} \tfrac{1}{2}\int_0^1 [\hat{y}^{+}(\alpha) + \hat{y}^{-}(\alpha)]\, d\alpha.\end{gathered} \tag{3}$$

This study considers three cases: (1) the implementation of fuzzy optimization methods of Tanaka and Zimmermann for the case of **fuzzy inequalities** where the right-hand side value is an aspiration; (2) the implementation of fuzzy optimization methods using surprise functions for the case of **fuzzy right-hand side values** where the inequality is crisp but the right-hand side value is fuzzy; and (3) the implementation of possibilistic optimization methods (penalty on the weighted expected average of the constraint violations) for the case of **possibilistic right-hand side values** where the inequality is crisp but the right-hand side value is a possibilistic real number.

Fuzzy inequalities mean that the crisp right-hand side values are targets. Fuzzy right-hand side values mean that various values of the right-hand side as defined by the fuzzy set have different preferences as measured by the $\alpha - level$. Possibilistic right-hand side values mean that the entity described by the right-hand side exists but the research and empirical evidence does not support a single real-valued number but a distribution of possible values.

The inequality is called *soft* when it means "come as close as possible" to the crisp right-hand side value. Typically, the highest degree of attaining the inequality is to maximize the $\alpha - level$ of the soft inequalities altogether to the same maximal degree. These are like chance constraints in stochastic optimization. In the context of fuzzy optimization, the method that is used is called flexible programming. While the original method was to obtain simultaneously the largest (single) $\alpha - level$ for all constraints, there is no reason why one could not maximize the weighted sum of $\alpha - levels$, one for each soft constraint where different weights mean different levels of importance of attaining the target. In addition, if certain constraints were required to attain at least a minimum level, these could be obtained by adjusting the target or the $\alpha - level$ associated with the particular constraint(s). This generalization is the essence of the surprise function approach. Regardless, soft constraints are handled by flexible programming methods.

Right-hand side values that are fuzzy numbers (the values described on the right-hand side have non-sharp boundaries) translate, mathematically, into a problem of attaining the highest level of feasibility as a sum of $\alpha - levels$ of each constraint. To do this, surprise functions are used that penalize constraint violations dynamically within the range of tolerances specified by the membership function, where the preferred values that are closest to one are not penalized and the least preferred (values near the edges of the support) are infinitely penalized. In-between, of course, the penalties are less than infinity and greater than zero. The surprise function acts as a barrier function.

Right-hand side values represented by a possibility distribution indicate that evidence at hand is incomplete. The entity described by the right-hand side exists. However, for whatever reason, the evidence as to what the specific crisp value the entity attains is incomplete and the distribution describes the best information available and its value measured by the level of confidence as given by its $\alpha - level$. Jamison and Lodwick (see [6], [9]) deal directly with possibility distributions of parameters by considering all constraint violations as allowable at an *a-priori* cost/penalty and minimizing the expected average, a generalization of expected value (see [6]), and as such *use methods that consider all possible outcomes as a weighted expected average penalty.* That is, possibilistic uncertainty in optimization takes into account violations as penalties on all **outcomes** (of the state of the constraints). It optimizes over sets of (possibility) **distributions** so that it is possibilistic optimization. The Jamison and Lodwick approach is a possibilistic generalization of the recourse models in stochastic optimization where violations of constraints are considered as allowable up to a maximum but at a cost. In this case, something akin to recourse models in stochastic optimization, robust optimization, or mean/variance optimization, is used where constraint violations are allowed at a penalty and the objective cost is the average expected value of the penalty.

1.2 Application of Fuzzy and Possibilistic Optimization to Radiation Therapy

The use of particle beams to treat tumors is called the radiation therapy problem (RTP). Beams of particles, usually photons or electrons, are oriented at various angles and with varying intensities to deposit dose (energy/unit mass) to the tumor. The idea is to deposit as much dose as possible to the tumor while sparing normal tissue.

