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Materials Incorporating Volume Fraction  
with Detailed Computations**

Lynn Schreyer Bennethum

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# Compressibility Moduli for Porous Materials Incorporating Volume Fraction with Detailed Computations

Lynn S. Bennethum \*

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## Abstract

Traditional deformation parameters such as drained,unjacketed, and pseudo-bulk compressibilities are developed for a saturated porous media assuming they can be uniquely determined by measuring the liquid pressure, confining pressure, and changes in volume. Physically however, the liquid volume fraction (porosity for a saturated porous medium) plays an important role, yet it is not directly measurable. In this paper the compressibilities are defined in terms of the experiments used to evaluate them, and then mathematically related to the compressibilities of the liquid and solid phases and the volume fraction. The results can then be used to determine the effects volume fraction and compressibilities of each phase have on deformation properties of the porous media. This theory is then shown to be a generalization of previous works and comparisons are made with previously derived relationships of Zimmerman, Biot, and Gassmann.

This version contains details of computations in Appendix A.

**Key words:** thermodynamics, pressure, compressibility, mixture theory, compressibility moduli

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\*University of Colorado at Denver, Center for Computational Mathematics, Campus Box 170, 1250 14th St. Suite 600, P.O. Box 173364, Denver, CO 80217-3364. Lynn.Bennethum@cudenver.edu

# 1 Introduction

We consider a porous material composed of two phases, a liquid and a solid. For such porous materials several compressibility moduli are generally defined [29]: drained, undrained, jacketed, unjacketed, etc. These moduli are defined by the experiment used to measure them and they are then used in the effective stress principle to model deformation and in large-scale tectonic calculations [27]. Previous theoretical formulations for the compressibility moduli (e.g. Zimmerman [31], Biot [8, 9]), assume that volume is a function of the liquid pressure and the differential pressure (the difference between the overburden pressure and the liquid pressure). Because this theoretical foundation is widely cited and used [28, 25, 6, 7], it seems appropriate to understand the assumptions which underlie these theories.

The purpose of this paper is to (1) understand the relationship between the compressibilities of the individual phases and the compressibilities which are measurable (drained, undrained, etc.), (2) set up a rigorous framework which will allow extension to more complicated porous materials (e.g. swelling porous materials such as highly active clay soils, concrete, and cartilage), (3) understand the implicit assumptions made in previous work, and (4) demonstrate the mathematics required for such calculations. We do this by generalizing the classical work of Biot [8, 9] and Zimmerman [31] by exploring the consequences of methodically incorporating an internal variable which greatly influences the behavior of soils: the porosity or volume fraction of

liquid in the medium. This is motivated in Section 4 of this paper, and consequently it will be assumed that the compressibility moduli are not only a function of liquid and overburden pressure, but also of the volume fraction. This variable is not directly measurable; although changes in liquid content are measurable, the amount of liquid in a porous material at any given time is not. Perhaps this is the reason its influence has been neglected.

The approach we take is a thermodynamic approach which is more commonly used in physical chemistry [13]. Among the thermodynamical modeling approaches used for porous media, one can choose to classify the models into two camps: one in which the energy of the medium is used, e.g. [15, 8], and one in which the energy of each phase is assumed [11, 23, 22, 3]. The first approach is sometimes termed the *continuum formulation* and the latter the *micromechanical approach* [17], though the term *micro* may be a misnomer since both approaches can be used at the macroscale. By carefully defining the energy of each phase within the conservation of energy and summing the energy equation over the phases, one can determine the relationship between the energy for the porous material and the energy of each phase [2, 23]. Thus the two formulations are related and it is a matter of preferred philosophy as to which formulation is chosen (see also [17]). In this paper we opt for the latter framework - choosing to consider the energy of each phase individually.

In either framework, the choice of independent variables upon which the energy depends is postulated - either specifically as a function representing work, energy, or a potential, or more generally (as is done in this paper)

as a non-specified relationship in which energy or work is assumed only to depend upon a chosen list of independent variables and the specific form is determined via an exploitation of the second law of thermodynamics [14, 19, 5] and then refined experimentally. This second approach gives a very general framework upon which many other models are a specific case. Thus one can, after postulating the appropriate form of the energy, model elastic, inelastic, viscoelastic, etc. behavior.

To obtain the stated goals, and for pedagogical reasons, we make the following assumptions: (1) only uniform pressures and deformations are considered and hence shear stresses are neglected; this implies the porous material is homogeneous and isotropic, (2) there is only a liquid and a solid phase with no trapped liquid, (3) temperature is uniform and held constant (temperature is not considered an independent variable), (4) the pressure and internal energy of the liquid phase are only functions of the density of the liquid phase and not of the liquid volume fraction (thus the results will not hold for swelling porous materials, where the behavior of the liquid phase is strongly dependent on the volume fraction), and (5) the pressure and internal energy of the solid phase are functions of the density of the solid phase and the volume fraction. This last assumption implies that the solid-phase pressure is dependent upon not only the solid phase density, but also on the porosity. These assumptions are for this paper only, and with the exception of assumption (2), each of these assumptions can be relaxed under the framework presented.

In the next section we present the conservation of mass and linear momentum in the mixture theoretic formulation. This allows us to precisely define pertinent variables and makes it clear as to the setting for the following derivations. In Section 3, the compressibility moduli are defined in terms of intensive and extensive variables. In Section 4 the moduli are related to the pressures of the individual phases, the compressibilities of the individual phases, and the volume fraction. In Section 5 we discuss and compare the results with Zimmerman's theory [31], Biot's theory [8, 9], and Gassmann [18] and we end in Section 6 with a few concluding remarks.

## 2 Conservation of Mass and Linear Momentum

In this section we present the conservation of mass and linear momentum in the mixture theoretic formulation. We make the assumption of phase separation, i.e. we are working in a formulation where these equations hold for each phase (liquid and solid) with appropriate interaction terms. The interaction terms can be described rigorously in terms of integration over the liquid-solid interface by volume averaging [20, 23, 30], but a constitutive equation for the interaction terms is still needed to close the system. If  $\alpha = l, s$  where  $l$  denotes the liquid and  $s$  denotes the solid phase, then via volume averaging the conservation of mass (or continuity equation) is given by [23, 16]

$$\frac{\partial(\varepsilon^\alpha \rho^\alpha)}{\partial t} + \nabla \cdot (\varepsilon^\alpha \rho^\alpha \mathbf{v}^\alpha) = 0, \quad \alpha = l, s \quad (1)$$

where  $\varepsilon^\alpha$  is the volume fraction of phase  $\alpha$  in a Representative Elementary Volume (REV) [1] so that  $\varepsilon^l + \varepsilon^s = 1$ ,  $\rho^\alpha$  is the averaged (true) density of phase  $\alpha$  (mass of  $\alpha$  per volume of  $\alpha$ ), and  $\mathbf{v}^\alpha$  is the velocity of phase  $\alpha$  (length per unit time). This equation assumes there is no mass transfer between phases. If one wants to incorporate chemical reactions then one needs a continuity equation for each species and the right-hand side must include terms accounting for the rate of chemical reactions. Likewise if one wants to incorporate phase transformation, a term on the right-hand side must be added to account for this as well.

One form of the conservation of linear momentum for phase  $\alpha$  is [23, 16]

$$\varepsilon^\alpha \rho^\alpha \frac{D^\alpha \mathbf{v}^\alpha}{Dt} - \nabla \cdot (\varepsilon^\alpha \mathbf{t}^\alpha) - \varepsilon^\alpha \rho^\alpha \mathbf{g} = \varepsilon^\alpha \rho^\alpha \widehat{\mathbf{T}}_\beta^\alpha, \quad \alpha = l, s, \quad (2)$$

where  $\mathbf{t}^\alpha$  is the partial Cauchy stress tensor for phase  $\alpha$ ,  $\mathbf{g}$  is gravity, and  $\varepsilon^\alpha \rho^\alpha \widehat{\mathbf{T}}_\beta^\alpha$  represents the rate momentum is transferred from phase  $\beta$  ( $\beta = l, s \neq \alpha$ ) to phase  $\alpha$  [23]. Usually the constraint  $\varepsilon^l \rho^l \widehat{\mathbf{T}}_s^l + \varepsilon^s \rho^s \widehat{\mathbf{T}}_l^s = \mathbf{0}$  is imposed, and it implies that the momentum equation must hold for the porous material as a single continuum; but this assumes that the interface has no momentum, so this restriction would not hold if the liquid-solid interface itself could support a stress [20].

A similar expression can be written for the conservation of energy in terms of the energy for each phase [23, 4]. In this case the internal energy,  $u^\alpha$ , is a function of entropy and specific volume, and since entropy is not a tangible quantity, a Legendre transformation [13] is normally performed:

$A^\alpha = u^\alpha - T\eta^\alpha$  where  $T = \partial u^\alpha / \partial \eta^\alpha$  is temperature,  $\eta^\alpha$  is entropy (per unit mass of  $\alpha$ ), and  $A^\alpha$  is the Helmholtz potential [13]. The advantage of using the Helmholtz potential is that this potential is a function of temperature and specific volume instead of entropy and specific volume [13].

At this stage one usually postulates a form of the energy and from here derives constitutive equations which do not violate the second law of thermodynamics. We will assume that:

$$A^l = A^l(\rho^l, \dots) \quad A^s = A^s(\rho^s, \varepsilon^l, \dots).$$

The choice of other independent variables depends on the particular model, and by choosing the other independent variables appropriately, one may obtain a linear elastic, elastic, visco-elastic, or plastic model. The work which follows is independent of the choice of additional variables, assuming the additional variables are held fixed while measuring the compressibility moduli.

### 3 Thermodynamic Definition of Compressibility

Compressibility moduli are defined in multiple ways, so for completeness we show the equivalence of the definitions here. Generally, compressibility measures the change in volume with respect to pressure. A material which has no change in volume under any change of pressure is termed *incompressible*. This is an idealized material, as all materials are to some extent compressible, but in some practical applications one material may be con-



sidered incompressible either relative to other materials or because pressure changes are so small that an incompressible assumption is appropriate.

### 3.1 Compressibility of a Single-Phase Material

For a single-phase, isotropic material, compressibility is defined as

$$C = \frac{1}{\rho} \frac{d\rho}{dp} \quad (3)$$

where  $\rho$  is density and  $p$  is pressure (positive under compression). Letting  $v = 1/\rho$  be the specific volume, one can switch to the more directly measurable independent variable,  $V$ , the total volume, as

$$C = v \frac{dv}{dp} \frac{d\rho}{dv} \quad (4)$$

$$= -\frac{1}{v} \frac{dv}{dp} \quad (5)$$

$$= -\frac{1}{V} \left. \frac{\partial V}{\partial p} \right|_M \quad (6)$$

where  $M$  is the total mass. Note that when switching from intensive to extensive independent variables we pick up an extra independent variable (in this case, mass). For an isotropic solid, the pressure is defined in terms of the trace of the stress tensor as  $p^s = -\frac{1}{3}(t_{11} + t_{22} + t_{33})$  where  $\mathbf{t}$  is the Cauchy stress tensor. For a linearly elastic solid the bulk modulus of compression is defined to be [21], p. 202:

$$\kappa = -\frac{dp}{d(\text{tre})} \quad (7)$$

where  $\mathbf{e}$  is the strain tensor,  $\text{tre}$  is the volumetric strain, given by the trace of the strain tensor,  $e_{11} + e_{22} + e_{33}$  [26], and  $\kappa = \frac{2}{3}\mu + \lambda$  where  $\mu$  and

$\lambda$  are the Lamé moduli. Let's assume that we have small strain so that  $\text{tr} \mathbf{e} = (V - V_0)/V_0$  where  $V$  is the total volume and  $V_0$  is the reference volume. Then we have

$$\kappa \approx -\frac{dp}{d\left(\frac{V-V_0}{V_0}\right)} = -V_0 \frac{dp}{dV} \quad (8)$$

so that  $\kappa \approx 1/C^s$ . Often if the solid is linearly elastic  $\kappa$  is defined as [21]

$$-\frac{p}{\text{tr} \mathbf{e}} \approx -\frac{p - p_0}{\left(\frac{V-V_0}{V_0}\right)}$$

i.e. we can replace the derivative with the quotient. For this work we use

$$C = \frac{1}{\rho} \frac{d\rho}{dp} = -\frac{1}{V} \left. \frac{\partial V}{\partial p} \right|_M \quad (9)$$

$$K = 1/C = \rho \frac{dp}{d\rho} = -v \frac{dp}{dv} = -V \left. \frac{\partial p}{\partial V} \right|_M, \quad (10)$$

where  $v = 1/\rho$ . Thus we assume we are always dealing with an isotropic solid. For a non-isotropic medium compressibility is not in general defined, except where possible through the stress-strain constitutive relationship.

