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STABILIZATION ARISING FROM PGEM: A REVIEW AND FURTHER DEVELOPMENTS

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ABSTRACT. The aim of this paper is twofold. First, we review the recent Petrov-Galerkin enriched method (PGEM) to stabilize numerical solutions of BVP's in primal and mixed forms. Then, we extend such enrichment technique to a mixed singularly perturbed problem, namely, the generalized Stokes problem, and focus on a stabilized finite element method arising in a natural way after performing static condensation. The resulting stabilized method is shown to lead optimal convergences, and afterward, it is numerically validated.

1. INTRODUCTION

The Stokes problem emanates from modeling creeping flows and incompressible elasticity. The problem fits into the abstract mixed method formulation [9, 11]. Mixed methods have various applications, among them modeling deformation of beams, arches, plates and shells. The approximation of these problems using standard finite element polynomials faces the challenge of satisfying stability conditions known as inf-sup conditions [9]. These stability conditions restrict which pairs of approximation (primal and dual variables) are allowed. Convenient pairs, such as equal-order interpolations, are in general prohibited.

Stabilized methods address the limitations of mixed methods [28, 27]. Introduced for advective-diffusive problems [15, 19], stabilized methods are built to enhance stability without affecting consistency. This is accomplished by adding terms based on residuals of the equations involving the trial functions while the test functions have different forms varying from least-squares to adjoint operators. For the Stokes problem these methods have been

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proved convergent for almost all pairs of interpolation [28, 25]. The drawback of stabilized methods is the choice of stability constants associated with the additional terms. In many applications the value of these constants affects the numerical results.

To shed some light on how to produce the stability constant for the Stokes problem, in [29] a relationship between the enrichment of a piecewise linear velocity field with a bubble function (MINI element [4]) and the stabilized method from [28] was first pointed out. The MINI element produces this stabilized method with a stability constant which is a function of the bubble shape and value. This gives us a recipe for getting the stability constant, namely, we pick a form of the bubble function and this gives us a specific value of the stability constant. This relationship has been extended to the generalized Stokes problem in [7] and to the advective-diffusive problem in [10, 5].

The relationship discovery left an open problem, namely how to choose optimal bubbles to produce the most accurate stabilized approximation. This question has been addressed introducing the residual-free-bubbles concept [14, 24, 13, 12]. The idea is to construct the bubbles by approximating a local problem dictated by the equations of the global problem. The bubbles solve a PDE problem governed by the residual of the piecewise polynomial component of the solution. The local problem is subject to a zero boundary condition (except for some problems defined in L^2), and this yields good solutions in some applications. However, the zero boundary condition limits the capability of the approximation in some cases. For example in the reactive dominated diffusive model, the residual-free bubbles method oscillates near a boundary layer. To address this shortcoming we introduced the so-called Petrov-Galerkin Enriched Method (PGEM) [21, 20] which is discussed in the next section.

The remainder of the paper is as follows: a review of PGEM is given in Section 2, in Section 3 PGEM is extended to the generalized Stokes problem, including an a priori error analysis, and in Section 4 we present some numerical results confirming the theoretical results.

1.1. **Notations.** Let Ω be an open bounded domain in $\mathbb{R}^2$ with polygonal boundary. As usual, $(\cdot\,,\cdot)_D$ stands for the inner product in $L^2(D)$ (or in $L^2(D)^2$, when necessary), and we denote by $\|\cdot\|_{s,D}$ ($|\cdot|_{s,D}$) the norm (seminorm) in $H^s(D)$ (or $H^s(D)^2$, if necessary). By $\{\mathcal{T}_h\}_{h>0}$ we denote a family of regular triangulations of Ω, built up using triangles K with boundary $\partial K = F_1 \cup F_2 \cup F_3$, $h_K := diam(K)$ and $h := \max\{h_K : K \in \mathcal{T}_h\}$.