The process begins with the patient's computed tomography (CT) scan. Each CT image is examined to identify and contour the tumor and normal structures. The image is subsequently vectorized. Likewise, candidate beams are discretized into beamlets where each beamlet is the width of a CT pixel. A pixel is the mathematical entity or structure (a square in the two-dimensional case and a cube in three dimensions) that is used to represent a unit area or volume of the body at a particular location. For this study, we restrict ourselves to two-dimensional problems so that the analysis is done on a series of images that cover the tumor, each two-dimensional. In our experiments, we used only one image per tumor. There were two tumors considered, head and prostate. For each, head and prostate, the experiment consists of four pixel resolutions ($64 \times 64, 128 \times 128, 256 \times 256$, and 512×512) and one set of beams each at 10 equally spaced angles. Since we constrain the dosage at each pixel, the complexity of the problem goes from a maximum of 64^2 to 512^2 potential constraints. However, since all pixels are not in the paths of the radiation beams that hit the tumor and some are outside the body, we *a-priori* set the delivered dosages at these pixels to zero and remove them from our analysis. This corresponds to blocking the beam which is always done in practice. The identification of a set of beam angles and weights that provide a lethal dose to the tumor cells while sparing healthy tissue with a resulting dose distribution acceptable and approved by the attendant radiation oncologist is called a *treatment plan*. A dose transfer matrix A^T (representing how one unit of radiation intensity in each beamlet is deposited in pixels - for historical reasons we use a transpose to emphasize its origin as the discrete version of the inverse Radon transform), called here the *attenuation matrix*, specific to the patient's geometry, is formed where columns of A^T correspond to the beamlets and rows represent pixels. A component of a column of the matrix A^T is non-zero if the corresponding beamlet intersects a pixel in which case the value is the positive fraction of the area of the intersection of the pixels with the beamlet (otherwise it is zero). The beams are then attenuated according to a factor dependent on the distance from where the beam enters the body to a pixel within the body and the type of tissue at that pixel. The variables are vectors x that represent the beamlet intensities.

There are a variety of ways of treating this problem without uncertainty. Pixels may be constrained individually or grouped into one constraint. Under idealized assumptions (see [2] or [4]) the problem without uncertainty has the following form:

$$\min\ z = f(x) \tag{4}$$
$$\text{subject to}\ A^T x \leq b \tag{5}$$
$$0 \leq x \leq U. \tag{6}$$

In the RTP literature there is no agreement on what the objective function $f(x)$ should be. For example, one finds the following objective functions: minimize total radiation, maximize minimum tumor dosage, minimize radiation to critical structure(s), minimize the probability of healthy tissue complication, maximize the probability of delivering a tumorcidal dose, or minimize maximum critical structure dose. Typically, oncologists consider the RTP as one of coming as close as possible to values specified by a radiation oncologist but within a given interval. This is the approach we use.

1.3 Uncertainty Optimization of the RTP

The basic RTP translated into a mathematical programming problem is:

$$\begin{aligned}
\min\ z &= f(x) = c^T x \\
\text{subject to} &: \\
\text{body dosage } Bx &\leq b_{body} \\
\text{critical organ}_i \text{ dosage } C_i x &\leq c_i \quad i = 1, \dots, N \\
\text{maximum tumor dosage } Tx &\geq t_{\min} \\
\text{minimum tumor dosage } Tx &\leq t_{\max} \\
0 &\leq x \leq U,
\end{aligned} \tag{7}$$

where the rows of B are body pixels, C_i are critical organ i pixels, and T are tumor pixels, respectively derived from the attenuation matrix A^T associated with a particular patient's CT. Moreover, for simplicity we have chosen a particular objective function, $f(x) = c^T x$ which is to minimize total weighted radiation ($c > 0$). Let

$$b = \begin{bmatrix} b_{body} \\ c_1 \\ \vdots \\ c_N \\ -t_{\min} \\ t_{\max} \end{bmatrix}$$

and

$$A = \begin{bmatrix} B \\ C_1 \\ \vdots \\ C_N \\ -T \\ T \end{bmatrix}.$$

Then the RTP is

$$\begin{aligned} \min\ z &= c^T x \\ \text{subject to}\quad Ax &\leq b \\ 0 &\leq x \leq U. \end{aligned}$$

We thus have two RTP mathematical representations that correspond to fuzzy optimization given below, equations (8) and (9), and one representation corresponding to possibilistic optimization, equation (10), where the specific target preferences of $\tilde{\leq}$, the fuzzy right-hand side levels $\tilde{b}$, and the possibilistic values of $\hat{b}$ are specified values given by a radiation oncologist.

We distinguish three optimization problems.

1. Fuzzy inequalities leading to flexible fuzzy optimization

$$\begin{aligned} \text{opt}\ z &= c^T x \\ \text{subject to}\quad Ax &\tilde{\leq} b \ \text{(soft constraint)} \\ 0 &\leq x \leq U \ \text{(hard constraint)}. \end{aligned} \tag{8}$$

2. Fuzzy number right-hand side values leading to surprise function fuzzy optimization

$$\begin{aligned} \text{opt}\ z = c^T x \\ \text{subject to}\quad Ax \leq \tilde{b} \ \text{(fuzzy number)} \\ 0 \leq x \leq U \ \text{(hard constraint)}. \end{aligned} \tag{9}$$

We emphasize that since $\tilde{b}$ is fuzzy, $Ax \leq \tilde{b}$ translates into a fuzzy set (as we shall see below) and the semantic is that we have crisp values belonging to this fuzzy set (at different degrees). Thus the constraint set is crisp.