### 3.2 Compressibility in Porous Media

In porous materials, there are several definitions for compressibility, depending on whether one takes the measurement allowing the liquid phase to drain or not, or whether one measures the liquid pressure or loading pressure, or the change in volume of the porous matrix, or the change in volume of the liquid phase only. If the liquid phase is allowed to drain, the experimental setup for determining these compressibilities is given in Figure 1. The porous

material, which has a total volume composed of the sum of the volume of the liquid phase,  $V^l$ , and solid phase  $V^s$ , is confined by an impermeable membrane except at one location where the liquid is allowed to leave the system. If the liquid phase is continuous so that the liquid can flow freely, then the pressure of the liquid in the porous material is given by  $p^l$  and this can either be imposed or measured. The uniform pressure on the system is sometimes referred to as the *confining pressure*, or as we term here, the *total pressure*,  $p$ . It can be shown by summing the momentum equations that  $p = \varepsilon^l p^l + \varepsilon^s p^s$  where  $\varepsilon^l$  is the volume fraction of the liquid phase,  $V^l/(V^l + V^s)$ , and  $\varepsilon^s$  is the volume fraction of the solid phase,  $1 - \varepsilon^l$  [5]. There are four variables which vary and are directly observable: the total pressure,  $p$ , liquid pressure,  $p^l$ , volume of the liquid in the porous material,  $V^l$ , and the volume of the bulk material,  $V = V^l + V^s$ . Experimentally one usually prescribes two of these variables and measures the response of the other two. If the liquid phase is not allowed to drain, then the compressibility is termed undrained and is determined by measuring the change in pressure,  $p$ , with respect to (bulk) volume,  $V$ .

There are many bulk compressibilities, and here we mention a few. In measuring the *drained compressibility* the liquid is allowed to drain from the system as it is being compressed so that the liquid pressure is held constant. Thus measurements are made at constant liquid pressure with no solid leaving

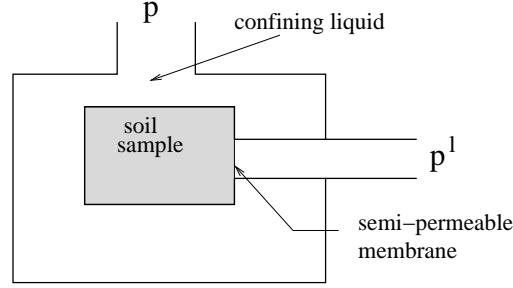


Figure 1: Experiment for compressibility

the system:

$$C_d^{-1} = K_d = -V \left. \frac{\partial p}{\partial V} \right|_{p^l, M^s} \quad (11)$$

where  $M^s$  is the total mass of the solid phase. Thus we are implicitly assuming that  $p = p(V, p^l, M^s)$ , i.e. that the external pressure is a function of the total volume, liquid pressure, and the mass of the solid phase.

To obtain the *unjacketed compressibility* the soil sample in Figure 1 is not confined so that the bulk liquid pressure (and hence confining pressure) and pore liquid have the same pressure ( $p^l = p$ ). Thus measurements are made keeping  $p^l - p$  and  $M^s$  fixed:

$$(C')^{-1} = K' = -V \left. \frac{\partial p}{\partial V} \right|_{p^l - p, M^s}. \quad (12)$$

In this setting it is implicitly assumed that  $p = p(V, p^l - p, M^s)$ . Note that this is not the compressibility of the solid phase as the volume is the volume of the matrix (not just the solid volume), and  $p$  is not necessarily equal to  $p^s$ .

The *undrained compressibility* is measured allowing no fluid or solid to

leave the system, so that the total mass of fluid and solid is held fixed. Thus

$$C_u^{-1} = K_u = -V \left. \frac{\partial p}{\partial V} \right|_{M^l, M^s}. \quad (13)$$

This implicitly assumes  $p = p(V, M^l, M^s)$ , i.e. the external pressure is a function of the total volume and the total mass of the the liquid and solid.

Other compressibilities include, respectively, the *drained pore compressibility*, *pseudo-bulk compressibility or poroelastic expansion coefficient*, and *pore compressibility* [31]

$$C_p^{-1} = K_p = -V^l \left. \frac{\partial p}{\partial V^l} \right|_{p^l, M^s} \quad (14)$$

$$C_{bp}^{-1} = K_{bp} = -V \left. \frac{\partial p^l}{\partial V} \right|_{p, M^s} \quad (15)$$

$$C_{pp}^{-1} = K_{pp} = -V^l \left. \frac{\partial p^l}{\partial V^l} \right|_{p, M^s}. \quad (16)$$

Note that  $C_{pp}^{-1}$  is not the same as the liquid compressibility. Liquid compressibility involves the change in liquid pressure with respect to liquid volume keeping the mass of the liquid fixed. Here the pore compressibility is evaluated keeping the overburden pressure and the mass of the solid fixed, which imposes a different boundary condition on the compressibility experiment.

## 4 Compressibility Moduli as a Function of Liquid and Solid Compressibilities

The goal of this section is to determine the relationship between the porous media compressibility moduli and the compressibility of the liquid and solid phase. To do this we add an additional independent variable, the volume fraction of the liquid phase,  $\varepsilon^l$ . As stated in the introduction, it is assumed that the solid-phase pressure is a function of the density of the solid phase as well as the volume fraction, and that the liquid pressure is a function of the liquid density only:

$$p^s = p^s(\varepsilon^s, \rho^s) \quad p^l = p^l(\rho^l). \quad (17)$$

Note that since  $p^\alpha = (\rho^\alpha)^2 \frac{\partial A^\alpha}{\partial \rho^\alpha}$  [13, 5],  $p^\alpha$  must be a function of the same independent variables as the Helmholtz potentials,  $A^\alpha$ . If  $A^\alpha$  has additional independent variables then so would  $p^\alpha$ , but as long as these additional independent variables (e.g., deviatoric part of strain, temperature,...) are held fixed during the compressibility experiments, the following results still hold.

Equation (17) allows us to capture the effects of the solid deforming into the pores without changing density. The assumption that the liquid pressure is not a function of volume fraction means that the following results will not hold for swelling soils such as montmorillonite where the thermodynamic properties (including pressure) is a function of the proximity to the solid phase, or equivalently, that the thermodynamic properties are a function of the volume fraction. This is the topic of future work.

The following results require quite a bit of calculations. Here we present only one calculation in detail. The remaining results require similar calculations, and are presented in Appendix A.

For our sample calculation consider the drained compressibility moduli (11). Noting that

$$p = \varepsilon^l p^l + \varepsilon^s p^s, \quad (18)$$

(which is obtained by summing the linear momentum equation of each phase, (2)), we see that by (17) the total pressure is a function of three intensive variables:  $\rho^l$ ,  $\rho^s$ , and  $\varepsilon^l$ . Thus  $p$  must be a function of four extensive variables. For the drained compressibility we choose the four variables to be consistent with  $\rho^l$ ,  $\rho^s$ , and  $\varepsilon^l$ :  $V^s$ ,  $M^s$ ,  $V^l$ , and  $M^l$ . We note that we could also take different combination of these variables. We could assume  $p^l = p^l(M^l, V^l)$ , or  $p^l = p^l(\rho^l)$ , or, assuming we can solve for the mass of the liquid phase, we could write  $M^l = M^l(p^l, V^l)$ . Strictly speaking  $p^l(\rho^l) \neq p^l(M^l, V^l)$  but we do not introduce additional notation and make it clear what function we are referring to by making the independent variables clear.

For the drained compressibility moduli we choose  $p = p(V^s, M^s, V^l, p^l)$  as the choice of independent variables. This gives the following for the total differential

$$\begin{aligned} dp = & \left. \frac{\partial p}{\partial V^s} \right|_{M^s, V^l, p^l} dV^s + \left. \frac{\partial p}{\partial M^s} \right|_{V^s, V^l, p^l} dM^s + \left. \frac{\partial p}{\partial V^l} \right|_{V^s, M^s, p^l} dV^l \\ & + \left. \frac{\partial p}{\partial p^l} \right|_{V^s, M^s, V^l} dp^l. \end{aligned} \quad (19)$$

Note that at this point we can take the partial derivative of both sides with respect to *any* combination of the given independent variables. Mathematically we just take the partials with respect to each total change (the coefficients are held fixed). This is explained well by Callen [13]. Taking the partial derivative of both sides with respect to the total volume  $V$  keeping  $M^s$  and  $p^l$  fixed (as is done experimentally in determining the drained compressibility where it is assumed the total solid mass does not change and the liquid pressure is held constant) we have [13, 24]

$$V \left. \frac{\partial p}{\partial V} \right|_{p^l, M^s} = V \left. \frac{\partial p}{\partial V^s} \right|_{M^s, V^l, p^l} \left. \frac{\partial V^s}{\partial V} \right|_{p^l, M^s} + V \left. \frac{\partial p}{\partial V^l} \right|_{V^s, M^s, p^l} \left. \frac{\partial V^l}{\partial V} \right|_{p^l, M^s}. \quad (20)$$

Consider first  $\left. \frac{\partial V^l}{\partial V} \right|_{p^l, M^s}$ . We assume this term is measurable (observable) since changes in  $V^l$  and  $V$  are measurable, so let us denote it as  $g$ . The function  $g$  is a function of only  $V^l$  and  $V$  for this experiment, and so in terms of intensive variables it is a function of the volume fraction,  $\varepsilon^l$ . It may be a function of other independent variables as well, and this shall be discussed in detail later. Thus we have

$$g(\varepsilon^l) = \left. \frac{\partial V^l}{\partial V} \right|_{p^l, M^s}. \quad (21)$$

Since  $V^s = V - V^l$  we also have

$$\left. \frac{\partial V^s}{\partial V} \right|_{p^l, M^s} = 1 - g(\varepsilon^l). \quad (22)$$

We now need to evaluate the two remaining terms:  $\left. \frac{\partial p}{\partial V^s} \right|_{M^s, V^l, p^l}$  and  $\left. \frac{\partial p}{\partial V^l} \right|_{V^s, M^s, p^l}$ . We first convert these terms to intensive variables (the pressure is not a function of



the quantity of material). Since the mass of the solid is held fixed throughout all these experiments, we take advantage of this by dividing each extensive quantity by the total solid mass. Using such facts as  $\rho^s$  representing the mass of solid per volume of solid we get  $V/M^s = 1/(\varepsilon^s \rho^s)$ , and similarly  $V^l/M^s = (V^l/V)/(M^s/V) = \varepsilon^l/(\varepsilon^s \rho^s)$ . So for example, if  $V^l$  is held fixed in the partial, then  $V^l/M^s$  is equivalently held fixed and

$$-V \left. \frac{\partial p}{\partial V^s} \right|_{M^s, V^l, p^l} = -\frac{1}{\varepsilon^s \rho^s} \left. \frac{\partial p}{\partial \left( \frac{1}{\rho^s} \right)} \right|_{\frac{\varepsilon^l}{\varepsilon^s \rho^s}, p^l} = \frac{\rho^s}{\varepsilon^s} \left. \frac{\partial p}{\partial \rho^s} \right|_{\frac{\varepsilon^l}{\varepsilon^s \rho^s}, p^l}. \quad (23)$$

Similarly

$$\begin{aligned} -V \left. \frac{\partial p}{\partial V^l} \right|_{M^s, V^s, p^l} &= -\frac{1}{\varepsilon^s \rho^s} \left. \frac{\partial p}{\partial \left( \frac{\varepsilon^l}{\varepsilon^s \rho^s} \right)} \right|_{\frac{1}{\rho^s}, p^l} = -\frac{1}{\varepsilon^s} \left. \frac{\partial p}{\partial \left( \frac{\varepsilon^l}{\varepsilon^s} \right)} \right|_{\rho^s, p^l} \\ &= -\frac{1}{\varepsilon^s} \left. \frac{\partial p}{\partial \varepsilon^l} \right|_{\rho^s, p^l} \frac{\partial \varepsilon^l}{\partial \left( \frac{\varepsilon^l}{\varepsilon^s} \right)} = -\varepsilon^s \left. \frac{\partial p}{\partial \varepsilon^l} \right|_{\rho^s, p^l}. \end{aligned} \quad (24)$$