We denote by $\mathcal{E}_\Omega$ the set of internal edges of $\mathcal{T}_h$ and for $K \in \mathcal{T}_h$ we denote by $\mathcal{E}(K)$ the set of its sides. The characteristic length of $F \in \mathcal{E}_\Omega$ is $h_F = |F|$, $\boldsymbol{n}$ is the normal outward vector on ∂K, $\partial_{\boldsymbol{s}}$ and $\partial_{\boldsymbol{n}}$ are the tangential and normal derivative operators, respectively, and $\mathbf{I}$ is the $\mathbb{R}^{2\times 2}$ identity matrix. Also, for $K \in \mathcal{T}_h$ and $F \in \mathcal{E}_\Omega$ we define the following neighborhoods:

$$\omega_K := \cup\{K' \in \mathcal{T}_h : K \cap K' \neq \emptyset\} \quad , \quad \omega_F := \cup\{K' \in \mathcal{T}_h : F \in \mathcal{E}(K')\}\,.$$

For $F \in \mathcal{E}_\Omega$ we denote by $[\![v]\!]_F$ the jump of a function v across F. Further, we introduce the standard linear finite element space

$$V_h \;=\; \{v \in C^0(\overline{\Omega}) : v|_K \in \mathbb{P}_1(K)\,,\, \forall K \in \mathcal{T}_h\}\,. \tag{1}$$

Finally, $H^1(\mathcal{T}_h)$ and $H^1_0(\mathcal{T}_h)$ stand for the spaces of functions whose restriction to $K \in \mathcal{T}_h$ belongs to $H^1(K)$ and $H^1_0(K)$, respectively, and we present a finite dimensional space $E_h \subset H^1(\mathcal{T}_h)$, called multiscale space, such that $V_h \cap E_h = \{0\}$, which will be used to enrich the trial space and will be problem dependent.

2. A review of Petrov-Galerkin Enriched Methods

Petrov-Galerkin enriched methods (PGEM) are designed to give superior accuracy along with enhanced stability. The method is based on the variational formulation of a specific model and is obtained by approximating the trial function by piecewise polynomials enriched with multiscale functions; the test function is approximated by piecewise polynomials enriched with bubble functions. This difference between the approximations of the test and trial functions is part of the Petrov-Galerkin framework.

We have zero boundary conditions on element edges (or faces in 3D) by selecting bubbles as enrichment of test functions. This enables static condensation. As a result, a differential equation for the enrichment function holds for each element and the multiscale enrichment can be condensed as a function of the piecewise component of the solution and the data. Once the expression of the multiscale component of the solution is available we then substitute it into the equation tested by the piecewise polynomial component. The method that arises is a stabilized method with several improvements. Among these, we can quote

- the enrichment produces an additional stability without compromising consistency in a different manner than standard stabilized methods;
- the accuracy is improved by letting the multiscale enrichment be different than zero on the element boundaries;

- the additional stabilized terms may have a different form than the apparent canonical modifications using least-squares or adjoint operators.

The latter is the key on accuracy comparisons with stabilized methods.

We start our review by looking at the first PGEM in reactive-diffusive problems [21, 20]. Let us start by recalling the model: find u such that

$$\begin{aligned} \sigma u - \triangle u &= f \quad \text{in } \Omega\,, \\ u &= 0 \quad \text{on } \partial\Omega\,, \end{aligned} \tag{2}$$

where $\sigma \in \mathbb{R}^+$ denotes the reactive constant and f is a given data.

The usual variational formulation for this problem is given by: Find $u \in H_0^1(\Omega)$ such that:

$$A\big(u,v\big) = (f,v)_\Omega \qquad \forall\, v \in H_0^1(\Omega)\,, \tag{3}$$

where

$$A\big(u,v\big) := \sigma(u,v)_\Omega + (\nabla u, \nabla v)_\Omega\,. \tag{4}$$

We take the trial enrichment to be in E_h and the test enrichment to be in $H_0^1(K)$. These are enrichment to piecewise linear u_1 and v_1, respectively. The PGEM methods becomes: find $u_1 + u_e$ such that

$$A\big(u_1+u_e, v_1+v_b\big) = (f, v_1+v_b)_\Omega \quad \forall\, v_1+v_b \in V_h \oplus H_0^1(\mathcal{T}_h). \tag{5}$$

Considering $v_1 = 0$ we have an equation in each element as follows:

$$\mathcal{L}u_e = -\sigma u_1 + \Delta u_1 + f = -\sigma u_1 + f, \tag{6}$$

where we used the linearity of u_1 in K.

This needs a boundary condition. One possibility is to set zero as the boundary condition which would reduce the method to the residual-free-bubble method. We explore new possibilities to allow the enrichment to be non-zero on the boundary. For this particular model the enrichment solves an equation governed by an operator projected along the boundary, namely,

$$\bar\sigma u_e - \partial_{ss} u_e = \frac{\bar\sigma}{\sigma}\,(f - \sigma u_1) \quad \text{and } u_e = 0 \quad \text{at the nodes,} \tag{7}$$

where $\bar\sigma$ represents σ multiplied by a suitable constant.

