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Adaptive Selection of Face Coarse Degrees of Freedom in the BDDC and the FETI-DP Iterative Substructuring Methods

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Abstract

We propose a class of method for the adaptive selection of the coarse space of the BDDC and FETI-DP iterative substructuring methods. The methods work by adding coarse degrees of freedom constructed from eigenvectors associated with intersections of selected pairs of adjacent substructures. It is assumed that the starting coarse degrees of freedom are already sufficient to prevent relative rigid body motions in any selected pair of adjacent substructures. A heuristic indicator of the the condition number is developed and a minimal number of coarse degrees of freedom is added to decrease the indicator under a given threshold. It is shown numerically on 2D elasticity problems that the indicator based on pairs of substructures with common edges predicts the actual condition number reasonably well, and that the method can select adaptively the hard part of the problem and concentrate computational work there to achieve good convergence of the iterations at a modest cost.

1 Introduction

The BDDC and FETI-DP methods are iterative substructuring methods that use coarse degrees of freedom associated with corners and edges (in 2D) or faces

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(in 3D, further on just faces) between substructures, and they are currently the most advanced versions of the BDD and FETI families of methods. The BDDC method by Dohrmann [3] is a Neumann-Neumann method of Schwarz type [4]. The BDDC method iterates on the system of primal variables reduced to the interfaces between the substructures, and it can be understood as a further development of the BDD method by Mandel [17]. The FETI-DP method by Farhat et al. [5,6] is a dual method that iterates on a system for Lagrange multipliers that enforce continuity on the interfaces, and it is a further development of the FETI method by Farhat and Roux [7]. Algebraic relations between FETI and BDD methods were pointed out by Rixen et al. [21], Klawonn and Widlund [10], and Fragakis and Papadrakakis [8]. A common bound on the condition number of both the FETI and the BDD method in terms of a single inequality in was given in [10]. In the case of corner constraints only, methods same as BDDC were derived as primal versions of FETI-DP by Fragakis and Papadrakakis [8], who have also observed that the eigenvalues of BDD and a certain version of FETI are identical, and by Cros [2]. Mandel, Dohrmann, and Tezaur [19] have formulated the methods in terms of general coarse degrees of freedom without treating the corner degrees of freedom in a special way, proved that the eigenvalues of the preconditioned operators of BDDC and FETI-DP are the same except possibly for zero and one, and obtained a simplified and fully algebraic version (i.e., without any undetermined constants) of a common condition number estimate for BDDC and FETI-DP, related to an estimate by Klawon and Widlund [10] for BDD and FETI. The present work builds on the algebraic condition estimate from [19]. Simpler proofs of the equality of eigenvalues were recently obtained by Li and Widlund [15] and by Brenner and Sung [1], who also gave an example when BDDC has an eigenvalue equal to one but FETI-DP does not. However, in practice, there are other eigenvalues very close to one, and the performance of the FETI-DP and BDDC methods is essentially identical [19].

In this paper, we build on the algebraic estimate from [19] to develop an adaptive fast method. This estimate can be computed from the matrices in the method as the solution of an eigenvalue problem. By restricting the eigenproblems onto pairs of adjacent substructures, we obtain a reliable heuristic indicator of the condition number. Finally, we show how to use the eigenvectors, which are supported on subsets of the intersections of adjacent substructures, to build coarse degrees of freedom that result in an optimal decrease of the heuristic condition number indicator. We show on numerical examples that the indicator is quite close and that our adaptive approach can result in the concentration of computational work in a small troublesome part of the problem, which leads to a good convergence behavior at a small added cost.

We also give a new, very simple variational setting of BDDC, which is an abstract version of partial subassembly described in [16], and a short proof of

the condition number bound. However, we do not adopt the change of variables introduced in [16], which would seem to require more work in our setting.

Related work on adaptive coarse space selection has focused on the global problem of selecting the smallest number of corners to prevent coarse mechanisms (Lesoinne [14]) and the smallest number of (more general) coarse degrees of freedom to assure asymptotically optimal convergence estimates (Klawonn and Widlund [11]). This required considering potentially large chains of substructures, based on the global behavior of the structure. Our goal is different. We assume that the starting coarse degrees of freedom (i.e., those present before the adaptive selection of additional ones) are already sufficient to prevent relative rigid body motions of any two adjacent substructures that we compute the indicator and additional coarse degrees of freedom from. Our methodology is local in nature, involving only two substructures at a time. This methodology is quite general and dimension independent. In this paper, the application is 2D, the selected pairs of adjacent substructures are those sharing a common edge, and the starting coarse degrees of freedom are all displacements at substructure corners. A brief presentation of the main result of this paper, without proofs, is contained in the conference paper [20].

The paper is organized as follows. In Sec. 2, we present the fundamentals of the BDDC and FETI-DP methods and the condition number bound. In Sec. 3, we derive an efficient eigenvalue formulation for computing the condition number bound. In Sec. 4, we develop the heuristic condition number indicator. In Sec. 5, we describe the method of selection of additional coarse degrees of freedom. In Sec. 6, we report on numerical results, which confirm the validity of the heuristic indicator and demonstrate the efficiency of the proposed algorithm.

2   BDDC and FETI-DP

In this section, we first introduce substructuring concepts on a model problem (Sec. 2.1). Then we define BDDC in a simple variational form and give the bound on the condition number (Sec. 2.2). In Sec. 2.3, we show how the spaces in the variational form arise in terms of linear algebra only, using only algebraic properties of the matrices involved. The matrix formulation is enough to formulate the BDDC and the FETI-DP algorithms in matrix form in Sec. 2.4.

All spaces are Euclidean spaces of column vectors. We make no distinction between a linear operator and its matrix. The symbol ^{T} denotes the transpose. For a symmetric positive semidefinite matrix S, $\|u\|_S$ denotes the seminorm $\sqrt{u^T S u}$. For a symmetric positive semidefinite bilinear form c, $\|u\|_c$ denotes the seminorm $\sqrt{c\,(u,u)}$.