So substituting into (20) we now have

$$K_d = -V \left. \frac{\partial p}{\partial V} \right|_{p^l, M^s} = -g(\varepsilon^l) \varepsilon^s \left. \frac{\partial p}{\partial \varepsilon^l} \right|_{\rho^s, p^l} + (1 - g(\varepsilon^l)) \frac{\rho^s}{\varepsilon^s} \left. \frac{\partial p}{\partial \rho^s} \right|_{\frac{\varepsilon^l}{\varepsilon^s \rho^s}, p^l}. \quad (25)$$

To determine  $\partial p/\partial \varepsilon^l$  and  $\partial p/\partial \rho^s$ , we begin with  $p = \varepsilon^l p^l(\rho^l) + \varepsilon^s p^s(\varepsilon^s, \rho^s)$  and take the total differential:

$$\begin{aligned} dp &= \left. \frac{\partial p}{\partial \varepsilon^l} \right|_{\rho^l, \rho^s} d\varepsilon^l + \left. \frac{\partial p}{\partial \rho^l} \right|_{\varepsilon^l, \rho^s} d\rho^l + \left. \frac{\partial p}{\partial \rho^s} \right|_{\varepsilon^l, \rho^l} d\rho^s \\ &= (p^l - p^s) d\varepsilon^l + \varepsilon^l \frac{dp^l}{d\rho^l} d\rho^l + \varepsilon^s \left. \frac{\partial p^s}{\partial \varepsilon^s} \right|_{\rho^s} d\varepsilon^s + \varepsilon^s \left. \frac{\partial p^s}{\partial \rho^s} \right|_{\varepsilon^s} d\rho^s. \end{aligned} \quad (26)$$

Now to determine  $\partial p/\partial \varepsilon^l$  keeping  $\rho^s$  and  $p^l$  fixed, we take the partial of (26) with respect to  $\varepsilon^l$  to get

$$\left. \frac{\partial p}{\partial \varepsilon^l} \right|_{\rho^s, p^l} = (p^l - p^s) + \varepsilon^s \left. \frac{\partial p^s}{\partial \varepsilon^s} \right|_{\rho^s} (-1). \quad (27)$$

Similarly, taking the partial of (26) with respect to  $\rho^s$  keeping  $\varepsilon^l/(\varepsilon^s\rho^s)$  and  $p^l$  fixed we get

$$\left. \frac{\partial p}{\partial \rho^s} \right|_{\frac{\varepsilon^l}{\varepsilon^s\rho^s}, p^l} = (p^l - p^s) \left. \frac{\partial \varepsilon^l}{\partial \rho^s} \right|_{\frac{\varepsilon^l}{\varepsilon^s\rho^s}} + \varepsilon^s \left. \frac{\partial p^s}{\partial \varepsilon^s} \right|_{\rho^s} \left. \frac{\partial \varepsilon^s}{\partial \rho^s} \right|_{\frac{\varepsilon^l}{\varepsilon^s\rho^s}} + \varepsilon^s \left. \frac{\partial p^s}{\partial \rho^s} \right|_{\varepsilon^s}. \quad (28)$$

The term  $\left. \frac{\partial \varepsilon^l}{\partial \rho^s} \right|_{\frac{\varepsilon^l}{\varepsilon^s\rho^s}, p^l}$  is evaluated as follows. Since  $\varepsilon^l/(\varepsilon^s\rho^s)$  is held fixed, let's denote it as  $k$ , a constant. Then

$$\rho^s = \frac{\varepsilon^l}{k\varepsilon^s} \quad (29)$$

$$\left. \frac{\partial \rho^s}{\partial \varepsilon^l} \right|_k = \frac{1}{k(\varepsilon^s)^2} = \frac{\rho^s}{\varepsilon^l\varepsilon^s}, \quad (30)$$

so  $\partial \varepsilon^l / \partial \rho^s = \varepsilon^l \varepsilon^s / \rho^s$ . This also gives us  $\partial \varepsilon^s / \partial \rho^s = -\partial \varepsilon^l / \partial \rho^s$ . Substituting this result into (28) we have

$$\left. \frac{\partial p}{\partial \rho^s} \right|_{\frac{\varepsilon^l}{\varepsilon^s\rho^s}, p^l} = \frac{\varepsilon^l \varepsilon^s}{\rho^s} (p^l - p^s) - \frac{\varepsilon^l (\varepsilon^s)^2}{\rho^s} \left. \frac{\partial p^s}{\partial \varepsilon^s} \right|_{\rho^s} + \varepsilon^s \left. \frac{\partial p^s}{\partial \rho^s} \right|_{\varepsilon^s}. \quad (31)$$

Using (27) and (31) in (25) and simplifying yields the final form of the drained compressibility coefficient:

$$\begin{aligned} K_d &= g(\varepsilon^l) \left[ -\varepsilon^s (p^l - p^s) + (\varepsilon^s)^2 \left. \frac{\partial p^s}{\partial \varepsilon^s} \right|_{\rho^s} \right] \\ &+ (1 - g(\varepsilon^l)) \left[ \varepsilon^l (p^l - p^s) - \varepsilon^l \varepsilon^s \left. \frac{\partial p^s}{\partial \varepsilon^s} \right|_{\rho^s} + K^s \right] \end{aligned} \quad (32)$$

where  $K^s$  is the compressibility moduli for the solid phase as defined in (10):

$$K^s = \rho^s \left. \frac{\partial p^s}{\partial \rho^s} \right|_{\varepsilon^s}. \quad (33)$$

Note that in general, because  $p^s = p^s(\rho^s, \varepsilon^s)$  (i.e. the solid pressure is determined by density and volume fraction) the solid phase compressibility is a

function of volume fraction. However as a first approximation for the case where the liquid and solid are not highly interactive it would seem appropriate to assume  $K^s = K^s(\rho^s)$ .

From (32) we see that the drained compressibility is a function of the pressure difference  $p^l - p^s$ , the *configurational compression*,  $\partial p^s / \partial \varepsilon^s |_{\rho^s}$ , the volume fraction,  $\varepsilon^l$ , and the compressibility of the solid phase,  $K^s$ .

Repeating the above calculation for other compressibility moduli we obtain (see Appendix A)

$$\begin{aligned} K' &= -V \left. \frac{\partial p}{\partial V} \right|_{p^l - p, M^s} \\ &= \bar{g}(\varepsilon^l) \varepsilon^s \left. \frac{\partial p^s}{\partial \varepsilon^s} \right|_{\rho^s} + (1 - \bar{g}(\varepsilon^l)) \left[ -\varepsilon^l \left. \frac{\partial p^s}{\partial \varepsilon^s} \right|_{\rho^s} + \frac{1}{\varepsilon^s} K^s \right] \end{aligned} \quad (34)$$

$$\begin{aligned} K_u &= -V \left. \frac{\partial p}{\partial V} \right|_{M^l, M^s} \\ &= \tilde{g}(\varepsilon^l) \left[ -\varepsilon^s (p^l - p^s) + (\varepsilon^s)^2 \left. \frac{\partial p^s}{\partial \varepsilon^s} \right|_{\rho^s} + K^l \right] \\ &\quad + (1 - \tilde{g}(\varepsilon^l)) \left[ \varepsilon^l (p^l - p^s) - \varepsilon^l \varepsilon^s \left. \frac{\partial p^s}{\partial \varepsilon^s} \right|_{\rho^s} + K^s \right] \end{aligned} \quad (35)$$

$$\begin{aligned} K_p &= -V^l \left. \frac{\partial p}{\partial V^l} \right|_{p^l, M^s} \\ &= -\varepsilon^l \varepsilon^s (p^l - p^s) + \varepsilon^l (\varepsilon^s)^2 \left. \frac{\partial p^s}{\partial \varepsilon^s} \right|_{\rho^s} \\ &\quad + \left( \frac{1}{g(\varepsilon^l)} - 1 \right) \left[ (\varepsilon^l)^2 (p^l - p^s) - (\varepsilon^l)^2 \varepsilon^s \left. \frac{\partial p^s}{\partial \varepsilon^s} \right|_{\rho^s} + \varepsilon^l K^s \right] \\ &= \frac{\varepsilon^l}{g(\varepsilon^l)} K_d \end{aligned} \quad (36)$$

$$\begin{aligned}
K_{bp} &= V \left. \frac{\partial p^l}{\partial V} \right|_{p, M^s} \\
&= \hat{g}(\varepsilon^l) \left[ -\frac{\varepsilon^s}{\varepsilon^l} (p^l - p^s) + \frac{(\varepsilon^s)^2}{\varepsilon^l} \left. \frac{\partial p^s}{\partial \varepsilon^s} \right|_{\rho^s} \right] \\
&\quad + (1 - \hat{g}(\varepsilon^l)) \left[ (p^l - p^s) - \varepsilon^s \left. \frac{\partial p^s}{\partial \varepsilon^s} \right|_{\rho^s} + \frac{1}{\varepsilon^l} K^s \right] \tag{37}
\end{aligned}$$

$$\begin{aligned}
K_{pp} &= -V^l \left. \frac{\partial p^l}{\partial V^l} \right|_{p, M^s} \\
&= -\varepsilon^s (p^l - p^s) + (\varepsilon^s)^2 \left. \frac{\partial p^s}{\partial \varepsilon^s} \right|_{\rho^s} \\
&\quad + \left( \frac{1}{\hat{g}(\varepsilon^l)} - 1 \right) \left[ \varepsilon^l (p^l - p^s) - \varepsilon^l \varepsilon^s \left. \frac{\partial p^s}{\partial \varepsilon^s} \right|_{\rho^s} + K^s \right] \\
&= \frac{\varepsilon^l}{\hat{g}(\varepsilon^l)} K_{bp} \tag{38}
\end{aligned}$$

where

$$g(\varepsilon^l) = \left. \frac{\partial V^l}{\partial V} \right|_{p^l, M^s} \tag{39}$$

$$\tilde{g}(\varepsilon^l) = \left. \frac{\partial V^l}{\partial V} \right|_{M^l, M^s} \tag{40}$$

$$\bar{g}(\varepsilon^l) = \left. \frac{\partial V^l}{\partial V} \right|_{p^l - p, M^s} \tag{41}$$

$$\hat{g}(\varepsilon^l) = \left. \frac{\partial V^l}{\partial V} \right|_{p, M^s} . \tag{42}$$

What we have done is to essentially add an independent variable so that the partial derivatives are well-defined. To explain this consider (25). The experimental definition of the drained compressibility assumes that pressure,  $p$ , is a function of  $V$ ,  $p^l$ , and  $M^s$ ; i.e. that it is a function of 3 variables, some

of which are extensive. In reality pressure is a function of 4 variables (some of which are extensive), e.g.  $V$ ,  $p^l$ ,  $M^s$ ,  $V^s$ , or 3 intensive variables, e.g.  $\rho^s$ ,  $\rho^l$  or  $p^l$ , and  $\varepsilon^l$ . To relate the experimental values (e.g., the compressibilities) with the mathematically well-defined quantities (defined in terms of three intensive variables) we assume that  $g(\varepsilon^l)$ ,  $\tilde{g}(\varepsilon^l)$ , etc., are well-defined. Strictly, they may not be, since they have the same shortcomings as the definitions of the compressibility moduli. But also like the compressibility moduli, they are measurable, and though they may not be uniquely defined (they may take on different values depending on e.g.  $M^l$ ), by measuring the compressibility moduli and e.g.  $g(\varepsilon^l)$ , we can obtain the well-defined parameters  $\partial p^s / \partial \varepsilon^s|_{\rho^s}$ ,  $p^l - p^s$ . It is these well-defined parameters which can then be used in a mathematical model to arrive at a system of equations.

Note that although  $g(\varepsilon^l)$ ,  $\bar{g}(\varepsilon^l)$ , and  $\hat{g}(\varepsilon^l)$  are theoretically measurable,  $\tilde{g}(\varepsilon^l)$  is not measurable (there is no mass leaving the system to be measured). This limits the application of this theory in regards to composite materials, where in general no mass leaves the system.

## 5 Comparison with Zimmerman and Biot

In this section we strive to understand relationships (32) and (34)-(38) by comparing these results to what has been done by Zimmerman, [31], Biot [8], and Gassmann [18, 27].