3. Possibilistic right-hand values leading to penalized possibilistic optimization

$$\begin{array}{rcll} \text{opt } z & = & c^T x & \\ \text{subject to } \quad Ax & \leq & \hat{b} & \text{(possibilistic distribution)} \\ 0 & \leq & x \leq U & \text{(hard constraint).} \end{array} \tag{10}$$

We emphasize that the right-hand side of the inequality constraint, $\hat{b}$, is a possibilistic distribution. Thus, the constraint set, $\{\hat{y} \mid \hat{y} = Ax - \hat{b} \leq 0\}$, over which we are optimizing is a set of possibilistic distributions. For this case, right-hand side value $\hat{b}$ arises from the semantics of the problem. For example, it may arise because the information concerning the value of $\hat{b}$ (the value of safe or lethal dosages) is inconclusive (rather than its boundary being non-sharp which is the case were the right-hand side is fuzzy or the inequality is fuzzy).

2 Fuzzy Optimization

The two types of fuzzy optimization (8) and (9) are distinct. The first (Tanaka and Zimmermann) is the oldest of the approaches and its semantics are soft constraints, relaxation of the meaning of less than or equal to. The second (Neumaier) approach is one in which the right-hand values are fuzzy sets.

2.1 Tanaka and Zimmermann

Two researchers, first [15] and then [18], independently implemented the ideas of [1] in the context of mathematical programming problems. The idea of [1] was that constraint inequalities (and the objective function) were targets or goals. Thus, in the context of fuzzy uncertainty, membership functions (resulting fuzzy sets) of all constraints (and objective) were intersected. The fuzzy intersections are the "and" operation which, for membership functions, are minima. More recent research has studied the use of t-norms replacing the standard minimum definition of intersection though this leads to crisp nonlinear programming equivalents (see [8]) whereas the Zimmermann approach remains a linear programming problem. The optimization problem is thus to maximize the resulting function that is obtained by intersecting all constraint functions (and objective function).

The concrete implementation of this in optimization models, as given by [15] and then [18], deals with soft inequalities which in turn are translated into hard inequalities whose right hand sides are $\alpha - level$ values. In particular, we assume that if the i^{th} constraint is satisfied, that is, is less than or equal to b_i, we are 100% satisfied (membership function value of one). If the i^{th} constraint is greater than $b_i + d_i$, we are 100% dissatisfied (membership function value of 0). In-between, we linearly interpolate, that is, the right-hand side value is a fuzzy trapezoidal number $\tilde{b}_i = 0/0/b_i/b_i + d_i$. Assuming that the fuzzy optimization problem arose from the standard (crisp) linear programming problem,

$$\begin{aligned} \min\ z &= c^T x \\ \text{subject to} Ax &\leq b \\ 0 &\leq x \leq U, \end{aligned}$$

we have, due to a flexible meaning of the inequality,

$$\begin{aligned} \min\ c^T x &\tilde{\leq} Z_0 \\ \text{subject to} Ax &\tilde{\leq} b \\ 0 &\leq x \leq U, \end{aligned}$$

where the membership function associated with the i^{th} constraint (including the objective) is,

$$\mu_i(x) = \begin{cases} 1 & \text{for } \sum_{j=1}^{N} a_{ij}x_j \leq b_i \\ 1 - \frac{1}{d_i}\left(\sum_{j=1}^{N} a_{ij}x_j - b_i\right) & \text{for } b_i \leq \sum_{j=1}^{N} a_{ij}x_j \leq b_i + d_i \\ 0 & \text{otherwise.} \end{cases} \tag{11}$$

The symmetric optimization approach of [1] as translated into the crisp linear programming problem of [18] is:

$$\max\ z = \alpha$$

$$\text{subject to} \quad \alpha d_i + \sum_{j=1}^{N} a_{ij} x_j \leq b_i + d_i$$
$$0 \leq x \leq U.$$

We modify the above formulation and allow the original objective function to remain as part of the objective of the transformed problem to which we add (according to [18]) a (penalized) maximization of the $\alpha - level$. This is the so-called asymmetric fuzzy optimization problem. Thus for the RTP and our experiments, the revised fuzzy (flexible) Tanaka and Zimmermann linear program is:

$$\begin{aligned} \max\ z &= \alpha - c^T x \\ \text{subject to} \quad \alpha d_i + \sum_{j=1}^{N} a_{ij} x_j &\leq b_i + d_i \\ 0 &\leq x \leq U, \qquad 0 \leq \alpha \leq 1. \end{aligned} \tag{12}$$