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2.1 Substructuring for a Model Problem

To fix ideas, we first show how the spaces and operators we will work with arise in the standard substructuring setting for a model elliptic problem, cf., e.g., [22]. Consider the plane stress problem on a bounded polygonal domain $\Omega \subset \mathbb{R}^2$, decomposed into N nonoverlapping polygonal subdomains Ω_i, $i = 1, ..., N$ (Fig. 1). Subdomain Ω_i will be called substructure i. Each substructure is the union of Lagrangean Q1 finite elements, and the nodes of the elements on the interfaces between substructures coincide. The boundary of Ω_i is denoted by $\partial \Omega_i$. Define

$$\Gamma = \bigcup_{i=1}^{N} \partial \Omega_i, \quad \Gamma_i = \Gamma \cap \partial \Omega_i.$$

Each node is associated with two degrees of freedom. The interface Γ is composed of substructure edges, which are regarded as sets open in Γ, and of the substructure vertices (the endpoints of the edges), called corners. By subassembly, we obtain local stiffness matrices of the substructures. Let S_i be Schur complement obtained by eliminating all degrees of freedom in the local stiffness matrix of substructure i, other than on Γ_i.

In the remainder of this paper, we only need the following concepts of substructuring rather than the specific properties of the model problem. The space of all vectors of local degrees of freedom on Γ_i is denoted by W_i, and we have $S_i : W_i \rightarrow W_i$. The matrices S_i are assumed to be symmetric and positive semidefinite. The space of all vectors of global degrees of freedom on Γ is denoted by U. The vectors of the local substructure degrees of freedom $w_i \in W_i$ and the vector of the global degrees of freedom $u \in U$ are related by $w_i = R_i u$, where R_i is the restriction operator (a zero-one matrix). Clearly,

$$R_i : U \rightarrow W_i, \quad R_i R_i^T = I, \quad i = 1, \dots, N. \tag{1}$$

Let

$$W = W_1 \times \cdots \times W_N,$$

and consider vectors and matrices in the block form

$$w = \begin{bmatrix} w_1 \\ \vdots \\ w_N \end{bmatrix}, \quad w \in W, \quad R = \begin{bmatrix} R_1 \\ \vdots \\ R_N \end{bmatrix}, \quad S = \begin{bmatrix} S_1 & & \\ & \ddots & \\ & & S_N \end{bmatrix}.$$

The problem to be solved is the constrained minimization of energy,

$$\frac{1}{2} a(w, w) - g^T w \rightarrow \min \text{ subject to } w \in \widehat{W}, \tag{2}$$

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where

$$\widehat{W} = \operatorname{range} R, \quad R : U \to W, \tag{3}$$

is the space of all vectors of degrees of freedom on the substructures that coincide on the interfaces, and

$$a(w, z) = w^T S z, \quad w, z \in W. \tag{4}$$

Equivalently, (2) can be written in the assembled form as the system of linear algebraic equations

$$Au = R^T g, \text{ where } A = R^T S R. \tag{5}$$

Consider the matrix

$$B = [B_1, \dots, B_N],$$

defined as follows. Each row B corresponds to a degree of freedom common to a pair of substructures i and j. The entries of the row are zero except for one $+1$ in the block i and one -1 in the block j, so that the condition

$$Bw = 0 \iff w \in \widehat{W},$$

that is

$$\operatorname{range} R = \operatorname{null} B. \tag{6}$$

An important ingredient of substructuring methods is the averaging operator

$$E = RR^T D_P, \tag{7}$$

where $D_P : W \to W$ is a given weight matrix such that the decomposition of unity property holds, $R^T D_P R = I$. Clearly, E is a projection. In terms of substructuring, E is an averaging operator that maps the substructure local degrees of freedom to global degrees of freedom.

In the remainder of Sec. 2, except for (18), we do not rely on the substructuring framework. Rather, only the algebraic properties expressed by the numbered equations (1)–(7) are assumed. We proceed in an abstract manner, using the substructuring concepts for illustration only.

From (6), $Bw = 0$ implies $(I - E)w = 0$, so there exist a matrix B_D such that $I - E = B_D^T B$, hence

$$B_D^T B + RR^T D_P = I. \tag{8}$$

In computations, we use the matrix B_D constructed from B as

$$B_D = [D_{D1}B_1, \dots, D_{DN}B_N],$$

where the matrices D_{Di} are determined from D_P, see [12,19].

The operator B was originally introduced to formulate the FETI-DP algorithm (Sec. 2.4) and it is not needed for the formulation of the BDDC algorithm, but here we use it to take advantage of the property (8).

2.2 Variational Setting of BDDC and Condition Number Bound

We now formulate the BDDC method in a particularly simple abstract variational form, which is inspired by a view of the Neumann-Neumann methods, going back to [4]. We also give a simplified proof of the condition number bound from [19].

We wish to solve the abstract linear problem

$$u \in \widehat{W} : a(u,v) = \langle f, v \rangle, \quad \forall v \in \widehat{W}, \tag{9}$$

where $\widehat{W}$ is a finite dimensional linear space, a is a symmetric bilinear form positive definite on $\widehat{W}$, and $f \in \widehat{W}'$ is the right hand side, with $\langle f, v \rangle$ denoting the value of the functional f at v. The operator $A : \widehat{W} \mapsto \widehat{W}'$ associated with a is defined by

$$a(u,v) = \langle Au, v \rangle, \quad \forall u, v \in \widehat{W}.$$

Suppose that the bilinear form a is defined and symmetric positive semidefinite on a larger space $W \supset \widehat{W}$. The BDDC preconditioner $P : \widehat{W}' \to \widehat{W}$ is defined by

$$P : r \longmapsto u = Ew, \quad w \in \widetilde{W} : \quad a(w, z) = \langle r, Ez \rangle, \quad \forall z \in \widetilde{W}. \tag{10}$$

where E is projection from $\widetilde{W}$ onto $\widehat{W}$, and the space $\widetilde{W}$ is such that

$$\widehat{W} \subset \widetilde{W} \subset W.$$

Assumption 1 $a(\cdot, \cdot)$ *is positive definite on* $\widetilde{W}$.

It should be noted that the idea to restrict the bilinear form $a(\cdot, \cdot)$ from W to $\widetilde{W}$ is closely related to the subassembly in [16].