## 5.1 Comparison with Zimmerman

In [31], Zimmerman implicitly makes a couple of assumptions. One is that the volumetric strain is only a function of the confining pressure,  $p$ , and the pore pressure,  $p^l$ , see equations (1.9) and (1.10). In so doing he is neglecting the dependence on either the mass of the solid or the solid density. This may be reasonable if the solid phase is incompressible relative to the soil mixture. He further assumes that theunjacketed compressibility,  $C'$ , is the solid compressibility,  $C^s$ , p. 11:

The crucial step in this derivation is the recognition [Geertsma, 1957] that this state [unjacketed] of stress and strain within the matrix is exactly the same as that which would occur if the pores were hypothetically filled up with matrix material, and the boundary conditions on the outer surface were left unchanged. In this latter case, the total bulk strain is equal to  $-C_r dP$ , so that the bulk volume change is given by  $dV_b = -C_r V_b^i dP$ .

In the above  $C_r$  is the compressibility of the rock, i.e.  $C_r = \frac{1}{K^s}$ , and  $V_b^i$  is the initial bulk volume, i.e.  $V_b^i = V_0$ . Comparing this result with (34) we see that we can recover this result if  $\bar{g}(\varepsilon^l) = \varepsilon^l$ , i.e. that the change in liquid volume with respect to bulk volume in theunjacketed state is the volume fraction. This may or may not be true since  $\bar{g}(\varepsilon^l)$  is not well-defined in regards to what is held fixed. To see this consider the following two ways of expressing  $V^l$ :

$$V^l = \varepsilon^l V \tag{43}$$

$$V^l = V - V^s. \quad (44)$$

Using the first equation, (43), we get

$$\left. \frac{\partial V^l}{\partial V} \right|_{\varepsilon^l} = \varepsilon^l \quad (45)$$

and using (44) we get

$$\left. \frac{\partial V^l}{\partial V} \right|_{V^s} = 1 \quad (46)$$

so that the second assumption implicitly assumes that the change of liquid volume with respect to bulk volume is proportional to the volume fraction.

Using assumption (45) for  $g(\varepsilon^l)$ ,  $\tilde{g}(\varepsilon^l)$ ,  $\bar{g}(\varepsilon^l)$ , and  $\hat{g}(\varepsilon^l)$  we obtain the following:

$$K_d = \varepsilon^s K^s \quad (47)$$

$$K' = K^s \quad (48)$$

$$K_u = \varepsilon^l K^l + \varepsilon^s K^s \quad (49)$$

$$K_{bp} = \frac{\varepsilon^s}{\varepsilon^l} K^s \quad (50)$$

$$K_{pp} = K_{bp}. \quad (51)$$

Of course it is highly unlikely that  $g(\varepsilon^l) = \tilde{g}(\varepsilon^l) = \bar{g}(\varepsilon^l) = \hat{g}(\varepsilon^l)$ , as they require different variables to be held fixed during the experiment, and the above relations are just too simple. For example, experimentally it is known that  $K_d$  and  $K^s$  are in general orders of magnitude different [31], and that (49) is considered only as an upper bound (constant strain) for composite materials. But with this simplification we recover all of Zimmerman's relations

(equations (2.2), (2.6), (2.7), and (2.8) from [31]):

$$C_{pp} = C_p - C^s \quad (52)$$

$$C_{bp} = C_d - C^s \quad (53)$$

$$C_p = \frac{1}{\varepsilon^l}(C_d - C^s) \quad (54)$$

$$C_{pp} = \frac{1}{\varepsilon^l}(C_d - (1 + \varepsilon^l)C^s). \quad (55)$$

## 5.2 Comparison with Biot

Comparison with Biot's work is not as straight forward, since Biot's formulation works with a single potential (as oppose to a potential for each phase). We begin with a thermodynamic definition of pressure in the mixture theoretic framework. Recall that for a single phase fluid [13],

$$p = -\frac{\partial A}{\partial v} = \rho^2 \frac{\partial A}{\partial \rho}. \quad (56)$$

For a mixture of phases, it has been shown that the pressure (as defined to be 1/3 the trace of the Cauchy stress tensor at equilibrium) is [11, 22, 5]

$$p^\alpha = -\left. \frac{\partial A^\alpha}{\partial v^\alpha} \right|_{\varepsilon^l, T, \rho^\beta, \dots} = (\rho^\alpha)^2 \left. \frac{\partial A^\alpha}{\partial \rho^\alpha} \right|_{\varepsilon^l, T, \rho^\beta, \dots} \quad (57)$$

where  $\alpha = l, s$  for the liquid and solid phases, respectively, and  $\rho^\beta$  is the density of phase  $\beta \neq \alpha$ .

Biot [8] assumed there exists a potential energy density,  $U$ , for which (using the notation of this paper)  $dU = -pd\varepsilon - p^l d\zeta$  and that this is an exact differential. This implies:

$$p = -\left. \frac{\partial U}{\partial \varepsilon} \right|_{\zeta} \quad (58)$$



$$p^l = - \left. \frac{\partial U}{\partial \zeta} \right|_{\epsilon} \quad (59)$$

where  $\epsilon$  is the volumetric strain and  $\zeta$  is “the increment of water volume per unit volume of soil”. It does not seem to be exactly clear what  $\zeta$  is as several definitions have been used. Biot and Willis [9] used  $\zeta = -\nabla \cdot (\epsilon^l \mathbf{u}^{l,s})$  where  $\mathbf{u}^{l,s} = \mathbf{u}^l - \mathbf{u}^s$  is the displacement of the fluid phase relative to the solid; Rice and Cleary [28] used  $\zeta = \frac{\epsilon^l \rho^l - \epsilon_0^l \rho_0^l}{\rho_0^l}$ ; and Berryman [6] used  $\zeta = \frac{V^l - V_0^l}{V}$ . Note that Rice and Cleary’s and Berryman’s definition coincide if the density of the liquid phase is constant, which is an assumption assumed in Biot’s work [8]. To make our comparison we will use Rice and Cleary’s definition since for a compressible fluid the change in mass is more accurately measurable (see also [32]). In order to make this comparison, we note that  $U$  must have the same units as pressure, or that it has units of energy/volume. In the mixture theoretic formulation set up in Section 2, the Helmholtz potential has units of energy per unit mass. By summing the conservation of energy equation for each phase and neglecting velocity, one obtains the Helmholtz potential for the mixture [16]:

$$A = \epsilon^l \rho^l A^l + \epsilon^s \rho^s A^s \quad (60)$$

which has units of energy per unit volume of mixture. Thus  $U$  and  $A$  have the same units.

Let’s look at Biot’s definition of confining pressure first. Using the standard small volumetric deformation assumption and that the mass of the solid

is fixed ( $M_0^s = M^s$ ), we have

$$\epsilon \approx \frac{V - V_0}{V} = 1 - \frac{V_0}{V} = 1 - \frac{\frac{V_0}{M_0^s}}{\frac{V}{M^s}} = 1 - \frac{\frac{1}{\epsilon_0^s \rho_0^s}}{\frac{1}{\epsilon^s \rho^s}} = 1 - \frac{\epsilon^s \rho^s}{\epsilon_0^s \rho_0^s}. \quad (61)$$

Thus from (58) we have Biot's thermodynamic definition of the confining pressure as

$$p = - \left. \frac{\partial U}{\partial \epsilon} \right|_{\zeta} = \left. \frac{\partial U}{\partial \left( \frac{\epsilon^s \rho^s}{\epsilon_0^s \rho_0^s} \right)} \right|_{\frac{\epsilon^l \rho^l}{\rho_0^l}} = \epsilon_0^s \rho_0^s \left. \frac{\partial U}{\partial (\epsilon^s \rho^s)} \right|_{\frac{\epsilon^l \rho^l}{\rho_0^l}} = \epsilon_0^s \rho_0^s \left. \frac{\partial U}{\partial \rho} \right|_{\frac{\epsilon^l \rho^l}{\rho_0^l}} \quad (62)$$

where  $\rho = \epsilon^l \rho^l + \epsilon^s \rho^s$  is the average density of the porous material. Note that the last equality holds because  $\epsilon^l \rho^l$  is held fixed in the partial.

In Appendix B, we derive the thermodynamic definition of pressure for the mixture formulation. Under the assumption that  $A(\epsilon^l, \rho^l, \rho^s) = \epsilon^l \rho^l A^l(\epsilon^l, \rho^l) + \epsilon^s \rho^s A^s(\epsilon^l, \rho^s)$ , we have (Appendix B):

$$p = - \left. \frac{\partial(vA)}{\partial v} \right|_{\frac{\epsilon^l \rho^l}{\rho}} = \rho^2 \left. \frac{\partial(vA)}{\partial \rho} \right|_{\frac{\epsilon^l \rho^l}{\rho}} = \epsilon^l \rho^2 \left. \frac{\partial(vA)}{\partial \rho} \right|_{\frac{\epsilon^l \rho^l}{\rho}, \rho^s} + \epsilon^s \rho^2 \left. \frac{\partial(vA)}{\partial \rho} \right|_{\frac{\epsilon^s \rho^s}{\rho}, \rho^l}. \quad (63)$$

Comparing (63) with Biot's definition, (62), we see that one of the differences is with the partial derivative portion of the pressure definitions: they are off by the factor of the specific volume,  $v = 1/\rho$ , which is inside the partial in (63). Bringing it out of the partial (which would be justified if the porous material were incompressible, which is what Biot assumed) gives a partial which is the same as that in the Biot's definition of pressure. The definitions are then off by the coefficient in front of the partial ( $\rho$  versus  $\epsilon_0^s \rho_0^s$ ), and by what is held fixed ( $\epsilon^l \rho^l / \rho$  versus  $\epsilon^l \rho^l / \rho_0^l$ ).

Now let's look at Biot's definition of the liquid pressure, (59). It is assumed that the mass of the solid is held fixed throughout, so keeping  $\epsilon$  fixed, which is equivalent to keeping the total volume,  $V$ , fixed, is also equivalent to keeping  $V/M^s = \frac{1}{\epsilon^s \rho^s}$  fixed, which in turn is equivalent to keeping  $\epsilon^s \rho^s$  fixed. Using Rice and Cleary's [28] definition for  $\zeta$  we have:

$$p^l = \left. \frac{\partial U}{\partial \left( \frac{\epsilon^l \rho^l - \epsilon_0^l \rho_0^l}{\rho_0^l} \right)} \right|_{\epsilon^s \rho^s} = \rho_0^l \left. \frac{\partial U}{\partial (\epsilon^l \rho^l)} \right|_{\epsilon^s \rho^s} = \rho_0^l \left. \frac{\partial (\epsilon^l \rho^l U^l + \epsilon^s \rho^s U^s)}{\partial (\epsilon^l \rho^l)} \right|_{\epsilon^s \rho^s} \quad (64)$$

If we assume that  $\rho^s$  is constant (incompressible solid), then keeping  $\epsilon^s \rho^s$  fixed is equivalent to keeping  $\epsilon^s$  (or equivalently,  $\epsilon^l$ ) fixed. With both  $\epsilon^s$  and  $\rho^s$  fixed,  $U^s$  is fixed (it is assumed to not be a function of  $\rho^l$ ). If  $U^l$  is not a function of  $\rho^s$ , then (64) simplifies to

$$p^l \approx \rho_0^l \left. \frac{\partial (\rho^l U^l)}{\partial \rho^l} \right|_{\epsilon^s}. \quad (65)$$

Comparing this expression with the thermodynamic definition used in the mixture theoretic formulation (57):

$$p^l = (\rho^l)^2 \left. \frac{\partial A^l}{\partial \rho^l} \right|_{\epsilon^l}$$

we see that if the density of the liquid and solid phase are constant (as Biot assumed) the two definitions agree.

### 5.3 Comparison with Gassmann

Gassmann derived an equation relating the drained and undrained compressibility. There are two distinct equations attributed to Gassmann, one derived

in Gassmann's original work [18, 10]:

$$K_u = K^s \frac{K_d + Q}{K^s + Q} \quad \text{where} \quad Q = \frac{K^l(K^s - K_d)}{\varepsilon^l(K^s - K^l)} \quad (66)$$

which can be re-written as:

$$\frac{K^s}{K^s - K_u} = \frac{K^s}{K^s - K_d} + \frac{K^l}{\varepsilon^l(K^s - K^l)} \quad (67)$$

and one termed a form of *Gassmann equations* [27]

$$\frac{K_u}{K^s - K_u} = \frac{K_d}{K^s - K_d} + \frac{K^l}{\varepsilon^l(K^s - K^l)}, \quad (68)$$

which is used to predict the resulting increase of  $K_u$  of saturated rock from the effective bulk modulus of dry rock (normally approximated as  $K_d$ ) at low frequencies [27].