2.2 Surprise

A goal satisfaction problem is one in which a set of goals is to be attained. When the goals are in fact uncertain (fuzzy), then a best compromise can be found by minimization of the surprise function associated with the fuzzy goal. Each fuzzy constraint is considered to have the form as $Ax \leq \tilde{b}$, where the $\tilde{b}_i$ are fuzzy target values. However, instead of maximizing the $\alpha - levels$ of **all** constraints to the same degree as in Tanaka and Zimmermann, the surprise function approach seeks the best compromise solution in the constraints and maximizes the **overall** combined $\alpha - levels$ by applying a *dynamic* penalty to violations of constraints as measured by *surprise functions* (see [14]) as follows.

$$\sum_{j=1}^{N} a_{ij} x_j \leq \tilde{b}_i$$
$$\Updownarrow$$
$$\sum_{j=1}^{N} a_{ij} x_j = \tilde{\xi}_i$$

where

$$\mu_i(\xi) = pos(\tilde{b}_i \geq \xi). \tag{13}$$

Note that we are using a possibility distribution to define a fuzzy membership function and the membership functions are well-defined. For trapezoidal and triangular fuzzy numbers $\tilde{b}_i$, obtaining $\mu_i(\xi)$ is straightforward. Next, we obtain the surprise functions from the membership functions, $\mu_i(\xi)$, using the following equation:

$$s_i(\xi) = ((\mu_i(\xi))^{-1} - 1)^2. \tag{14}$$

Thus, the fuzzy optimization using surprise functions is:

$$\begin{aligned} \min z &= \sum_{i=1}^{total\ pixels} s_i(\sum_{j=1}^{N} a_{ij} x_j) \\ \text{subject to } 0 &\leq x \leq U. \end{aligned} \tag{15}$$

It has been proved that the objective function is convex and, of course, quadratic (see [14]), so that we have a quadratic programming problem with simple bound constraints. Our implementation modified (16) by minimizing total dosage to the objective and adding the hard constraint that a minimal dosage is delivered to the tumor pixels. The latter was done to improve the quality of the solutions. Moreover, in the implementations we have modified the original formulation to include the minimization of total radiation in the objective function. Adding $c^T x$ to the objective function does not change its quadratic form. That is,

$$\begin{aligned} \min z &= c^T x + \sum_{i=1}^{total\ pixels} s_i(\sum_{j=1}^{N} a_{ij} x_j) \\ \text{subject to} \sum_{j=1}^{N} T_{ij} x_j &\geq t_{\min} \\ 0 &\leq x \leq U. \end{aligned} \tag{16}$$

3 Possibilistic Optimization

Possibilistic optimization arises when the optimization model has at least one parameter that is a possibility distribution. When the right-hand side of the constraints consist of possibilistic distributions, the constraints may be violated at a cost. In this case we use a revised version of the Jamison and Lodwick possibilistic optimization [6] as follows:

$$
\begin{array}{lc}
\min & z = c^T x + p_B EA\left(\max\left(0, Bx - \hat{b}_{body}\right)\right) + \\
& \Sigma_{i=1}^{n} p_{C_i} EA\left(\max\left(0, C_i x - \hat{c}_i\right)\right) + p_T EA\left|Tx - \hat{t}\right| \\
\text{subject to} & 0 \le x \le U,
\end{array}
\tag{17}
$$

where, for the possibilistic distribution $\hat{f}$,

$$
EA\left(\hat{f}\right) = \frac{1}{2}\int_0^1 \left(\hat{f}^-(\alpha) + \hat{f}^+(\alpha)\right) d\alpha \tag{18}
$$

is its *expected average* (see [6]). We derive the translation of the possibilistic optimization model of the RTP for trapezoidal possibilistic right-hand side values.

Let the body and critical organ right hand-side values be defined a trapezoidal fuzzy numbers $\hat{b}_{body} = 0/0/35/40$ and $\hat{b}_{C_i} = 0/0/25/30$, respectively. Let the tumor tolerances be defined by the triangular fuzzy number $\hat{b}_T = 56/60/64$. Semantically, this means that the best evidence from research and/or empirical experience indicates that as long as the body pixels receive 35 units or less, the radiation oncologist is 100% satisfied with dosage at the body pixels. However, body pixels that receive more than 40 units are completely unacceptable to the oncologist. The same interpretation holds for the critical organs. For the tumor pixels, the radiation is to be held within the range of 56 and 64 with 60 being the ideal dosage.