Let

$$\omega = \sup_{w \in \widetilde{W}} \frac{\|Ew\|_a^2}{\|w\|_a^2}. \tag{11}$$

Theorem 2 *The abstract BDDC preconditioner (10) satisfies*

$$\kappa = \frac{\lambda_{\max}(PA)}{\lambda_{\min}(PA)} \leq \omega.$$

PROOF. Define the operator $G : \widehat{W} \to \widetilde{W}$ by

$$G : u \mapsto w, \quad \frac{1}{2} a(w, w) \to \min, \text{ s.t. } w \in \widetilde{W}, \ u = Ew. \tag{12}$$

Since a is positive definite on $\widetilde{W}$, G is well defined. Define the bilinear form b on $\widehat{W}$ by $b(u, v) = a(Gu, Gv)$. Now let u and w be as in (10). Since w is the solution of

$$\frac{1}{2} a(w, w) - \langle r, Ew \rangle \to \min, \quad w \in \widetilde{W},$$

it follows that u is the solution of

$$u \in \widehat{W} : \quad b(u, v) = \langle r, v \rangle, \quad \forall v \in \widehat{W}. \tag{13}$$

It remains to compare $\|u\|_a^2$ and $\|u\|_b^2$. Let $u \in \widehat{W}$ and define $w = Gu$. Then, from the minimization property (12) and the fact that $Eu = u \in \hat{W} \subset \widetilde{W}$, it follows that

$$\|u\|_b^2 = \|w\|_a^2 \le \|u\|_a^2.$$

On the other hand,

$$\|u\|_a^2 = \|Ew\|_a^2 \le \omega \|w\|_a^2 = \omega \|u\|_b^2,$$

which concludes the proof. $\square$

This formulation of BDDC is remarkably simple, involving only the bilinear form a, the space $\widetilde{W}$, and the projection E onto the subspace $\widehat{W}$. However, in the application to substructuring, W and $\widehat{W}$ are given, and the space $\widetilde{W}$ and the projection E are to be chosen. We need further notation to describe how these choices are made and how the space $\widetilde{W}$ will be selected adaptively.

With the specific form of the projection E from (7) and using (4), we can restate the condition number bound.

Theorem 3 *The condition number bound (11) satisfies*

$$\omega = \sup_{w \in \widetilde{W}} \frac{\left\| R R^T D_P w \right\|_S^2}{\|w\|_S^2} = \sup_{w \in \widetilde{W}} \frac{\left\| B_D^T B w \right\|_S^2}{\|w\|_S^2} \tag{14}$$

PROOF. The proof is similar as in [19, Theorem 25]. The proposition follows from the fact that the operator $R R^T D_P$ is a projection in the space $\widetilde{W}$ and the norm of a nontrivial projection in an inner product space depends only on the angle between its range and its nullspace [9]. From (8), this angle is the same for $R R^T D_P$ and $B_D^T B = I - R R^T D_P$. $\square$

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At this point, we still do not need to make an explicit reference to substructuring. Instead, from now on, we assume the properties (3)–(8), and that R is one-to-one. In particular, R is an isomorphism between U and $\widehat{W}$.

The space $\widetilde{W}$ is constructed using so-called coarse degrees of freedom. The coarse degrees of freedom can be, e.g., values at corners, or averages on edges. The space $\widetilde{W}$ is then given by the requirement that the coarse degrees of freedom on adjacent substructures coincide; for this reason, the terms coarse degrees of freedom and constraints are used interchangeably.

To choose the space $\widetilde{W}$, suppose we are given a space X and a linear operator $C : W \rightarrow X$ and define

$$\widetilde{W} = \left\{ w \in W : \exists u \in \widehat{W} : Cw = Cu \right\}.$$

By construction, $\widehat{W} \subset \widetilde{W}$. The values Cw will be called local coarse degrees of freedom. The space $\widetilde{W}$ consists of all vectors w whose local coarse degrees of freedom Cw are in the same space as the local coarse degrees of freedom of vectors from the subspace $\widehat{W}$; that is, for $w \in \widetilde{W}$, the local coarse degrees of freedom on adjacent substructures coincide. To represent their common values, i.e., the global coarse degrees of freedom of vectors $u \in \widehat{W}$, suppose there is a space U_c and linear operators $Q_P^T : U \rightarrow U_c, \quad R_c : U_c \rightarrow X$ such that R_c is one-to-one, and

$$CR = R_c Q_P^T. \tag{15}$$

The space $\widetilde{W}$ then satisfies

$$\widetilde{W} = \{ w \in W : \exists u_c \in U_c : Cw = R_c u_c \}, \tag{16}$$

and from (15), for $w \in \widetilde{W}$, the unique u_c that satisfies $Cw = R_c u_c$ is given by

$$u_c = Q_P^T v, \quad w = Rv.$$

An equivalent way to specify the space $\widetilde{W}$ is by the choice of an operator Q_D^T such that

$$\widetilde{W} = \left\{ w \in W : Q_D^T B w = 0 \right\}, \tag{17}$$

which will be needed in the adaptive algorithm. We will only need to construct Q_P for a given Q_D in a special case, see (44) below.

We will need a more specific construction of the matrix C in the substructuring

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framework. We build a block diagonal matrix C satisfying (15) by

$$C = \begin{bmatrix} C_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & C_N \end{bmatrix}, \quad C_i = R_{ci} Q_P^T R_i^T. \tag{18}$$

Then (15) follows from (18) and (1).

Here is an interpretation. The matrix C_i maps a vector of local degrees of freedom on substructure i to a vector of local coarse degrees of freedom on the substructure, and R_{ci} restricts a vector of all global coarse degree of freedom to a vector of local coarse degree of freedom on substructure i. A global coarse degree of freedom is given by a row of Q_P. The operator Q_P^T acts on vectors of global degrees of freedom in U and it selects global coarse degrees of freedom in U_c as linear combinations of global degrees of freedom.

In the model problem, there are corner coarse degrees of freedom, which are values at the corner, and edge coarse degrees of freedom, which are linear combinations of values on the edge.

2.4   The Iterative Algorithms

In this section, we formulate the methods how they were implemented in terms of matrices in our computations, following [19]. The original implementation of BDDC in [3,18] is mathematically equivalent but it requires a more complicated substructuring notation and it treats the corner coarse degrees of freedom and edge coarse degrees of freedom in the definition of $\widetilde{W}$ in different ways. Here, all coarse degrees of freedom are treated the same. The equivalence of the formulations of BDDC in this section and in Sec. 2.2 follows from [19, Lemma 7]. For another mathematically equivalent implementation of BDDC, involving a change of variables, see [15].

The BDDC method is the method of preconditioned conjugate gradients for the assembled system (5) and the preconditioner P defined by

$$Pr = R^T D_P \left(\Psi u_c + z \right),$$

where u_c is the solution of the coarse problem

$$\Psi^T S \Psi u_c = \Psi^T D_P^T R r \tag{19}$$

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and z is the solution of

$$
\begin{aligned}
Sz + C^T\mu &= D_P^T Rr, \\
Cz &= 0,
\end{aligned}
\tag{20}
$$

which is a collection of independent substructure problems. The coarse basis functions Ψ are defined by energy minimization,

$$
\operatorname{tr}\Psi^T S\Psi \to \min \text{ subject to } C\Psi = R_c,
$$

which is equivalent to the system

$$
\begin{bmatrix} S & C^T \\ C & 0 \end{bmatrix} \begin{bmatrix} \Psi \\ \Lambda \end{bmatrix} = \begin{bmatrix} 0 \\ R_c \end{bmatrix}.
\tag{21}
$$

Using corner coarse degrees of freedom, which allows to eliminate a part of the second block in a simple manner, the problems (20) and (21) can be solved without solving any indefinite systems [19].