Interestingly both these results can be recovered by neglecting porosity change through the approximation  $g(\varepsilon^l) = \tilde{g}(\varepsilon^l) = \varepsilon^l$ . This allows us to use equations (47) and (49) and then both (67) and (68) are satisfied.

It should be noted that although setting  $g(\varepsilon^l) = \tilde{g}(\varepsilon^l) = \varepsilon^l$  is one simplifying assumption which recovers the expressions derived by Zimmerman, Gassmann, and Bishop, there are probably other simplifying expressions which also recover these relationships. This one seemed the obvious one to try since in previous work changing volumetric fractions were not considered.

As a side note, Brown and Korrington [12] also claimed to have derived a generalization of Gassmann's equations, however equations (6) and (7) of [12] are inconsistent. See Appendix C.

## **6 Discussion**

Under the assumptions stated in this paper, e.g. two-phase porous media (liquid and solid) where the liquid phase energy is not a function of the density of the solid phase, and the solid phase energy is not a function of the density of the liquid phase, and that only compression (and not shear) experiments are performed, the goals of this paper were to:

1. Understand the relationship between the compressibilities of the individual phases and the compressibilities which are measurable (drained, undrained, etc.). This was done and the general results are given by equations (32) and (34)-(38).
2. Set up a rigorous framework which allows this framework to be extended to more complicated porous materials (e.g. swelling porous materials such as clay soils and cartilage) by relaxing some of the assumptions prescribed in this paper.
3. Understand the implicitly assumed assumptions made in previous works, which we accomplished by re-examining the list of independent variables, adding the volume fraction as an independent variable and then showing that this generalized previous works of Zimmerman, Biot, and Gassmann.
4. Demonstrate the mathematics required for such calculations, which we did using multi-variable calculus.

One result of this paper was to determine the relationship between quantities which are unambiguously defined thermodynamically (e.g. pressures of each phase, compressibilities of each phase) and quantities which are measurable (compressibility coefficients). To analyze these relationships let's consider two simplifying cases: One in which the solid phase is incompressible and one in which both the solid and liquid phases are compressible and see how much information can be obtained about the system from the measurable quantities.

For an incompressible solid, the volume of the solid phase is fixed, and thus the partials of liquid volume with respect to solid volumes are now well-defined since  $V = V^l + V^s$ . This gives

$$g(\varepsilon^l) = \tilde{g}(\varepsilon^l) = \bar{g}(\varepsilon^l) = \hat{g}(\varepsilon^l) = 1. \quad (69)$$

This simplifies expressions (32) and (34)-(38) immensely. With a little manipulation, we get:

$$\varepsilon^l = \frac{K_p}{K_d} = \frac{K_{pp}}{K_{bp}} \quad (70)$$

$$\left. \frac{\partial p^s}{\partial \varepsilon^s} \right|_{\rho^s} = \frac{K' K_d}{K_d - K_p} \quad (71)$$

$$p^l - p^s = K' - \frac{(K_d)^2}{K_d - K_p} \quad (72)$$

Thus in this case it is possible to determine the well-defined quantities in terms of experimentally measurable quantities, assuming the quantity in the denominator,  $K_d - K_p = \varepsilon^s K_d$ , is not too small.

Going to the case where both the liquid and solid phases are compressible, the expressions become more complicated. We can still obtain the volume fraction:

$$\varepsilon^l = \frac{g(\varepsilon^l)K_p}{K_d} = \frac{\widehat{g}(\varepsilon^l)K_{pp}}{K_{bp}} \quad (73)$$

and from the unjacketed compressibility coefficient,  $K'$ , (34) we get:

$$\left. \frac{\partial p^s}{\partial \varepsilon^s} \right|_{\rho^s} = \frac{\varepsilon^s K' - (1 - \bar{g}(\varepsilon^l))K^s}{\varepsilon^s(\bar{g}(\varepsilon^l) - \varepsilon^l)}, \quad (74)$$

from which the volume fraction can be eliminated using (73). Note that if  $\bar{g}(\varepsilon^l) \approx \varepsilon^l$ , then this expression is ill-posed (in this case  $K' = K^s$  and so (74), gives 0 in both the denominator and numerator). From the definition of  $\bar{g}(\varepsilon^l)$  we see that  $\bar{g}(\varepsilon^l) \approx \varepsilon^l$  when the volume fraction does not change much when liquid content (porosity) changes and  $p^l - p$  is held fixed (see Section 5.1), i.e. when both phases have roughly the same compressibility (for an equal change in pressure, the volume changes for each phase are roughly the same). This makes sense in that if the compressibilities are about the same it is difficult to distinguish between the phases by performing macroscopic experiments.

Finally, using the above results and the expression for the drained compressibility, we obtain an expression for the difference in liquid and solid pressure:

$$\begin{aligned} (p^l - p^s)(g(\varepsilon^l) - \varepsilon^l) &= -K_d + \left[ \frac{g(\varepsilon^l) - \varepsilon^l}{\bar{g}(\varepsilon^l) - \varepsilon^l} \right] [\varepsilon^s K' - (1 - \bar{g}(\varepsilon^l))K^s] \\ &\quad + (1 - g(\varepsilon^l))K^s. \end{aligned} \quad (75)$$

The volume fractions  $\varepsilon^l$  and  $\varepsilon^s$  could be eliminated via (73). Note that we again have problems if  $g(\varepsilon^l) \approx \varepsilon^l$  or if  $\bar{g}(\varepsilon^l) \approx \varepsilon^l$ .

In this manuscript we have explored, in a thermodynamically rigorous way, the relationships between compressibility coefficients, the compressibilities of the materials, the volume fractions, and the pressures of each phase. Future work includes relating the above to the Terzaghi stress principle, extending the above results for a swelling porous material, and to use experiments to determine whether the derived results differ significantly from classical results.

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## Nomenclature

In general, a subscript Greek letter indicates a macroscale quantity of a phase. Subscript 0 indicates a quantity at the reference state. A caret over the symbol,  $\hat{\phantom{x}}$ , is used to emphasize that the quantity represents a transfer from either another phase.



$A^\alpha$	Helmholtz free energy density of $\alpha$ -phase, [energy $\alpha$ / mass $\alpha$ ].
$C$	compressibility for a single phase (3), [length <sup>2</sup> /force]
$C_d$	drained compressibility, (11), [length <sup>2</sup> /force]
$C'$	unjacketed compressibility, (12), [length <sup>2</sup> /force]
$C_u$	undrained compressibility, (13), [length <sup>2</sup> /force]
$C_p$	drained pore compressibility, (14), [length <sup>2</sup> /force]
$C_{bp}$	pseudo-bulk compressibility, (15), [length <sup>2</sup> /force]
$C_{pp}$	pore compressibility, (16), [length <sup>2</sup> /force]
$e^s$	macroscale strain tensor of solid phase, [-]
$g$	gravity, [length/time <sup>2</sup> ]
$g(\varepsilon^l)$	measure of changing liquid content, see (21), [-]
$\tilde{g}(\varepsilon^l)$	measure of changing liquid content, see (40), [-]
$\bar{g}(\varepsilon^l)$	measure of changing liquid content, see (41), [-]
$\hat{g}(\varepsilon^l)$	measure of changing liquid content, see (42), [-]
$K$	inverse of compressibility, (10), [force/length <sup>2</sup> ]
$K_d$	drained modulus, (11), [force/length <sup>2</sup> ]
$K'$	unjacketed modulus, (12), [force/length <sup>2</sup> ]
$K_u$	undrained modulus, (13), [force/length <sup>2</sup> ]
$K_p$	drained pore modulus, (14), [force/length <sup>2</sup> ]
$K_{bp}$	pseudo-bulk modulus, (15), [force/length <sup>2</sup> ]
$K_{pp}$	pore modulus, (16), [force/length <sup>2</sup> ]
$M$	total mass, [mass]
$M^\alpha$	total mass of phase $\alpha$ , [mass $\alpha$ ]
$p$	total or confining pressure, (16) and (63), [force/length <sup>2</sup> ]
$p^\alpha$	pressure of phase $\alpha$ , (57), [force/length <sup>2</sup> ]
$t^\alpha$	Cauchy stress tensor of $\alpha$ -phase, [force/length <sup>2</sup> ]
$T$	temperature
$\hat{\mathbf{T}}_\beta^\alpha$	gain of momentum of phase $\alpha$ due to mechanical interactions with phase $\beta$ [length/time <sup>2</sup> ]
$u^\alpha$	internal energy, [energy $\alpha$ /mass $\alpha_j$ ]
$v$	specific volume, [volume $\alpha$ /mass $\alpha$ ]
$v^\alpha$	specific volume of phase $\alpha$ , [volume $\alpha$ /mass $\alpha$ ]
$\mathbf{v}^\alpha$	velocity of phase $\alpha$ , [length/time]
$V$	total volume = $V^s + V^l$ , [volume]
$V^\alpha$	total volume of phase $\alpha$ , [volume $\alpha$ ]

- $\epsilon$  volumetric strain, (61), [-]  
 $\varepsilon^\alpha$  volume fraction of  $\alpha$ -phase in Representative Elementary Volume (REV)  $=|\delta V_\alpha|/|\delta V|$   
 $\eta^\alpha$  entropy density, [ $\alpha$  entropy/(mass  $\alpha$ -time)]  
 $\kappa$  bulk modulus of compression for a solid, (7), [force/length<sup>2</sup>]  
 $\rho^\alpha$  mass density of phase  $\alpha$ , [mass  $\alpha$ / volume  $\alpha$ ]  
 $\zeta$  the increment of water volume per unit volume of soil as defined by Biot.

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## 7 Appendix A: Additional Calculations for Compressibility Moduli

In this section we give the intermediate calculations for all compressibilities except  $K_d$ , whose calculations are given in the body of this paper. All calculations have the same logic as the derivation of the drained compressibility moduli.

### Unjacketed

In this experiment the overburden pressure is equal to the liquid pressure,  $p = p^l$ , or  $p - p^l$  is held fixed. Also, by  $p = \varepsilon^l p^l + \varepsilon^s p^s$  we have that  $p^s = p^l = p$ .

This gives

$$K' = -V \left. \frac{\partial p}{\partial V} \right|_{p^l-p, M^s} = -V \left. \frac{\partial p^s}{\partial V} \right|_{p^l-p, M^s} \quad (76)$$

Throughout the paper we assumed that  $p^s = p^s(\varepsilon^s, \rho^s)$ , which in extensive terms corresponds to  $p^s = p^s(V^l, V^s, M^s)$ . Thus, corresponding to (20), we have

$$V \left. \frac{\partial p^s}{\partial V} \right|_{p^l-p, M^s} = V \left. \frac{\partial p^s}{\partial V^l} \right|_{M^s, V^s} \left. \frac{\partial V^l}{\partial V} \right|_{p^l-p, M^s} + V \left. \frac{\partial p^s}{\partial V^s} \right|_{M^s, V^l} \left. \frac{\partial V^s}{\partial V} \right|_{p^l-p, M^s}. \quad (77)$$

Using the fact that  $M^s$  is fixed we have

$$\begin{aligned} V \left. \frac{\partial p^s}{\partial V^l} \right|_{M^s, V^s} &= \frac{1}{\varepsilon^s \rho^s} \left. \frac{\partial p^s}{\partial \left( \frac{\varepsilon^l}{\varepsilon^s \rho^s} \right)} \right|_{\frac{1}{\rho^s}} = \frac{1}{\varepsilon^s} \left. \frac{\partial p^s}{\partial \left( \frac{\varepsilon^l}{\varepsilon^s} \right)} \right|_{\rho^s} = \frac{1}{\varepsilon^s} \left. \frac{\partial p^s}{\partial \varepsilon^s} \right|_{\rho^s} \frac{\partial \varepsilon^s}{\partial \left( \frac{\varepsilon^l}{\varepsilon^s} \right)} \\ &= -\varepsilon^s \left. \frac{\partial p^s}{\partial \varepsilon^s} \right|_{\rho^s}. \end{aligned} \quad (78)$$

where in the second equality we took advantage of  $\rho^s$  being fixed.