Remark 1 *We want to emphasize that, for a triangular* ***possibilistic*** *distribution* $\hat{b}_T = 56/60/64$*, a dosage of 60 units to a tumor cell means that the best information at hand indicates that this is a lethal dose whereas, for a triangular* ***fuzzy*** *membership function* $\tilde{b}_T = 56/60/64$*, a dose of 60 units to a tumor cell will kill the tumor cell with certainty. The semantics are clearly different!*

Thus, the left and right membership functions are, respectively, for the body:

$$
\begin{aligned}
\hat{b}^-_{body}(\alpha) &= 0 \\
\hat{b}^+_{body}(\alpha) &= 40 - 5\alpha.
\end{aligned}
$$

The left and right membership functions for the critical organs are

$$\begin{aligned}\hat{b}_{C_i}^{-}(\alpha) &= 0\\ \hat{b}_{C_i}^{+}(\alpha) &= 30-5\alpha,\end{aligned}$$

and the left and right membership functions for the tumor are

$$\begin{aligned}\hat{b}_{T}^{-}(\alpha) &= 56+4\alpha\\ \hat{b}_{T}^{+}(\alpha) &= 64-4\alpha.\end{aligned}$$

Since the Jamison and Lodwick approach is more complex it will be derived. It will be shown that every possibilistic model that arises from a crisp linear programming problem has a closed form representation as a convex nonlinear programming problem with simple bound constraints. The general case will be obvious from the specifics of the RTP application given below. The penalty functions for the Jamison and Lodwick method associated with the RTP are as follows, where $p > 0$ are vector weights.

1. Body:

$$\begin{aligned}&p_{body}EA\left(\max\left(0, Bx-\hat{b}_{body}\right)\right) =\\ &\tfrac{1}{2}p_{body}\left(\int_0^1 \max\left(0, \hat{b}_{body}^{-}(\alpha)-Bx\right)d\alpha + \int_0^1 \max\left(0, Bx-\hat{b}_{body}^{+}(\alpha)\right)d\alpha\right) =\\ &\tfrac{1}{2}p_{body}\int_0^1 \max\left(0, Bx-(40-5\alpha)\right)d\alpha,\end{aligned} \tag{19}$$

since $\hat{b}_{body}^{-}(\alpha)-Bx=-Bx\leq 0$.

2. Critical organs

$$\begin{aligned}&p_{C_i}EA\left(\max\left(0, C_ix-\hat{b}_{C_i}\right)\right) =\\ &\tfrac{1}{2}p_{C_i}\left(\int_0^1 \max\left(0, \hat{b}_{C_i}^{-}(\alpha)-C_ix\right)d\alpha + \int_0^1 \max\left(0, C_ix-\hat{b}_{C_i}^{+}(\alpha)\right)d\alpha\right) =\\ &\tfrac{1}{2}p_{C_i}\int_0^1 \max\left(0, C_ix-(30-5\alpha)\right)d\alpha,\end{aligned} \tag{20}$$

since $\hat{b}_{C_i}^{-}(\alpha)-C_ix=-C_ix\leq 0$.

3. Tumor

$$p_T EA\left(\left|Tx - \hat{b}_T\right|\right) =$$
$$\tfrac{1}{2}p_T\left(\int_0^1 \max\left(0, \hat{b}_T^-(\alpha) - Tx\right)d\alpha + \int_0^1 \max\left(0, Tx - \hat{b}_T^+(\alpha)\right)d\alpha\right) =$$
$$\tfrac{1}{2}p_T\left(\int_0^1 \max\left(0, 56 + 4\alpha - Tx\right)d\alpha + \int_0^1 \max\left(0, Tx - (64 - 4\alpha)\right)d\alpha\right). \tag{21}$$

We will use the following substitution for the maximum in the above integrals:

$$\max(0,x) = \frac{1}{2}\{\sqrt{x^2 + \in} + x\} \tag{22}$$
$$\text{where } \in \; > 0 \text{ is very small.}$$