For completeness, we state the FETI-DP method using the ingredients we have introduced already. Again, this formulation treats all coarse degrees of freedom in the same way and allows for a substatial simplification; the original formulations [6,5,12] treat the corner degrees of freedom separately. The FETI-DP method solves the saddle point problem

$$
\min_{w\in\widetilde{W}} \max_{\lambda} \mathcal{L}(w,\lambda) = \max_{\lambda} \min_{w\in\widetilde{W}} \mathcal{L}(w,\lambda),
$$

where

$$
\mathcal{L}(w,\lambda) = \frac{1}{2}w^T Sw - w^T f + w^T B^T \lambda,
$$

by iterating on the dual problem

$$
\frac{\partial \mathcal{F}(\lambda)}{\partial \lambda} = F\lambda - h = 0,
$$

where

$$
\mathcal{F}(\lambda) = \min_{w\in\widetilde{W}} \mathcal{L}(w,\lambda),
$$

by preconditioned conjugate gradients, with the preconditioner

$$
M = B_D S B_D^T.
$$

See [19] for further details.

Theorem 4 ([19, Theorem 26]) *The eigenvalues of the preconditioned operators PA of BDDC and MF of FETI-DP are the same except possibly for eigenvalues of zero and one. All other eigenvalues are larger or equal to one.*

Zero eigenvalues in FETI-DP originate from redundant constraints, i.e., linearly dependent rows of B. For a simplified proof of Theorem 4, see [15]. For a further study of the relation between FETI-DP and BDDC, see [1].

3 Eigenvalue Formulation of the Condition Number Bound

Clearly, the problem of finding the condition bound (14) can be written as an eigenvalue problem on $\widetilde{W}$. The goal of this section is to formulate this problem as a generalized eigenvalue problem on the space W with one of the matrices positive definite, which makes the eigenvalues convenient to compute numerically.

We do not advocate computing the condition bound ω numerically in practice. Instead, this section serves as a preparation for the development of a local indicator of the condition number in the following sections.

Denote by Π the orthogonal projection in W onto $\widetilde{W}$. The stationary points of the Rayleigh quotient $\left\| B_D^T B w \right\|_S^2 / \|w\|_S^2$ are the eigenvectors and the values of the Rayleigh quotient at the stationary points are the eigenvalues of the generalized eigenvalue problem on W,

$$\Pi B^T B_D S B_D^T B \Pi w = \lambda \Pi S \Pi w. \tag{22}$$

The maximization problem (14) now becomes the problem to find the maximal eigenvalue of (22). However, the matrices on both sides are in general singular, and the eigenvector is in general determined only up to a component in null Π, so we may look only for eigenvectors in range $\Pi = \widetilde{W}$.

Lemma 5 *Let S, T, Π be square matrices of the same size, Π an orthogonal projection, $t \neq 0$, and $\lambda \neq 0$ scalars, and u a vector. Then*

(1) It holds that
$$\Pi T \Pi w = \lambda \Pi S \Pi w \text{ and } w \in \text{range } \Pi \tag{23}$$
if and only
$$\Pi T \Pi w = \lambda \left(\Pi S \Pi + t \left(I - \Pi \right) \right) w. \tag{24}$$
(2) If S is symmetric and positive definite on $\text{range } \Pi$ and $t > 0$, then $\Pi S \Pi + t \left(I - \Pi \right)$ is symmetric positive definite.

PROOF.

(1) Assume that (23) holds. Then $\left(I - \Pi \right) w = 0$, and (24) follows. On the other hand, if (24) holds, then $\left(I - \Pi \right) w \in \text{range } \Pi$ because $t \neq 0$ and $\lambda \neq 0$, so $\left(I - \Pi \right) w = 0$, and, consequently, (23) holds.

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(2) Let S be symmetric and positive definite on range Π and $v = w + z$, $w \in \text{range}\,\Pi$, $z \in \text{null}\,\Pi$. Then

$$v^T \left(\Pi S\Pi + t\left(I - \Pi\right)\right) v = w^T \Pi S\Pi w + tz^T \left(I - \Pi\right) z$$
$$= w^T Sw + tz^T z > 0,$$

unless $w = z = 0$. $\square$

Theorem 6 *Let $t > 0$. Then the nonzero eigenvalues and the eigenvectors of (22) are same as those of the generalized eigenvalue problem*

$$\Pi B^T B_D S B_D^T B\Pi w = \lambda\left(\Pi S\Pi + t\left(I - \Pi\right)\right) w. \tag{25}$$

The matrix on the left-hand side of (25) is symmetric positive semidefinite and the matrix on the right-hand side is symmetric positive definite. In particular, the upper bound ω on the condition number from (14) is the maximal eigenvalue of (25).

PROOF. The equivalence of (22) and (25) follows from Lemma 5 (1). Positive definiteness of the matrix $\Pi S\Pi + t\left(I - \Pi\right)$ follows from Lemma 5 (2) and the assumption that S is positive definite on $\widetilde{W}$. $\square$

In practice, we choose t to be roughly the same magnitude as S. Note that if the eigenvalue is computed only approximately, the result will in general depend on the parameter t.

We need an explicit formula for the projection Π.

Lemma 7 *If C and R_c have linearly independent rows, then the orthogonal projection in W onto $\widetilde{W}$ is*

$$\Pi = I - \begin{bmatrix} C^T & 0 \end{bmatrix} \begin{bmatrix} CC^T & R_c \\ R_c^T & 0 \end{bmatrix}^{-1} \begin{bmatrix} C \\ 0 \end{bmatrix}. \tag{26}$$

PROOF. The equation $v = \Pi u$ is equivalent to

$$v \in \widetilde{W}, \quad u - v \perp \widetilde{W}, \tag{27}$$

which, from the definition of $\widetilde{W}$ in (16), can be written as

$$\exists v_c : Cv = R_c v_c$$
$$\forall w : \left(Cw = R_c w_c \Rightarrow (u - v)^T w = 0\right) \tag{28}$$