Now we repeat the process for  $V \partial p^s / (\partial V^s)$ :

$$\begin{aligned} V \left. \frac{\partial p^s}{\partial V^s} \right|_{V^l, M^s} &= \frac{1}{\varepsilon^s \rho^s} \left. \frac{\partial p^s}{\partial \left( \frac{1}{\rho^s} \right)} \right|_{\frac{\varepsilon^l}{\varepsilon^s \rho^s}} = \frac{1}{\varepsilon^s \rho^s} \left. \frac{\partial p^s}{\partial \rho^s} \right|_{\frac{\varepsilon^l}{\varepsilon^s \rho^s}} \frac{\partial \rho^s}{\partial \left( \frac{1}{\rho^s} \right)} \\ &= -\frac{\rho^s}{\varepsilon^s} \left. \frac{\partial p^s}{\partial \rho^s} \right|_{\frac{\varepsilon^l}{\varepsilon^s \rho^s}} \end{aligned} \quad (79)$$

We now tackle  $\partial p^s / \partial \rho^s$ . So beginning with  $p^s = p^s(\varepsilon^s, \rho^s)$  we have

$$\begin{aligned} \left. \frac{\partial p^s}{\partial \rho^s} \right|_{\frac{\varepsilon^l}{\varepsilon^s \rho^s}} &= \left. \frac{\partial p^s}{\partial \varepsilon^s} \right|_{\rho^s} \left. \frac{\partial \varepsilon^s}{\partial \rho^s} \right|_{\frac{\varepsilon^l}{\varepsilon^s \rho^s}} + \left. \frac{\partial p^s}{\partial \rho^s} \right|_{\varepsilon^s} \left. \frac{\partial \rho^s}{\partial \rho^s} \right|_{\frac{\varepsilon^l}{\varepsilon^s \rho^s}} \\ &= \left. \frac{\partial p^s}{\partial \varepsilon^s} \right|_{\rho^s} \left( -\frac{\varepsilon^l \varepsilon^s}{\rho^s} \right) + \left. \frac{\partial p^s}{\partial \rho^s} \right|_{\varepsilon^s} \end{aligned} \quad (80)$$

so that (79) becomes

$$V \frac{\partial p^s}{\partial V^s} \Big|_{V^l, M^s} = \varepsilon^l \frac{\partial p^s}{\partial \varepsilon^s} \Big|_{\rho^s} - \frac{1}{\varepsilon^s} K^s. \quad (81)$$

Substituting (78) and (81) into (77) we have

$$K' = \bar{g}(\varepsilon^l) \varepsilon^s \frac{\partial p^s}{\partial \varepsilon^s} \Big|_{\rho^s} + (1 - \bar{g}(\varepsilon^l)) \left[ -\varepsilon^l \frac{\partial p^s}{\partial \varepsilon^s} \Big|_{\rho^s} + \frac{1}{\varepsilon^s} K^s \right] \quad (82)$$

where

$$\bar{g}(\varepsilon^l) = \frac{\partial V^l}{\partial V} \Big|_{p^l - p, M^s} \quad (83)$$

### Undrained Compressibility

$$K_u = -V \frac{\partial p}{\partial V} \Big|_{M^l, M^s}. \quad (84)$$

Assuming pressure,  $p$ , is a function of  $M^l$ ,  $M^s$ ,  $V^l$ , and  $V^s$  we have, corresponding to (20),

$$V \frac{\partial p}{\partial V} \Big|_{M^l, M^s} = V \frac{\partial p}{\partial V^l} \Big|_{M^l, M^s, V^s} \frac{\partial V^l}{\partial V} \Big|_{M^l, M^s} + V \frac{\partial p}{\partial V^s} \Big|_{M^l, M^s, V^l} \frac{\partial V^s}{\partial V} \Big|_{M^l, M^s}. \quad (85)$$

Using the fact that  $M^s$  is fixed we have

$$\begin{aligned} V \frac{\partial p}{\partial V^l} \Big|_{V^s, M^l, M^s} &= \frac{1}{\varepsilon^s \rho^s} \frac{\partial p}{\partial \left( \frac{\varepsilon^l}{\varepsilon^s \rho^s} \right)} \Big|_{\rho^s, \frac{\varepsilon^l \rho^l}{\varepsilon^s}} \\ &= \frac{1}{\varepsilon^s} \frac{\partial p}{\partial \varepsilon^l} \Big|_{\rho^s, \frac{\varepsilon^l \rho^l}{\varepsilon^s}} \frac{\partial \varepsilon^l}{\partial \left( \frac{\varepsilon^l}{\varepsilon^s} \right)} = \varepsilon^s \frac{\partial p}{\partial \varepsilon^l} \Big|_{\rho^s, \frac{\varepsilon^l \rho^l}{\varepsilon^s}}, \end{aligned} \quad (86)$$



where in the second equality we took advantage of  $\rho^s$  being fixed. We now tackle  $\partial p/\partial \varepsilon^l$ . So beginning with (26), we have

$$\begin{aligned} \left. \frac{\partial p}{\partial \varepsilon^l} \right|_{\rho^s, \frac{\varepsilon^l \rho^l}{\varepsilon^s}} &= (p^l - p^s) + \varepsilon^l \frac{dp^l}{d\rho^l} \left. \frac{\partial \rho^l}{\partial \varepsilon^l} \right|_{\frac{\varepsilon^l \rho^l}{\varepsilon^s}} - \varepsilon^s \left. \frac{\partial p^s}{\partial \varepsilon^s} \right|_{\rho^s} \\ &= (p^l - p^s) - \frac{\rho^l}{\varepsilon^s} \frac{dp^l}{d\rho^l} - \varepsilon^s \left. \frac{\partial p^s}{\partial \varepsilon^s} \right|_{\rho^s} \end{aligned} \quad (87)$$

where we used  $\partial \rho^l / \partial \varepsilon^l |_{\varepsilon^l \rho^l / \varepsilon^s} = -\rho^l / (\varepsilon^l \varepsilon^s)$  which is obtained in a manner similar to (30). Substituting (87) into (86) we get

$$V \left. \frac{\partial p}{\partial V^l} \right|_{V^s, M^l, M^s} = \varepsilon^s (p^l - p^s) - K^l - (\varepsilon^s)^2 \left. \frac{\partial p^s}{\partial \varepsilon^s} \right|_{\rho^s}. \quad (88)$$

Now we repeat the process for  $V \partial p / (\partial V^s)$ :

$$\begin{aligned} V \left. \frac{\partial p}{\partial V^s} \right|_{V^l, M^l, M^s} &= \frac{1}{\varepsilon^s \rho^s} \left. \frac{\partial p}{\partial \left( \frac{1}{\rho^s} \right)} \right|_{\frac{\varepsilon^l}{\varepsilon^s \rho^s}, \frac{\varepsilon^l \rho^l}{\varepsilon^s \rho^s}} = \frac{1}{\varepsilon^s \rho^s} \left. \frac{\partial p}{\partial \rho^s} \right|_{\frac{\varepsilon^l}{\varepsilon^s \rho^s}, \rho^l} \frac{\partial \rho^s}{\partial \left( \frac{1}{\rho^s} \right)} \\ &= -\frac{\rho^s}{\varepsilon^s} \left. \frac{\partial p}{\partial \rho^s} \right|_{\frac{\varepsilon^l}{\varepsilon^s \rho^s}, \rho^l} \end{aligned} \quad (89)$$

We now tackle  $\partial p / \partial \rho^s$ . So beginning with (26), we have

$$\begin{aligned} \left. \frac{\partial p}{\partial \rho^s} \right|_{\frac{\varepsilon^l}{\varepsilon^s \rho^s}, \rho^l} &= (p^l - p^s) \left. \frac{\partial \varepsilon^l}{\partial \rho^s} \right|_{\frac{\varepsilon^l}{\varepsilon^s \rho^s}} + \varepsilon^s \left. \frac{\partial p^s}{\partial \varepsilon^s} \right|_{\rho^s} \left. \frac{\partial \varepsilon^s}{\partial \rho^s} \right|_{\frac{\varepsilon^l}{\varepsilon^s \rho^s}} + \varepsilon^s \left. \frac{\partial p^s}{\partial \rho^s} \right|_{\varepsilon^s} \\ &= (p^l - p^s) \frac{\varepsilon^l \varepsilon^s}{\rho^s} + \varepsilon^s \left. \frac{\partial p^s}{\partial \varepsilon^s} \right|_{\rho^s} \left( -\frac{\varepsilon^l \varepsilon^s}{\rho^s} \right) + \varepsilon^s \left. \frac{\partial p^s}{\partial \rho^s} \right|_{\varepsilon^s} \end{aligned} \quad (90)$$

so that (89) becomes

$$V \left. \frac{\partial p}{\partial V^s} \right|_{V^l, M^l, M^s} = -\varepsilon^l (p^l - p^s) + \varepsilon^l \varepsilon^s \left. \frac{\partial p^s}{\partial \varepsilon^s} \right|_{\rho^s} - K^s. \quad (91)$$

Substituting (88) and (91) into (85) we have

$$K_u = \tilde{g}(\varepsilon^l) \left[ -\varepsilon^s (p^l - p^s) + (\varepsilon^s)^2 \left. \frac{\partial p^s}{\partial \varepsilon^s} \right|_{\rho^s} + K^l \right]$$

$$+ (1 - \tilde{g}(\varepsilon^l)) \left[ \varepsilon^l (p^l - p^s) - \varepsilon^l \varepsilon^s \left. \frac{\partial p^s}{\partial \varepsilon^s} \right|_{\rho^s} + K^s \right] \quad (92)$$

where

$$\tilde{g}(\varepsilon^l) = \left. \frac{\partial V^l}{\partial V} \right|_{M^l, M^s} \quad (93)$$

### Drained Pore

In this experiment the liquid pressure is held fixed and the change in liquid volume is measured under changing overburden pressures. So we have

$$K_p = -V^l \left. \frac{\partial p}{\partial V^l} \right|_{p^l, M^s}. \quad (94)$$

Assuming pressure,  $p$ , is a function of  $p^l$ ,  $M^s$ ,  $V^l$  and  $V^s$ , we have, corresponding to (20),

$$V^l \left. \frac{\partial p}{\partial V^l} \right|_{p^l, M^s} = V^l \left. \frac{\partial p}{\partial V^l} \right|_{p^l, M^s, V^s} \frac{\partial V^l}{\partial V^l} \Big|_{p^l, M^s} + V^l \left. \frac{\partial p}{\partial V^s} \right|_{p^l, M^s, V^l} \frac{\partial V^s}{\partial V^l} \Big|_{p^l, M^s} \quad (95)$$

Using the fact that  $M^s$  is fixed we have

$$\begin{aligned} V^l \left. \frac{\partial p}{\partial V^l} \right|_{p^l, M^s, V^s} &= \frac{\varepsilon^l}{\varepsilon^s \rho^s} \left. \frac{\partial p}{\partial \left( \frac{\varepsilon^l}{\varepsilon^s \rho^s} \right)} \right|_{p^l, \frac{1}{\rho^s}} = \frac{\varepsilon^l}{\varepsilon^s} \left. \frac{\partial p}{\partial \left( \frac{\varepsilon^l}{\varepsilon^s} \right)} \right|_{p^l, \rho^s} \\ &= \frac{\varepsilon^l}{\varepsilon^s} \left. \frac{\partial p}{\partial \varepsilon^l} \right|_{p^l, \rho^s} \frac{\partial \varepsilon^l}{\partial \left( \frac{\varepsilon^l}{\varepsilon^s} \right)} = \varepsilon^l \varepsilon^s \left. \frac{\partial p}{\partial \varepsilon^l} \right|_{p^l, \rho^s}. \end{aligned} \quad (96)$$

where in the second equality we took advantage of  $\rho^s$  being fixed. We now tackle  $\partial p / \partial \varepsilon^l$ . So beginning with (26), we have

$$\left. \frac{\partial p}{\partial \varepsilon^l} \right|_{p^l, \rho^s} = (p^l - p^s) - \varepsilon^s \left. \frac{\partial p^s}{\partial \varepsilon^s} \right|_{\rho^s}. \quad (97)$$