Thus, the penalty, $b(x)$, for the body pixels is,

$$\begin{aligned}
b(x) &= \frac{1}{2}p_{body}\int_0^1 \max\left(0, Bx - 40 + 5\alpha\right)d\alpha \\
&= \frac{1}{4}p_{body}\int_0^1 \left\{\sqrt{(Bx - 40 + 5\alpha)^2 + \in} + Bx - 40 + 5\alpha\right\}d\alpha \\
&= \frac{1}{4}p_{body}\left\{Bx - 37.5 + \int_0^1 \sqrt{25\alpha^2 + 10(Bx - 40)\alpha + (Bx - 40)^2 + \in}d\alpha\right\} \\
&= \frac{1}{4}p_{body}\left\{Bx - 37.5 + 5\int_0^1 \sqrt{\alpha^2 + \frac{2}{5}(Bx - 40)\alpha + \frac{1}{25}\left[(Bx - 40)^2 + \in\right]}d\alpha\right\} \\
&= \frac{1}{4}p_{body}\left\{Bx - 37.5 + 5\int_0^1 \sqrt{\alpha^2 + b_1\alpha + c_1}d\alpha\right\} \\
&= \frac{1}{4}p_{body}\left\{Bx - 37.5 + 5\left[\sqrt{1 + b_1 + c_1}\left(\frac{1}{2} + \frac{1}{4}b_1\right) - \frac{1}{4}b_1\sqrt{c_1} + \right.\right. \\
&\quad \left.\left.\left(\frac{1}{8}b_1^2 - \frac{1}{2}c_1\right)\left(\ln\left(b_1 + 2\sqrt{c_1}\right) - \ln\left(2 + b_1 + 2\sqrt{1 + b_1 + c_1}\right)\right)\right]\right\},
\end{aligned} \tag{23}$$

where,

$$b_1 = \frac{2}{5}(Bx - 40) \tag{24}$$
$$c_1 = \frac{1}{25}(Bx - 40)^2 + \frac{1}{25}\in, \tag{25}$$

since

$$\int_0^1 \sqrt{\alpha^2 + b_1\alpha + c_1}d\alpha$$

$$
\begin{aligned}
&= \sqrt{(1+b_1+c_1)}\left(\frac{1}{2}+\frac{1}{4}b_1\right)-\frac{1}{4}b_1\sqrt{c_1} \\
&\quad +\left(\frac{1}{8}b_1^2-\frac{1}{2}c_1\right)\left[\ln\left(b_1+2\sqrt{c_1}\right)-\ln\left(2+b_1+2\sqrt{1+b_1+c_1}\right)\right].
\end{aligned}
$$

In a similar manner, we have, for the critical organ pixels,

$$
\begin{aligned}
C_i(x) &= \frac{1}{2}p_{C_i}\int_0^1 \max\left(0, C_i x-30+5\alpha\right)d\alpha \\
&= \frac{1}{4}p_{C_i}\left\{C_i x-27.5+5\int_0^1 \sqrt{\alpha^2+b_2\alpha+c_2}\,d\alpha\right\} \\
&= \frac{1}{4}p_{C_i}\left\{C_i x-27.5+5\left[\sqrt{1+b_2+c_2}\left(\frac{1}{2}+\frac{1}{4}b_2\right)-\frac{1}{4}b_2\sqrt{c_2}+\right.\right. \\
&\quad \left(\frac{1}{8}b_2^2-\frac{1}{2}c_2\right)\left(\ln\left(b_2+2\sqrt{c_2}\right)-\right. \\
&\quad \left.\left.\left.\ln\left(2+b_2+2\sqrt{1+b_2+c_2}\right)\right)\right]\right\}, \qquad (26)
\end{aligned}
$$

where

$$
\begin{aligned}
b_2 &= \frac{2}{5}(C_i x-30) \qquad (27) \\
c_2 &= \frac{1}{25}(C_i x-30)^2+\frac{1}{25}\in . \qquad (28)
\end{aligned}
$$

And the penalty function for the tumor pixels is,

$$
\begin{aligned}
t_1(x) &= \frac{1}{2}p_T\int_0^1 \max\left(0, Tx-64+4\alpha\right)d\alpha \\
&= \frac{1}{4}p_T\left\{Tx-62+4\int_0^1 \sqrt{\alpha^2+b_3\alpha+c_3}\,d\alpha\right\} \\
&= \frac{1}{4}p_T\left\{Tx-62+4\left[\sqrt{1+b_3+c_3}\left(\frac{1}{2}+\frac{1}{4}b_3\right)-\frac{1}{4}b_3\sqrt{c_3}+\right.\right. \\
&\quad \left(\frac{1}{8}b_3^2-\frac{1}{2}c_3\right)\left(\ln\left(b_3+2\sqrt{c_3}\right)-\right. \\
&\quad \left.\left.\left.\ln\left(2+b_3+2\sqrt{1+b_3+c_3}\right)\right)\right]\right\}, \qquad (29)
\end{aligned}
$$