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Writing (28) in block matrix notation, we have

$$\begin{bmatrix} C & -R_c \end{bmatrix} \begin{bmatrix} w \\ w_c \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} (u-v)^T & 0 \end{bmatrix} \begin{bmatrix} w \\ w_c \end{bmatrix} = 0,$$

and, consequently, (28) is equivalent to

$$\exists x : \begin{bmatrix} (u-v)^T & 0 \end{bmatrix} = x^T \begin{bmatrix} C & -R_c \end{bmatrix}.$$

Consequently, $v = \Pi u$ is equivalent to the system of equations

$$Cv = R_c v_c, \quad u - v = C^T x, \quad R_c^T x = 0.$$

Substituting from the second equation

$$v = u - C^T x \tag{29}$$

gives the system

$$\begin{bmatrix} CC^T & R_c \\ R_c^T & 0 \end{bmatrix} \begin{bmatrix} x \\ v_c \end{bmatrix} = \begin{bmatrix} Cu \\ 0 \end{bmatrix}. \tag{30}$$

It follows from the assumptions that the system (30) is regular. Now using (29), we have

$$v = u - \begin{bmatrix} C^T & 0 \end{bmatrix} \begin{bmatrix} x \\ v_c \end{bmatrix} = u - \begin{bmatrix} C^T & 0 \end{bmatrix} \begin{bmatrix} CC^T & R_c \\ R_c^T & 0 \end{bmatrix}^{-1} \begin{bmatrix} C \\ 0 \end{bmatrix} u$$

and (26) follows. $\square$

If a subspace iteration type method, such as the efficient conjugate gradient method for eigenvalues [13], is used for the generalized eigenvalue problem, the matrix of the projection Π is not needed; only matrix-vector products need to be evaluated. Each evaluation of Πu requires the solution the system (30). The dimension of this system is the same as the dimension of the system (21), which is solved to find the coarse basis functions.

4   Local Indicator of the Condition Number Bound

From now on, we need to assume the full substructuring framework from Sec. 2.1, though we are of course not limited to the model problem.

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As the indicator of the condition number, we propose the maximum of the bounds ω_{ij} from (25) computed by considering the domains consisting of one pair of adjacent substructures i, j at a time. For such pair, denote the vectors of degrees of freedom associated with the two substructures by

$$w_{ij} = \begin{bmatrix} w_i \\ w_j \end{bmatrix}, \quad w_i \in W_i, \quad w_j \in W_j,$$

and the submatrices associated with the pair of substructures by

$$S_{ij} = \begin{bmatrix} S_i & 0 \\ 0 & S_j \end{bmatrix}, \quad C_{ij} = \begin{bmatrix} C_i & 0 \\ 0 & C_j \end{bmatrix}. \tag{31}$$

Define B_{ij} be a submatrix of $\begin{bmatrix} B_i & B_j \end{bmatrix}$ that consists of a subset of all rows that have exactly one $+1$ and one -1. Then the condition $B_{ij}w_{ij} = 0$ enforces the equality of w_i and w_j at a subset $\mathcal{S}_{ij}$ of degrees of freedom at the intersection of substructures i and j. Define B_{Dij} as the submatrix of $\begin{bmatrix} D_{Di}B_i & D_{Dj}B_j \end{bmatrix}$ consisting of the same selection of rows as B_{ij}. Let R_{cij} be the submatrix of $\begin{bmatrix} R_{ci} \\ R_{cj} \end{bmatrix}$ consisting of all of its nonzero columns. Finally, let $\widetilde{W}_{ij}$ be the space of vectors with the coarse degrees of freedom same on the intersection of the substructures i and j,

$$\widetilde{W}_{ij} = \{ w_{ij} : \exists w_c : C_{ij}w_{ij} = R_{cij}w_c \}. \tag{32}$$

Then, analogously to (14), define

$$\omega_{ij} = \sup_{w_{ij} \in \widetilde{W}_{ij}} \frac{\left\| B_{Dij}^T B_{ij} w_{ij} \right\|_{S_{ij}}^2}{\|w_{ij}\|_{S_{ij}}^2} = \sup_{w_{ij} \in \widetilde{W}_{ij}} \frac{w_{ij}^T B_{ij}^T B_{Dij} S_{ij} B_{Dij}^T B_{ij} w_{ij}}{w_{ij}^T S_{ij} w_{ij}}. \tag{33}$$

and let

$$\widetilde{\omega} = \max_{ij \in \mathcal{A}} \omega_{ij}. \tag{34}$$

The choice of the set $\mathcal{A}$ of pairs of adjacent substructures and the subsets $\mathcal{S}_{ij}$ of their intersections will be specified in Sec. 6.

To guarantee that ω_{ij} is finite, we need

Assumption 8 $\forall w_{ij} \in \widetilde{W}_{ij} : S_{ij}w_{ij} = 0 \Rightarrow B_{ij}w_{ij} = 0.$

Assumption 8 is satisfied when the substructures i and j linked by only the coarse degrees of freedom do not form a mechanism, in other words, if the

coarse degrees of freedom are sufficient to constrain the rigid body modes of the two substructures into a single set of rigid body modes, which are then continuous across substructure boundary. Assumption 8 can be satisfied in some other cases as well, for example when two substructures in 3D share a straight edge, all degrees of freedom are displacement degrees of freedom, and the coarse degrees of freedom are displacements at the ends of the edge. Then $S_{ij}w_{ij} = 0$ allows for independent rigid body rotations of the substructures around the common edge, $\dim(\text{null } S_{ij}) = 7$, yet the values of w_{ij} on the common edge coincide.

The computation of ω_{ij} proceeds exactly as the computation of ω in Sec. 3. Let Π_{ij} be obtained from (26), where all matrices on the right-hand side are replaced by their restrictions on substructures i and j as in (31-32). Then ω_{ij} is the largest eigenvalue of the generalized eigenvalue problem

$$\Pi_{ij}B_{ij}^T B_{Dij}S_{ij}B_{Dij}^T B_{ij}\Pi_{ij}w_{ij} = \omega_{ij}\left(\Pi_{ij}S_{ij}\Pi_{ij} + t\left(I - \Pi_{ij}\right)\right)w_{ij}. \qquad (35)$$

However, the matrices on both sides of (35) are typically singular even if Assumption 8 is satisfied because of rigid body modes that move substructures i and j as a whole. To reduce (35) to an eigenvalue problem with the matrix on the right-hand side positive definite, we use matrices Z_i, Z_j that generate a superspace of rigid body modes of the two substructures:

$$\text{null } S_i \subset \text{range } Z_i, \quad \text{null } S_j \subset \text{range } Z_j \qquad (36)$$

The matrices Z_i and Z_j are often available from finite element software. To avoid using any other information than the system matrices, we can instead use the matrices of the coarse basis functions

$$Z_i = \Psi_i, \quad Z_j = \Psi_j$$

from (21), because the span of the coarse basis functions contains the rigid body modes. In this case, however, this will be more expensive, because there are typically more coarse basis functions for the two substructures than the number of the rigid body modes.