Substituting (97) into (96) we get

$$V^l \frac{\partial p}{\partial V^l} \Big|_{p^l, M^s, V^s} = \varepsilon^l \varepsilon^s (p^l - p^s) - \varepsilon^l (\varepsilon^s)^2 \frac{\partial p^s}{\partial \varepsilon^s} \Big|_{\rho^s}. \quad (98)$$

Now we repeat the process for  $V \partial p / (\partial V^s)$ :

$$\begin{aligned} V^l \frac{\partial p}{\partial V^s} \Big|_{p^l, M^s, V^l} &= \frac{\varepsilon^l}{\varepsilon^s \rho^s} \frac{\partial p}{\partial \left(\frac{1}{\rho^s}\right)} \Big|_{p^l, \frac{\varepsilon^l}{\varepsilon^s \rho^s}} = \frac{\varepsilon^l}{\varepsilon^s \rho^s} \frac{\partial p}{\partial \rho^s} \Big|_{p^l, \frac{\varepsilon^l}{\varepsilon^s \rho^s}} \frac{\partial \rho^s}{\partial \left(\frac{1}{\rho^s}\right)} \\ &= -\frac{\varepsilon^l \rho^s}{\varepsilon^s} \frac{\partial p}{\partial \rho^s} \Big|_{p^l, \frac{\varepsilon^l}{\varepsilon^s \rho^s}} \end{aligned} \quad (99)$$

We now tackle  $\partial p / \partial \rho^s$ . So beginning with  $p = p(p^l, \varepsilon^s, \rho^s)$  we have

$$\begin{aligned} \frac{\partial p}{\partial \rho^s} \Big|_{p^l, \frac{\varepsilon^l}{\varepsilon^s \rho^s}} &= (p^l - p^s) \frac{\partial \varepsilon^l}{\partial \rho^s} \Big|_{p^l, \frac{\varepsilon^l}{\varepsilon^s \rho^s}} + \varepsilon^s \frac{\partial p^s}{\partial \varepsilon^s} \Big|_{\rho^s} \frac{\partial \varepsilon^s}{\partial \rho^s} \Big|_{p^l, \frac{\varepsilon^l}{\varepsilon^s \rho^s}} + \varepsilon^s \frac{\partial p^s}{\partial \rho^s} \Big|_{\varepsilon^s} \frac{\partial \rho^s}{\partial \rho^s} \Big|_{p^l, \frac{\varepsilon^l}{\varepsilon^s \rho^s}} \\ &= \frac{\varepsilon^l \varepsilon^s}{\rho^s} (p^l - p^s) - \frac{\varepsilon^l (\varepsilon^s)^2}{\rho^s} \frac{\partial p^s}{\partial \varepsilon^s} \Big|_{\rho^s} + \varepsilon^s \frac{\partial p^s}{\partial \rho^s} \Big|_{\varepsilon^s} \end{aligned} \quad (100)$$

so that (99) becomes

$$V^l \frac{\partial p}{\partial V^s} \Big|_{p^l, M^s, V^l} = -(\varepsilon^l)^2 (p^l - p^s) + (\varepsilon^l)^2 \varepsilon^s \frac{\partial p^s}{\partial \varepsilon^s} \Big|_{\rho^s} - \varepsilon^l K^s. \quad (101)$$

Substituting (96) and (101) into (95) we have

$$\begin{aligned} K_p &= -\varepsilon^l \varepsilon^s (p^l - p^s) + \varepsilon^l (\varepsilon^s)^2 \frac{\partial p}{\partial \varepsilon^s} \Big|_{\rho^s} \\ &\quad + \left( \frac{1}{g(\varepsilon^l)} - 1 \right) \left[ (\varepsilon^l)^2 (p^l - p^s) - (\varepsilon^l)^2 \varepsilon^s \frac{\partial p^s}{\partial \varepsilon^s} \Big|_{\rho^s} + \varepsilon^l K^s \right] \end{aligned} \quad (102)$$

where

$$g(\varepsilon^l) = \frac{\partial V^l}{\partial V} \Big|_{p^l, M^s}. \quad (103)$$

Pseudo-Bulk

In this experiment the over-burden pressure is held fixed and the change in volume is measured under changing liquid pressures. So we have

$$K_{bp} = V \left. \frac{\partial p^l}{\partial V} \right|_{p, M^s}. \quad (104)$$

Throughout the paper we assume  $p^l = p^l(\rho^l)$ , which in extensive terms corresponds to  $p^l = p^l(V^l, M^l)$ . However working with  $M^l$  and  $V^l$  as independent variables is difficult since measuring the volume of the liquid phase exiting the system very closely determines the mass of the liquid phase exiting the system. So instead, we make use of  $p = p(p^l, V^l, V^s, M^s)$ . Assuming that this functional relationship is solvable for  $p^l$ , we have  $p^l = p^l(p, V^l, V^s, M^s)$ . It may be that since  $p^l$  is only a function of two macroscale variables ( $M^l$  and  $V^l$ ) that  $p^l$  may be a function of only 2 or 3 of the 4 independent variables, ( $p, V^l, V^s, M^s$ ), but assuming dependence on all 4 is the most general case. Proceeding corresponding to (20), we have

$$V \left. \frac{\partial p^l}{\partial V} \right|_{p, M^s} = V \left. \frac{\partial p^l}{\partial V^l} \right|_{p, V^s, M^s} \left. \frac{\partial V^l}{\partial V} \right|_{p, M^s} + V \left. \frac{\partial p^l}{\partial V^s} \right|_{p, V^l, M^s} \left. \frac{\partial V^s}{\partial V} \right|_{p, M^s} \quad (105)$$

Using the fact that  $M^s$  is fixed we have

$$\begin{aligned} V \left. \frac{\partial p^l}{\partial V^l} \right|_{p, V^s, M^s} &= \frac{1}{\varepsilon^s \rho^s} \left. \frac{\partial p^l}{\partial \left( \frac{\varepsilon^l}{\varepsilon^s \rho^s} \right)} \right|_{p, \frac{1}{\rho^s}} = \frac{1}{\varepsilon^s} \left. \frac{\partial p^l}{\partial \left( \frac{\varepsilon^l}{\varepsilon^s} \right)} \right|_{p, \rho^s} \\ &= \frac{1}{\varepsilon^s} \left. \frac{\partial p^l}{\partial \varepsilon^l} \right|_{p, \rho^s} \frac{\partial \varepsilon^l}{\partial \left( \frac{\varepsilon^l}{\varepsilon^s} \right)} = \varepsilon^s \left. \frac{\partial p^l}{\partial \varepsilon^l} \right|_{p, \rho^s}. \end{aligned} \quad (106)$$

where in the second equality we took advantage of  $\rho^s$  being fixed. We now

tackle  $\partial p^l / \partial \varepsilon^l$ . Using (26) and that  $p$  is fixed we have

$$0 = (p^l - p^s) + \varepsilon^l \left. \frac{\partial p^l}{\partial \varepsilon^l} \right|_{p, \rho^s} - \varepsilon^s \left. \frac{\partial p^s}{\partial \varepsilon^s} \right|_{\rho^s} \quad (107)$$

so that

$$\left. \frac{\partial p^l}{\partial \varepsilon^l} \right|_{p, \rho^s} = -\frac{1}{\varepsilon^l} (p^l - p^s) + \frac{\varepsilon^s}{\varepsilon^l} \left. \frac{\partial p^s}{\partial \varepsilon^s} \right|_{\rho^s} \quad (108)$$

Substituting (108) into (106) we get

$$V \left. \frac{\partial p^l}{\partial V^l} \right|_{p, V^l, M^s} = -\frac{\varepsilon^s}{\varepsilon^l} (p^l - p^s) + \frac{(\varepsilon^s)^2}{\varepsilon^l} \left. \frac{\partial p^s}{\partial \varepsilon^s} \right|_{\rho^s} \quad (109)$$

Now we repeat the process for  $V \partial p^l / (\partial V^s)$ :

$$\begin{aligned} V \left. \frac{\partial p^l}{\partial V^s} \right|_{p, V^l, M^s} &= \frac{1}{\varepsilon^s \rho^s} \left. \frac{\partial p^l}{\partial \left( \frac{1}{\rho^s} \right)} \right|_{p, \frac{\varepsilon^l}{\varepsilon^s \rho^s}} = \frac{1}{\varepsilon^s \rho^s} \left. \frac{\partial p^l}{\partial \rho^s} \right|_{p, \frac{\varepsilon^l}{\varepsilon^s \rho^s}} \frac{\partial \rho^s}{\partial \left( \frac{1}{\rho^s} \right)} \\ &= -\frac{\rho^s}{\varepsilon^s} \left. \frac{\partial p^l}{\partial \rho^s} \right|_{p, \frac{\varepsilon^l}{\varepsilon^s \rho^s}} \end{aligned} \quad (110)$$

We now tackle  $\partial p^l / \partial \rho^s$ . So beginning with  $p = p(\varepsilon^l, p^l, \rho^s)$  we have

$$\begin{aligned} 0 &= (p^l - p^s) \left. \frac{\partial \varepsilon^l}{\partial \rho^s} \right|_{p, \frac{\varepsilon^l}{\varepsilon^s \rho^s}} + \varepsilon^l \left. \frac{\partial p^l}{\partial \rho^s} \right|_{p, \frac{\varepsilon^l}{\varepsilon^s \rho^s}} + \varepsilon^s \left. \frac{\partial p^s}{\partial \varepsilon^s} \right|_{\rho^s} \left. \frac{\partial \varepsilon^s}{\partial \rho^s} \right|_{p, \frac{\varepsilon^l}{\varepsilon^s \rho^s}} + \varepsilon^s \left. \frac{\partial p^s}{\partial \rho^s} \right|_{\varepsilon^s} \left. \frac{\partial \rho^s}{\partial \rho^s} \right|_{p, \frac{\varepsilon^l}{\varepsilon^s \rho^s}} \\ &= \frac{\varepsilon^l \varepsilon^s}{\rho^s} (p^l - p^s) + \varepsilon^l \left. \frac{\partial p^l}{\partial \rho^s} \right|_{p, \frac{\varepsilon^l}{\varepsilon^s \rho^s}} - \frac{\varepsilon^l (\varepsilon^s)^2}{\rho^s} \left. \frac{\partial p^s}{\partial \varepsilon^s} \right|_{\rho^s} + \varepsilon^s \left. \frac{\partial p^s}{\partial \rho^s} \right|_{\varepsilon^s}, \end{aligned} \quad (111)$$

so that

$$\left. \frac{\partial p^l}{\partial \rho^s} \right|_{p, \frac{\varepsilon^l}{\varepsilon^s \rho^s}} = -\frac{\varepsilon^s}{\rho^s} (p^l - p^s) + \frac{(\varepsilon^s)^2}{\rho^s} \left. \frac{\partial p^s}{\partial \varepsilon^s} \right|_{\rho^s} - \frac{\varepsilon^s}{\varepsilon^l} \left. \frac{\partial p^s}{\partial \rho^s} \right|_{\varepsilon^s}. \quad (112)$$

Thus (110) becomes

$$V \left. \frac{\partial p^l}{\partial V^s} \right|_{p, V^l, M^s} = (p^l - p^s) - \varepsilon^s \left. \frac{\partial p^s}{\partial \varepsilon^s} \right|_{\rho^s} + \frac{1}{\varepsilon^l} K^s. \quad (113)$$

Substituting (109) and (113) into (105) we have

$$K_{bp} = \hat{g}(\varepsilon^l) \left[ -\frac{\varepsilon^s}{\varepsilon^l} (p^l - p^s) + \frac{(\varepsilon^s)^2}{\varepsilon^l} \frac{\partial p^s}{\partial \varepsilon^s} \Big|_{\rho^s} \right] + (1 - \hat{g}(\varepsilon^l)) \left[ (p^l - p^s) - \varepsilon^s \frac{\partial p^s}{\partial \varepsilon^s} \Big|_{\rho^s} + \frac{1}{\varepsilon^l} K^s \right], \quad (114)$$

where

$$\hat{g}(\varepsilon^l) = \frac{\partial V^l}{\partial V} \Big|_{p, M^s}. \quad (115)$$