where

$$b_3 = \frac{1}{2}(Tx - 64) \tag{30}$$

$$c_3 = \frac{1}{16}(Tx - 64)^2 + \frac{1}{16} \in . \tag{31}$$

$$\begin{aligned} t_2(x) &= \frac{1}{2}p_T \int_0^1 \max(0, -Tx + 56 + 4\alpha)\, d\alpha \\ &= \frac{1}{4}p_T \left\{ -Tx + 58 + 4 \int_0^1 \sqrt{\alpha^2 + b_4\alpha + c_4} d\alpha \right\} \\ &= \frac{1}{4}p_T \left\{ -Tx + 58 + 4 \left[\sqrt{1 + b_4 + c_4} \left(\frac{1}{2} + \frac{1}{4}b_4 \right) - \frac{1}{4}b_4\sqrt{c_4} + \right.\right. \\ &\quad \left(\frac{1}{8}b_4^2 - \frac{1}{2}c_4 \right) (\ln(b_4 + 2\sqrt{c_4}) - \\ &\quad \left.\left. \ln\left(2 + b_4 + 2\sqrt{1 + b_4 + c_4} \right)\right) \right] \right\}, \end{aligned} \tag{32}$$

where

$$b_4 = \frac{1}{2}(-Tx + 56) \tag{33}$$

$$c_4 = \frac{1}{16}(-Tx + 56)^2 + \frac{1}{16} \in . \tag{34}$$

The possibilistic RTP becomes:

$$\begin{aligned} \min z &= c^T x + b(x) + \sum_{i=1}^{K} C_i(x) + t_1(x) + t_2(x) \\ \text{subject to } 0 &\leq x \leq U. \end{aligned} \tag{35}$$

For the general problem

$$\begin{aligned} \min \quad & z = c^T x + pEA\left(\max\left(0, Ax - \hat{b} \right) \right) \\ \text{subject to} \quad & 0 \leq x \leq U. \end{aligned} \tag{36}$$

The objective function is nonlinear, but is closed-form and convex, making the problem a convex programming problem with simple bound constraints as is proved below. This means that it is amenable to fast and efficient optimization techniques and local optima are global.

Theorem 1 *The transformed Jamison and Lodwick possibilistic optimization problem arising from the general linear programming problem*

$$\begin{aligned} \min\ z &= c^T x \\ \textit{such that.} Ax &\leq \hat{b} \\ 0 &\leq x \leq U, \end{aligned}$$

with trapezoidal possibilistic right-hand side, $\hat{b}$, is a convex programming problem with simple bound constraints, whose objective function is given in closed-form.

Proof: *We know from [6] that the objective function is convex so what we must show is that $EA\left(\max\left(0, Ax - \hat{b}\right)\right)$ has a closed-form.*

$$\begin{aligned} EA\left(\max\left(0, Ax - \hat{b}\right)\right) &= \frac{1}{2}\int_0^1 \left[\max(0, Ax - b^-(\alpha)) + \max(0, Ax - b^+(\alpha))\right] d\alpha \\ &= \frac{1}{4}\left\{\int_0^1 \left[\sqrt{(Ax - b^-(\alpha))^2 + \in} + Ax - b^-(\alpha)\right] d\alpha \right. \\ &\quad \left. + \int_0^1 \left[\sqrt{(Ax - b^+(\alpha))^2 + \in} + Ax - b^+(\alpha)\right] d\alpha \right\}. \end{aligned}$$

If either $b^-(\alpha)$ and/or $b^+(\alpha)$ is constant, that is, one of the sides of the trapezoid is vertical, then the corresponding integral is closed-form since it does not involve α. Thus, without loss of generality, assume that the left side of the trapezoidal possibilistic number $\hat{b}$ is not vertical (the right side is done in the same way), that is,

$$b^-(\alpha) = \underline{m}\alpha + \underline{b}.$$

Then (integration is done component-wise and all arithmetic operations are component-wise),

$$\begin{aligned} &\int_0^1 \left[\sqrt{(Ax - b^-(\alpha))^2 + \in} + Ax - b^-(\alpha)\right] d\alpha \\ = &\int_0^1 \left[\sqrt{(Ax - \underline{m}\alpha - \underline{b})^2 + \in} + Ax - \underline{m}\alpha - \underline{b}\right] d\alpha \\ = &\ Ax - \frac{1}{2}\underline{m} - \underline{b} + \int_0^1 \sqrt{(Ax - \underline{m}\alpha - b^-)^2 + \in} d\alpha. \end{aligned}$$

The integral is

$$\begin{aligned} & \int_0^1 \sqrt{(Ax-\underline{m}\alpha-\underline{b})^2+\in} d\alpha \\ = & \; (\underline{m})^2 \int_0^1 \sqrt{\alpha^2+b\alpha+c} d\alpha, \end{aligned}$$

where

$$\begin{aligned} & = \frac{2\underline{b}-2Ax}{\underline{m}} \\ c & = \frac{(Ax)^2-Ax\underline{b}+\underline{b}^2+\in}{\underline{m}^2}. \end{aligned}$$