Let

$$Z_{ij} = \begin{bmatrix} Z_i \\ Z_j \end{bmatrix}.$$

Then

$$\text{null}\left(\Pi_{ij}S_{ij}\Pi_{ij} + t\left(I - \Pi_{ij}\right)\right) \subset \text{null } S_{ij} \subset \text{range } Z_{ij}$$

and we find a basis of $\text{null}\left(\Pi_{ij}S_{ij}\Pi_{ij} + t\left(I - \Pi_{ij}\right)\right)$ by computing first the nullspace of a much smaller symmetric positive semidefinite matrix,

$$\text{null}\left(Z_{ij}^T\left(\Pi_{ij}S_{ij}\Pi_{ij} + t\left(I - \Pi_{ij}\right)\right)Z_{ij}\right) = \text{range } K, \qquad (37)$$

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and applying the QR decomposition

$$Z_{ij}K = QR, \quad Q^T Q = I,$$

which gives

$$\text{range}\, Q = \text{null}\, \left(\Pi_{ij} S_{ij} \Pi_{ij} + t\, (I - \Pi_{ij}) \right).$$

Consequently,

$$\overline{\Pi}_{ij} = I - QQ^T$$

is the orthogonal projection onto range $(\Pi_{ij} S_{ij} \Pi_{ij} + t\,(I - \Pi_{ij}))$, and applying Lemma 5 again, we have:

Theorem 9 *The nonzero eigenvalues $\lambda_{ij,k}$ and the corresponding eigenvectors $w_{ij,k}$ of*

$$\Pi_{ij} B_{ij}^T B_{Dij} S_{ij} B_{Dij}^T B_{ij} \Pi_{ij} w_{ij,k} = \lambda_{ij,k} \Pi_{ij} S_{ij} \Pi_{ij} w_{ij,k}, \tag{38}$$

are the same as the nonzero eigenvalues and the corresponding eigenvectors of

$$X_{ij} w_{ij,k} = \lambda_{ij,k} Y_{ij} w_{ij,k}, \tag{39}$$

where

$$X_{ij} = \Pi_{ij} B_{ij}^T B_{Dij} S_{ij} B_{Dij}^T B_{ij} \Pi_{ij},$$
$$Y_{ij} = \left(\overline{\Pi}_{ij} \left(\Pi_{ij} S_{ij} \Pi_{ij} + t\,(I - \Pi_{ij}) \right) \overline{\Pi}_{ij} + t\, \left(I - \overline{\Pi}_{ij} \right) \right),$$

In addition, X_{ij} is symmetric positive semidefinite and Y_{ij} is symmetric positive definite.

The matrices in the eigenvalue problems (37) and (39) are dense. If it is more efficient to represent the matrices as dense or to use an eigensolver that requires only matrix-vector multiplications depends on the size of the substructures and the dimension of the problem.

5 Adaptive Selection of Coarse Degrees of Freedom

We start with the following well known result from linear algebra, formulated in a way suitable for our purposes.

Lemma 10 *Let $d\,(\cdot, \cdot)$ and $e\,(\cdot, \cdot)$ be symmetric positive semidefinite bilinear forms on a linear space Y of dimension n, $e\,(\cdot, \cdot)$ positive definite. Then the generalized eigenvalue problem in variational form*

$$u \in Y : d\,(u, v) = \lambda e\,(u, v), \quad \forall v \in V$$

has n linearly independent eigenvectors u_k and the corresponding eigenvalues $\lambda_k \geq 0$. Order $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq 0$. Then for any subspace $Y_k \subset Y$ of

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dimension k,

$$\max_{u \in Y_k, u \neq 0} \frac{d(u,u)}{e(u,u)} \geq \lambda_{k+1},$$

with equality if

$$Y_k = \{u \in V : e(u_l, u) = 0, \quad \forall l = 1, \ldots, k\}. \tag{40}$$

Write the space of vectors with the coarse degrees of freedom continuous across the interface between the substructures i and j as

$$\widetilde{W}_{ij} = \left\{ w_{ij} : Q_{Dij}^T B_{ij} w_{ij} = 0 \right\}, \tag{41}$$

similarly as in (17). Now the dual writing (41) of the coarse degrees of freedom allows us to garantee that ω_{ij} does not exceed a given target value τ by adding the minimal number of coarse degrees of freedom:

Theorem 11 *Let $w_{ij,k}^T$ be the eigenvectors and $\lambda_{ij,k}$ the eigenvalues from (38), where, without loss of generality, $\lambda_{ij,1} \geq \lambda_{ij,2} \geq \ldots$. Suppose $\ell_{ij} \geq 0$. Let the dual coarse degree of freedom selection matrix Q_{Dij}^T be augmented to become $\left[Q_{Dij}^T, q_{Dij,1}^T, \ldots, q_{Dij,\ell_{ij}}^T \right]$ with*

$$q_{Dij,k}^T = w_{ij,k}^T B_{ij}^T B_{Dij} S_{ij} B_{Dij}^T, \tag{42}$$

Then $\omega_{ij} = \lambda_{ij,\ell_{ij}+1}$, and $\omega_{ij} \geq \lambda_{ij,\ell_{ij}+1}$ for any other augmentation of Q_{Dij}^T by at most ℓ_{ij} columns. In particular, if $\lambda_{ij,\ell_{ij}+1} \leq \tau$ for all pairs of substructures i,j with a common edge, then $\widetilde{\omega} \leq \tau$.