### Pore Compressibility

In this experiment the over-burden pressure is held fixed and the change in liquid volume is measured under changing liquid pressures. So we have

$$K_{pp} = V^l \frac{\partial p^l}{\partial V^l} \Big|_{p, M^s}. \quad (116)$$

As in the previous case (pseudo-bulk) we assume  $p^l = p^l(p, V^l, V^s, M^s)$ . Proceeding corresponding to (20), we have

$$\begin{aligned} V^l \frac{\partial p^l}{\partial V^l} \Big|_{p, M^s} &= V^l \frac{\partial p^l}{\partial V^l} \Big|_{p, V^s, M^s} \frac{\partial V^l}{\partial V^l} \Big|_{p, M^s} + V^l \frac{\partial p^l}{\partial V^s} \Big|_{p, V^l, M^s} \frac{\partial V^s}{\partial V^l} \Big|_{p, M^s} \\ &= V^l \frac{\partial p^l}{\partial V^l} \Big|_{p, V^s, M^s} + \left( \frac{1}{\hat{g}(\varepsilon^l)} - 1 \right) V^l \frac{\partial p^l}{\partial V^s} \Big|_{p, V^l, M^s}. \end{aligned} \quad (117)$$

Using (109)

$$V^l \frac{\partial p^l}{\partial V^l} \Big|_{p, V^s, M^s} = \varepsilon^l V \frac{\partial p^l}{\partial V^l} \Big|_{p, V^s, M^s} = -\varepsilon^s (p^l - p^s) + (\varepsilon^s)^2 \frac{\partial p^s}{\partial \varepsilon^s} \Big|_{\rho^s}, \quad (118)$$

and using (113)

$$V^l \frac{\partial p^l}{\partial V^s} \Big|_{p, V^l, M^s} = \varepsilon^l V \frac{\partial p^l}{\partial V^s} \Big|_{p, V^l, M^s} = \varepsilon^l (p^l - p^s) - \varepsilon^l \varepsilon^s \frac{\partial p^s}{\partial \varepsilon^s} \Big|_{\rho^s} + K^s \quad (119)$$

Substituting (118) and (119) into (117) we have

$$K_{pp} = \left[ -\varepsilon^s(p^l - p^s) + (\varepsilon^s)^2 \frac{\partial p^s}{\partial \varepsilon^s} \Big|_{\rho^s} \right] + \left( \frac{1}{\hat{g}(\varepsilon^l)} - 1 \right) \left[ \varepsilon^l(p^l - p^s) - \varepsilon^l \varepsilon^s \frac{\partial p^s}{\partial \varepsilon^s} \Big|_{\rho^s} + K^s \right], \quad (120)$$

where

$$\hat{g}(\varepsilon^l) = \frac{\partial V^l}{\partial V} \Big|_{p, M^s}. \quad (121)$$

## 8 Appendix B: Thermodynamic Definition of Confining Pressure

How does the thermodynamic definition of the pressure of the porous medium,  $-\frac{\partial(\mathcal{V}A)}{\partial v}$  relate to the physical pressure in the porous medium,  $p = \varepsilon^l p^l + \varepsilon^s p^s$ ?

The thermodynamic definition at the macroscale is defined to be the change in total energy with respect to volume keeping mass fixed:  $p = -\frac{\partial A_T}{\partial V} \Big|_{M^l, M^s}$ .

Changing over to intensive variables leads to the  $p = -\frac{\partial(\mathcal{V}A)}{\partial v} \Big|_{\frac{\varepsilon^l \rho^l}{\rho}, \frac{\varepsilon^s \rho^s}{\rho}}$ . One

(major) problem is that the medium definition of pressure assumes only two independent variables (specific volume, mass fraction) whereas the energy

(per unit mass),  $A$ , is actually a function of three independent variables (density of each phase and volume fraction). In order to make this problem tractable we *assume that the energy of each phase is only a function of*

*volume fraction and density of its own phase*, so that although the intensive energy density of the porous medium,  $A = \varepsilon^l \rho^l A^l + \varepsilon^s \rho^s A^s$ , is still a function

of all three variables, each component is only a function of two independent

variables:  $A(\varepsilon^l, \rho^l, \rho^s) = \varepsilon^l \rho^l A^l(\varepsilon^l, \rho^l) + \varepsilon^s \rho^s A^s(\varepsilon^l, \rho^s)$ .

**Claim:**

Under the assumption that  $A(\varepsilon^l, \rho^l, \rho^s) = \varepsilon^l \rho^l A^l(\varepsilon^l, \rho^l) + \varepsilon^s \rho^s A^s(\varepsilon^l, \rho^s)$ ,

$$p = - \left. \frac{\partial(vA)}{\partial v} \right|_{\frac{\varepsilon^l \rho^l}{\rho}} = \rho^2 \left. \frac{\partial(vA)}{\partial \rho} \right|_{\frac{\varepsilon^l \rho^l}{\rho}} = \varepsilon^l \rho^2 \left. \frac{\partial(vA)}{\partial \rho} \right|_{\frac{\varepsilon^l \rho^l}{\rho}, \rho^s} + \varepsilon^s \rho^2 \left. \frac{\partial(vA)}{\partial \rho} \right|_{\frac{\varepsilon^s \rho^s}{\rho}, \rho^l} \quad (122)$$

**Proof:**

By summing the linear momentum equations for each phase we have that

$p = \varepsilon^l p^l + \varepsilon^s p^s$ . Using this as our starting point

$$\begin{aligned} p &= \varepsilon^l (\rho^l)^2 \left. \frac{\partial A^l}{\partial \rho^l} \right|_{\varepsilon^l, \rho^s} + \varepsilon^s (\rho^s)^2 \left. \frac{\partial A^s}{\partial \rho^s} \right|_{\varepsilon^s, \rho^l} \\ &= \rho^l \left. \frac{\partial(\varepsilon^l \rho^l A^l)}{\partial \rho^l} \right|_{\varepsilon^l, \rho^s} - \varepsilon^l \rho^l A^l + \rho^s \left. \frac{\partial(\varepsilon^s \rho^s A^s)}{\partial \rho^s} \right|_{\varepsilon^s, \rho^l} - \varepsilon^s \rho^s A^s \\ &= \rho^l \left. \frac{\partial A}{\partial \rho^l} \right|_{\varepsilon^l, \rho^s} + \rho^s \left. \frac{\partial A}{\partial \rho^s} \right|_{\varepsilon^s, \rho^l} - A \end{aligned} \quad (123)$$

where the last equality holds since  $\varepsilon^s \rho^s A^s$  is not a function of  $\rho^l$  and similarly  $\varepsilon^l \rho^l A^l$  is not a function of  $\rho^s$ . Now

$$\begin{aligned} \left. \frac{\partial A}{\partial \rho} \right|_{\frac{\varepsilon^l \rho^l}{\rho}, \rho^s} &= \left. \frac{\partial A}{\partial \varepsilon^l} \right|_{\rho^l, \rho^s} \left. \frac{\partial \varepsilon^l}{\partial \rho} \right|_{\frac{\varepsilon^l \rho^l}{\rho}, \rho^s} + \left. \frac{\partial A}{\partial \rho^l} \right|_{\varepsilon^l, \rho^s} \left. \frac{\partial \rho^l}{\partial \rho} \right|_{\frac{\varepsilon^l \rho^l}{\rho}, \rho^s} \\ &= \left. \frac{\partial A}{\partial \varepsilon^l} \right|_{\rho^l, \rho^s} \left( -\frac{\varepsilon^s}{\rho} \right) + \left. \frac{\partial A}{\partial \rho^l} \right|_{\varepsilon^l, \rho^s} \left( \frac{\rho^l}{\varepsilon^l \rho} \right) \end{aligned} \quad (124)$$

where quantities such as  $\left. \frac{\partial \varepsilon^l}{\partial \rho} \right|_{\frac{\varepsilon^l \rho^l}{\rho}, \rho^s}$  are calculated as follows: Using  $\rho = \varepsilon^l \rho^l + \varepsilon^s \rho^s$  we can solve for  $\rho^l$  to obtain

$$\rho^l = \frac{\rho - \rho^s + \varepsilon^l \rho^s}{\varepsilon^l}.$$



Since  $\frac{\varepsilon^l \rho^l}{\rho}$  is held fixed, let it be a constant,  $K$ , so that

$$\varepsilon^l \rho^l = K \rho$$

or, eliminating  $\rho^l$

$$\rho - \rho^s + \varepsilon^l \rho^s = K \rho.$$

Taking the partial of both sides with respect to  $\rho$  and re-arranging gives

$$\frac{\partial \varepsilon^l}{\partial \rho} \Big|_{\frac{\varepsilon^l \rho^l}{\rho}, \rho^s} = -\frac{\varepsilon^s}{\rho}.$$

Similar to (124) we also have

$$\frac{\partial A}{\partial \rho} \Big|_{\frac{\varepsilon^s \rho^s}{\rho}, \rho^l} = \frac{\partial A}{\partial \varepsilon^s} \Big|_{\rho^l, \rho^s} \left( -\frac{\varepsilon^l}{\rho} \right) + \frac{\partial A}{\partial \rho^s} \Big|_{\varepsilon^s, \rho^l} \left( \frac{\rho^s}{\varepsilon^s \rho} \right). \quad (125)$$

Eliminating  $\frac{\partial A}{\partial \rho^l} \Big|_{\varepsilon^l, \rho^s}$  and  $\frac{\partial A}{\partial \rho^s} \Big|_{\varepsilon^s, \rho^l}$  from (123) using (124) and (125) and simplifying we get

$$\begin{aligned} p &= \varepsilon^l \rho \frac{\partial A}{\partial \rho} \Big|_{\frac{\varepsilon^l \rho^l}{\rho}, \rho^s} + \varepsilon^s \rho \frac{\partial A}{\partial \rho} \Big|_{\frac{\varepsilon^s \rho^s}{\rho}, \rho^l} + \varepsilon^l \varepsilon^s \frac{\partial A}{\partial \varepsilon^l} \Big|_{\rho^l, \rho^s} + \varepsilon^l \varepsilon^s \frac{\partial A}{\partial \varepsilon^s} \Big|_{\rho^l, \rho^s} - A \\ &= \varepsilon^l \rho \frac{\partial A}{\partial \rho} \Big|_{\frac{\varepsilon^l \rho^l}{\rho}, \rho^s} + \varepsilon^s \rho \frac{\partial A}{\partial \rho} \Big|_{\frac{\varepsilon^s \rho^s}{\rho}, \rho^l} - A \\ &= \varepsilon^l \rho^2 \frac{\partial(vA)}{\partial \rho} \Big|_{\frac{\varepsilon^l \rho^l}{\rho}, \rho^s} + \varepsilon^s \rho^2 \frac{\partial(vA)}{\partial \rho} \Big|_{\frac{\varepsilon^s \rho^s}{\rho}, \rho^l}, \end{aligned}$$

where the second equality follows from  $\varepsilon^l = 1 - \varepsilon^s$ , and the third from using the product rule and the definition of  $v = \frac{1}{\rho}$ . And this completes the proof of the claim.

## 9 Appendix C: Brown and Korringa

In this section we briefly explain why equations (6) and (7) of [12] are inconsistent. In [12], the authors assume (as do many other authors) that the total volume is only a function of the liquid or pore pressure,  $p^l$ , and the pressure difference,  $p - p^l$ , where  $p$  is the confining pressure, which is a reasonable assumption. However in going from equation (6) to (7), the authors go from

$$dV = \left. \frac{\partial V}{\partial(p - p^l)} \right|_{p^l} d(p - p^l) + \left. \frac{\partial V}{\partial p^l} \right|_{p-p^l} dp^l$$

to

$$\left. \frac{\partial V}{\partial p} \right|_{M^l} dp = \left. \frac{\partial V}{\partial(p - p^l)} \right|_{p^l} d(p - p^l) + \left. \frac{\partial V}{\partial p^l} \right|_{p-p^l} dp^l$$

where  $M^l$  is the total mass of the liquid phase so that the term on the left side is directly related to the undrained compressibility. The mistake is that on the left side the authors assumed that  $V$  is a function of  $p$  and  $M^l$  (which is *not* consistent with the assumption of  $V$  being a function of the differential pressure and pore pressure) and do not take the partial of both sides with respect to the same quantity, see e.g. [24, 13]. Mathematically the second equation should be:

$$\left. \frac{\partial V}{\partial p} \right|_{M^l} = \left. \frac{\partial V}{\partial(p - p^l)} \right|_{p^l} \left. \frac{\partial(p - p^l)}{\partial p} \right|_{M^l} + \left. \frac{\partial V}{\partial p^l} \right|_{p-p^l} \left. \frac{\partial p^l}{\partial p} \right|_{M^l}.$$

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