But

$$\begin{aligned} \int_0^1 \sqrt{\alpha^2+b\alpha+c} d\alpha = & \sqrt{1+b+c}\left(\frac{1}{2}+\frac{1}{4}b\right)-\frac{1}{4}b\sqrt{c} \\ & +\left(\frac{1}{8}b^2-\frac{1}{2}c\right)\left(\ln\left(b+2\sqrt{c}\right)-\ln\left(2+b+2\sqrt{(1+c+b)}\right)\right) \end{aligned}$$

which proves the theorem.□

4 Numerical Experiments

Two sets of CT scans with tumors, head with a large tumor and two critical organs and prostate with a small tumor and three critical organs were used at four levels of resolution for one set of ten beam angles. The number of pixels were quadrupled in each experiment for our three methods. Below we list the run times (in seconds) on MATLAB using the optimization toolbox where T&Z stands for Tanaka and Zimmermann and J&L stands for Jamison and Lodwick. MATLAB is used for rapid prototyping and ease in interfacing with CT images. For large problems, T&Z ran out of memory. Moreover, surprise had a more than a 250-fold jump in time though there is only a 4-fold increase in the complexity (as measured by the number of constraints) for the prostate 512×512 resolution. When the complexity is moderately high, MATLAB's optimization toolbox does not perform consistently. Perhaps this due to memory allocation problems. Thus, the MATLAB implementational differences between the surprise approach and the other two methods have more to do with the size of the constraint matrices than with the algorithm itself. So, times listed below should be viewed in a relative sense. The experiments demonstrate that large fuzzy and possibilistic optimization problems can be solved efficiently so that solving problems with inherent fuzzy and possibilistic uncertainty is clearly tractable.

$HEAD$	Resolution	Constratins	Surprise	T&Z	$\alpha - level^*$	J&L
	64×64	977	1	19	0.7	902^{**}
	128×128	3,669	11	812	1.0	320
	256×256	14,159	120	8,235	0.5	4,848
	512×512	55,720	1,674			

TABLE 1: Runtimes (seconds) for a head tumor, 10 beam angles, two critical organs

$PROSTATE$	Resolution	Constraint	Surprise	T&Z	$\alpha - level^*$	J&L
	64×64	734	1	9	1.0	383
	128×128	2,875	5	816	0.4	2,825
	256×256	10,911	113	3,737	1.8^{***}	3,021
	512×512	43,127	$29,580^{**}$			

TABLE 2: Runtimes (seconds) for a prostate tumor, 10 beam angles three critical organs

* α-level of the joint constraints for the T&Z method
** Other programs running - memory allocation
*** Infeasible

The surprise approach seems to be the one that can handle large problems in MATLAB quickly. Clearly, to make Tanaka and Zimmermann as well as Jamison and Lodwick work effectively, different and more powerful systems such as C-Plex or GAMS need to be used. Moreover, the structures of the problems were not exploited. For the surprise approach, specialized quadratic programming with simple bound constraints algorithms were not used. The Jamison and Lodwick approach is convex with simple bound constraints. In addition, the times listed above are runtimes on a server and if there were others using MATLAB at the same time, memory allocation was even more of a problem.

5 Conclusion

It is clear that large fuzzy and possibilistic optimization problems can be implemented. Moreover, it is also clear that for fuzzy and possibilistic uncertainty, excellent computational approaches exist for large problems that directly model these uncertainties. Directly modeling optimization problems with possibilistic and fuzzy uncertainties is preferable to assuming that all uncertainty is probabilistic and, hence, using stochastic optimization to solve fuzzy or possibilistic optimization problems. The RTP is an example of a problem whose uncertainty is inherently fuzzy and also possibilistic and not probabilistic. It should be modeled directly as a fuzzy or possibilistic optimization problem.

Clear associated semantics for fuzzy and possibilistic optimization must be implemented and the means to implement the semantics were given. When semantics are crucial to the problem, there is a crucial difference between fuzzy and possibilistic optimization that must be maintained. One cannot use fuzzy optimization techniques (flexible programming or the surprise function approach) when indeed the distributions are inherently possibilistic even though a possibility distribution can be considered as a fuzzy membership function. It would be like solving a stochastic programming problem that is truly a recourse model using chance constraint techniques or vise versa.

Future research includes studying a larger suite of fuzzy and possibilistic optimization algorithms. Moreover, the implementation of algorithms that exploit the natural structure of the surprise function algorithm (quadratic programming with simple bound constraints), and the penalty approach (convex programming with simple bound constraints), to solve the RTP problem is a natural next step.

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