PROOF. Apply Lemma 10 with $Y = \widetilde{W}_{ij}$, $u = w_{ij}$, and

$$\frac{d(u,u)}{e(u,u)} = \frac{\left\| B_{Dij}^T B_{ij} w_{ij} \right\|_{S_{ij}}^2}{\| w_{ij} \|_{S_{ij}}^2} = \frac{w_{ij}^T B_{ij}^T B_{Dij} S_{ij} B_{Dij}^T B_{ij} w_{ij}}{w_{ij}^T S_{ij} w_{ij}}.$$

Then the orthogonality conditions in (40) become

$$\underbrace{w_{ij,k}^T B_{ij}^T B_{Dij} S_{ij} B_{Dij}^T}_{q_{Dij,k}^T} B_{ij} w_{ij} = 0, \quad k = 1, \ldots, \ell_{ij},$$

and V_k is $\widetilde{W}_{ij}$ after the augmentation of Q_{Dij}^T. $\quad\square$

It remains to construct the augmentation of the primal constraint matrix Q_P from the augmentation of Q_{Dij}^T. We need some more notation. Recall that by construction, every row of B_{ij} contains exactly one $+1$ and one -1. Let

U_{ij} be the space of vectors of global degrees of freedom on the intersection of substructures i and j that are constrained to coincide by the condition $B_{ij}w_{ij} = 0$. Then

$$B_{ij}w_{ij} = D_{ij}\left(I_{ij}w_i - I_{ji}w_j\right).$$

where $I_{ij} : W_i \to U_{ij}$ and $I_{ji} : W_j \to U_{ij}$ are $0-1$ matrices that restrict vectors of local degrees of freedom on substructure i and j, respectively, to degrees of freedom in the subset $\mathcal{S}_{ij}$, and D_{ij} is a diagonal matrix with diagoal entries ± 1. Then the condition $w_{ij} \in \widetilde{W}_{ij}$ can be written as

$$Q_{Dij}^T D_{ij}(I_{ij}w_i - I_{ji}w_j) = 0,$$

Let $R_{ij} : U \to U_{ij}$ be the $0-1$ matrix that restrict global vectors of degrees of freedom to the intersection of substructures i and j. Then the constraint defined by a column of Q_{Dij} can be written as

$$q_D^T R_{ij}^T D_{ij}I_{ij}w_i = q_D^T R_{ij}^T D_{ij}I_{ji}w_j, \tag{43}$$

where $q_D \in U$ has zeros everywhere outside of the subset $\mathcal{S}_{ij}$ of the intersection of the substructures i and j. The common value of both sides of (43) defines a coarse degree of freedom, and the corresponding column q_P of Q_P is is obtained by a permutation and a possible change of sign of the entries of q_D so that

$$q_P^T R_i^T = q_D^T R_{ij}^T D_{ij}I_{ij}. \tag{44}$$

The matrices R_{ci} and R_{cj} are augmented by one additional row.

Note that the new coarse degrees of freedom are defined as linear combinations of degrees of freedom from the given subset $\mathcal{S}_{ij}$ of the intersection of the adjacent substructures.

Now Lemma 10 and the formulation of the constraints allow us to add constraints that decrease the indicator ω_{ij} in an optimal manner. The proposed adaptive algorithm follows.

Algorithm 1 *To add coarse degrees of freedom to guarantee that $\omega_{ij} \leq \tau$, for a given a target value τ,*

(1) *compute the eigenvalues and eigenvectors of (39), starting from the largest eigenvalues, until the first ℓ_{ij} is found such that $\lambda_{ij,\ell_{ij}+1} \leq \tau$*
(2) *Add to Q_{Dij} the colums $q_{Dij,k}$ from (42).*
(3) *Obtain the corresponding columns of Q_P from (44) and update the constraint matrices C_i and R_{ci} from (18).*

The following two tests were done to check the correctness of our implementation.

(1) To make sure that the constraints were added correctly, we have confirmed that the two definitions of $\widetilde{W}_{ij}$ from (32) and (41) coincide,

$$\left\{ w_{ij} : Q_{Dij}^T B_{ij} w_{ij} = 0 \right\} = \left\{ w_{ij} : \exists u_c : C_{ij} w_{ij} = R_{cij} u_c \right\}.$$

In particular, we have computed matrices G and H such that

$$\text{range}\, G = \text{null}(Q_{Dij}^T B_{ij}) \quad \text{range}\, H = \begin{bmatrix} I & 0 \end{bmatrix} \text{null}\left(\begin{bmatrix} C_{ij} & -R_{cij} \end{bmatrix} \right),$$

and then we have checked numerically that range G = range H by testing that $GG^\dagger H = H$ and $HH^\dagger G = G$, where † is the Moore-Penrose pseudoinverse.

(2) We have confirmed that after the new coarse degrees of freedom are added, the new ω_{ij} equals $\lambda_{ij,\ell_{ij}+1}$ following Theorem 11.

6   Numerical Results

The method was tested on plane elasticity, discretized by Lagrange bilinear elements on a rectangular mesh decomposed into 16 substructure, with one edge between the substructures jagged (Fig. 1). The set $\mathcal{A}$ of pairs of adjacent substructures to compute the condition number indicator $\widetilde{\omega}$ by (34) was chosen as the set of all pairs of substructures with a common edge. The subsets $\mathcal{S}_{ij}$ to define new coarse degrees of freedom were taken as whole edges including corners. The starting set of coarse degrees of freedom consisted of all corner degrees of freedom. The matrices D_P in the averaging operator $E = RD_PR^T$ were diagonal, with the diagonal entries proportional to the diagonal entries of the substructure matrices before elimination of interiors. The computations were done in Matlab. The generalized eigenproblems on pairs of substructures were solved by the Matlab `eig` function with Choleski decomposition.

In Table 1, we show that the eigenvalues $\lambda_{ij,k}$ associated with edges between substructures clearly distinguish between the problematic edge and the others. Table 2 demonstrates that the addition of the coarse degrees of freedom created from the associated eigenvectors according to Theorem 11 decreases the condition number of the preconditioned system to approximately the prescribed value τ. Tables 3 and 4 contain the results of the same test for almost incompressible elasticity; here the iterations converge poorly or not at all without the additional coarse degrees of freedom, but adding the coarse degrees of freedom again decreases the condition number of the preconditioned system to approximately the prescribed value τ, and, in particular, the convergence of the iterations is recovered. This incompressible elasticity problem is particularly hard for an iterative method because standard bilinear elements were used instead of stable elements or reduced integration. The

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number of the added coarse degrees of freedom grows as the material approaches incompressibility. However, the purpose of this test was to show that the method can identify the problematic part of the problem and deal with it. We do not advocate the present method or the elements used for the solution of almost incompressible problems.

We have also simulated the effect of approximate eigensolvers. The eigenvector w_{ij} from (39) was replaced by $\widetilde{w}_{ij} = w_{ij} + \varepsilon r$, where r is a random vector with independent normally distributed entries, and ε was determined from the condition $|Y_{ij}\varepsilon r| = 0.5 |Y_{ij}w_{ij}|$, where $|\cdot|$ is the Euclidean norm. The eigenvalues λ_{ij} were replaced by the Rayleigh-Ritz values $\widetilde{w}_{ij}^T X_{ij} \widetilde{w}_{ij}^T / \widetilde{w}_{ij}^T Y_{ij} \widetilde{w}_{ij}^T$. The approximate eigenvectors and eigenvalues were then used in all computations instead of the exact ones. The results were essentially identical within display accuracy to those in Table 3.

7   Conclusion

Robustness of iterative substructuring methods in the presence of irregular mesh decomposition, singularities, and other adverse circumstances, is very important. We have described an adaptive algorithm that aims heuristically to achieve a predetermined convergence rate by adding coarse degrees of freedom using the solutions of eigenproblem associated with pairs of adjacent substructures. We have tested the performance of our algorithm on linear elasticity problems (both compressible and almost incompressible) in 2D with a regular mesh decomposition. Numerical tests have verified that the proposed condition number indicator is quite close to the actual condition number of the preconditioned problem and that the algorithm can find troublesome parts of the problem and concentrate computational work there to achieve good convergence at low cost. It was also verified numerically that the method still works well when the eigenproblems used to create the added coarse degrees of freedom are solved only approximately.

Future developments include identification of suitable 3D versions of the algorithm, tests on industrial type problems, and an efficient implementation with approximate eigensolvers. Our method is based on the assumption that the starting coarse degrees of freedom are already sufficient to prevent rigid body motions between certain pairs of adjacent substructures; the use of cheap approximate eigensolvers to obtain such starting coarse degrees of freedom should be also studied. A formulation of the present method in terms of the change of variable from [15] or in an abstract variational setting only would be also interesting. Finally, extensions of the adaptive approach to the multilevel case would be important to make possible the solution of problems that are both very large and numerically difficult.

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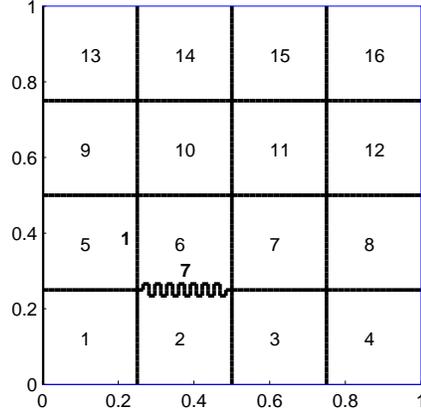
Fig. 1. Mesh with $H/h = 16$, 4×4 substructures, and one jagged edge between substructures 2 and 6. Zero displacement is imposed on the left edge. For compressible elasticity (Tables 1 and 2(a)) and tolerance $\tau = 10$, 7 coarse degrees of freedom at the jagged edge and 1 coarse degree of freedom at an adjacent edge are added automatically.

i	j	$\lambda_{ij,1}$	$\lambda_{ij,2}$	$\lambda_{ij,3}$	$\lambda_{ij,4}$	$\lambda_{ij,5}$	$\lambda_{ij,6}$	$\lambda_{ij,7}$	$\lambda_{ij,8}$
1	2	3.7	2.3	1.4	1.3	1.1	1.1	1.1	1.1
1	5	5.8	3.2	2.3	1.4	1.2	1.1	1.1	1.1
2	3	6.0	2.5	1.7	1.3	1.2	1.1	1.	1.1
2	6	21.7	19.5	17.8	14.9	14.5	11.7	11.2	9.7
3	4	3.3	2.3	1.4	1.3	1.1	1.1	1.1	1.1
3	7	7.1	5.1	3.2	1.8	1.4	1.3	1.2	1.1
4	8	5.9	3.4	2.6	1.4	1.2	1.1	1.1	1.1
5	6	12.0	4.9	4.4	1.8	1.6	1.3	1.3	1.2
5	9	5.9	3.4	2.6	1.4	1.3	1.3	1.1	1.1
6	7	8.7	4.9	3.9	1.8	1.5	1.3	1.2	1.1
6	10	7.3	4.8	3.4	1.8	1.4	1.3	1.2	1.1

Table 1

Several largest eigenvalues $\lambda_{ij,k}$ for several edges for the elasticity problem from Fig. 1 with $H/h = 16$. $(i, j) = (2, 6)$ is the jagged edge.

H/h	$Ndof$	τ	Nc	$\widetilde{\omega}$	κ	it
4	578		42	10.3	5.6	19
		10	43	5.2	4.0	18
		3	44	3.0	4.0	18
		2	58	2.0	2.8	15
16	8450		42	22	20	37
		10	50	8.7	9.9	29
		3	77	3.0	4.6	22
		2	112	2.0	2.6	15
64	132098		42	87	40	55
		10	89	9.8	9.9	36
		3	151	3.0	4.7	22
		2	174	2.0	2.9	17

Table 2

BDDC results for plane elasticity on a square with one jagged edge. The Lamé coefficients are $\lambda = 1$ and $\mu = 2$. H/h is the number of elements per substructure in one direction, $Ndof$ the number of degrees of freedom in the problem, τ the condition number tolerance as in Theorem 11, Nc the number of coarse degrees of freedom, $\widetilde{\omega}$ the apriori condition number indicator from (34), κ the approximate condition number computed from the Lanczos sequence in conjugate gradients, and it the number of BDDC iterations for relative residual tolerance 10^{-8}.

H/h	$Ndof$	τ	Nc	$\widetilde{\omega}$	κ	it
4	578		42	284.7	208.4	65
		10	68	8.2	8.6	28
		5	89	5.0	4.6	22
		3	114	2.9	2.6	16
16	8450		42	1012.0	1010.0	157
		10	87	9.9	9.6	29
		5	94	4.9	4.4	22
		3	126	3.0	2.9	19
64	132098		42	6909.8	1470.9	15
		10	183	9.8	9.7	37
		5	213	5.0	4.9	26
		3	274	3.0	3.0	20

Table 3

BDDC results for almost incompressible plane elasticity on a square with one jagged edge. The Lamé coefficients are $\lambda = 1000$ and $\mu = 2$. The headings are same as in Table 2.

H/h	$Ndof$	τ	Nc	$\widetilde{\omega}$	κ	it
4	578		42	2743.9	1875.04	158
		10	118	4.6	3.5	17
		5	118	4.6	3.5	17
		3	120	2.9	2.7	16
16	8450		42	9483.5	9389.9	113
		10	97	9.6	10.1	33
		5	120	5.0	5.0	24
		3	280	3.0	2.9	18
64	132098		42	29680.6	NA	∞
		10	218	9.8	9.6	40
		5	269	4.9	4.2	25
		3	313	3.0	2.9	18

Table 4

BDDC results for almost incompressible plane elasticity on a square with one jagged edge. The Lamé coefficients are $\lambda = 10000$ and $\mu = 2$. The headings are same as in Table 2.

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