Combining (6) and (7) we can solve for $u_e|_K$ to get

$$u_e = \mathcal{M}_K(f - \sigma u_1)\,, \tag{8}$$

in every $K \in \mathcal{T}_h$, where $\mathcal{M}_K$ is the solution operator associated to (6)-(7). This is then replaced in (5) to obtain a stabilized alike method. Note that (8) is a formal result that needs to be computed in detail. We do this by using basis functions for u_1 in the right-hand-sides of equations (6) and (7). For further details the interested reader is referred to [21, 20] and to [3] for an *a posteriori* error estimator.

Next, still keeping polynomial spaces enhanced with the solution of the local problem (8), a parabolic version of PGEM is proposed in [23] to deal with the unsteady reaction-diffusion problem. Stability is achieved for the reaction dominated case although persisting spurious oscillations show up as soon as small time step procedure is used. Consequently, it appears that overcoming such drawback demands replacing steady local enrichment (8) by its time-dependent version. In [30] this issue is addressed.

When applied to advective dominated problems, the PGEM aims to resolve internal and external exponential boundary layers. It stems from [22] that such cumbersome goal is accurately accomplished for external layers but not for internal ones since it is still highly mesh dependent. Therefore, it emerges from [22, 16] that compromising stability and flexibility leads to a non-conforming approach in which the RFB method is adopted for internal elements while the PGEM is set for elements touching external boundaries of Ω.

Turning back to mixed problems, a class of new stabilized finite element methods have been derived to tackle the Stokes model. Roughly, this is accomplished following through analogous steps as for the reaction-diffusion case, but now, just the velocity space is enhanced. Continuous piecewise linear space for the velocity and continuous piecewise linear or constant spaces for pressure have been made compatible in the sense of inf-sup condition by adding the multiscale function $\boldsymbol{u}_e$ to the linear contribution $\boldsymbol{u}_1$. The former solves the following elliptic problem

$$-\nu\Delta\boldsymbol{u}_e = \boldsymbol{f} - \nabla p_k,$$

where $\nu \in \mathbb{R}^+$ represents viscosity, p_k the polynomial pressure variable with order $k = 0, 1$ and $\boldsymbol{f}$ is given data. Concerning boundary conditions, however, we disregard the previous strategy and propose a quite different approach based on *a posteriori* error estimates. As a matter of fact, it can be shown that numerical errors are strongly related to the jumps of pressure and normal derivative of velocity on internal edges, and thus, we propose to correct them imposing the following boundary condition on $\boldsymbol{u}_e$: $\boldsymbol{u}_e = \boldsymbol{0}$ if $F \subset \partial\Omega$, else $\boldsymbol{u}_e$ solves

$$\begin{aligned} -\nu\, \partial_{\boldsymbol{ss}} \boldsymbol{u}_e &= \frac{1}{h_F} [\![\nu \partial_{\boldsymbol{n}} \boldsymbol{u}_1 \pm p_k \mathbf{I} \cdot \boldsymbol{n}]\!]_F \quad \text{in } F\,, \\ \boldsymbol{u}_e &= \boldsymbol{0} \quad \text{at the nodes.} \end{aligned} \tag{9}$$

It can be proved that the all derived methods achieve optimal convergence [1] and lead, naturally, to *a posteriori* error estimators [2].

A second example of a mixed problem is the Darcy model. In its mixed form it presents an additional variable besides the pressure, the so-called Darcy velocity, which is proportional to the gradient of pressure. Unlike the Stokes case, now we ought to enrich both velocity and pressure spaces in order to make the continuous piecewise linear and constant spaces compatible, and even more important, to end up with locally mass conservative methods [6]. Going through the enriching methodology, it turns out that the piecewise linear velocity and the constant pressure $(\boldsymbol{u}_1, p_0)$ have to be elementwise augmented with the function $(\boldsymbol{u}_e, p_e)$ which solves the Darcy problem:

$$\sigma\, \boldsymbol{u}_e + \nabla p_e \;=\; \boldsymbol{f} - \sigma \boldsymbol{u}_1, \quad \nabla\cdot \boldsymbol{u}_e = C_K \quad \text{in } K, \tag{10}$$

where C_K is a suitable constant. Concerning the boundary condition for (10) two different alternatives have been undertaken in [6] (leading to fix C_K). First, following the idea used for the Stokes case we set the boundary condition for $\boldsymbol{u}_e$ as:

$$\boldsymbol{u}_e \cdot \boldsymbol{n} = \frac{\alpha_F\, h_F}{\sigma} [\![p_0]\!]_F \quad \text{on each } F \subseteq \partial K \cap \mathcal{E}_\Omega, \tag{11}$$

where α_F is a positive constant close to one and independent of h which can vary on each F. We point out that such choice keeps final methods conforming while stability is achieved without losing the local mass conservation feature. Alternatively, we can consider $\boldsymbol{u}_e$ satisfying

$$\boldsymbol{u}_e \cdot \boldsymbol{n} = \frac{\alpha_F\, h_F}{\sigma} [\![p_0]\!]_F - \boldsymbol{u}_1 \cdot \boldsymbol{n} + \frac{1}{h_F} \int_F \boldsymbol{u}_1 \cdot \boldsymbol{n} \quad \text{on each } F \subseteq \partial K \cap \mathcal{E}_\Omega\,. \tag{12}$$

This second choice mixes the strategy of [21] and [1] and preserves all desirable properties of (11). Furthermore, analytical solutions arise easily avoiding additional computational costs due to two level calculations.

Applying the technique described above, in the next section we derive a new stabilized finite element method for the generalized Stokes problem taking care of the inf-sup condition and the boundary layer issue simultaneously.

3. An application to the generalized Stokes problem

Let $\boldsymbol{f} \in L^2(\Omega)^2$ and let us consider the following generalized Stokes problem: Find $(\boldsymbol{u}, p)$ such that

$$\begin{aligned} \boldsymbol{\mathcal{L}u} + \nabla p = \boldsymbol{f}, \quad \nabla\cdot \boldsymbol{u} = 0 \quad \text{in } \Omega\,, \\ \boldsymbol{u} = \boldsymbol{0} \quad \text{on } \partial\Omega\,, \end{aligned} \tag{13}$$

where $\boldsymbol{\mathcal{L}u} := \sigma\boldsymbol{u} - \nu\Delta\boldsymbol{u}$, and we recall that $\sigma, \nu \in \mathbb{R}^+$ denote the reaction term and the fluid viscosity, respectively. The usual variational formulation for problem (13) is given by: *Find* $(\boldsymbol{u}, p) \in \mathbf{V}\times Q := H_0^1(\Omega)^2 \times L_0^2(\Omega)$ *such that:*

$$\mathbf{B}\big((\boldsymbol{u}, p), (\boldsymbol{v}, q)\big) = \mathbf{F}\big(\boldsymbol{v}, q\big) \qquad \forall\,(\boldsymbol{v}, q) \in \mathbf{V}\times Q\,, \tag{14}$$

where

$$\mathbf{B}\big((\boldsymbol{u}, p), (\boldsymbol{v}, q)\big) := \sigma(\boldsymbol{u}, \boldsymbol{v})_\Omega + \nu(\nabla\boldsymbol{u}, \nabla\boldsymbol{v})_\Omega - (p, \nabla\cdot\boldsymbol{v})_\Omega + (q, \nabla\cdot\boldsymbol{u})_\Omega\,, \tag{15}$$

$$\mathbf{F}\big(\boldsymbol{v}, q\big) := (\boldsymbol{f}, \boldsymbol{v})_\Omega. \tag{16}$$

Hereafter, we will define the bilinear form $a(.,.)$ over $\mathbf{V}\times\mathbf{V}$ by

$$a(\boldsymbol{u}, \boldsymbol{v}) := \sigma(\boldsymbol{u}, \boldsymbol{v})_\Omega + \nu(\nabla\boldsymbol{u}, \nabla\boldsymbol{v})_\Omega\,. \tag{17}$$

In order to propose the Petrov-Galerkin method for (13), let $\mathbf{V}_h := [V_h \cap H_0^1(\Omega)]^2$, $Q_h := V_h \cap L_0^2(\Omega)$, where V_h is defined in (1). Then, we propose the following scheme for (13): *Find* $\boldsymbol{u}_1 + \boldsymbol{u}_e \in \mathbf{V}_h \oplus [E_h]^2$ *and* $p_1 \in Q_h$ *such that*

$$\mathbf{B}\big((\boldsymbol{u}_1 + \boldsymbol{u}_e, p_1), (\boldsymbol{v}_1 + \boldsymbol{v}_b, q_1)\big) = \mathbf{F}\big(\boldsymbol{v}_1 + \boldsymbol{v}_b, q_1\big),$$

for all $\boldsymbol{v}_1 + \boldsymbol{v}_b \in \mathbf{V}_h \oplus [H_0^1(\mathcal{T}_h)]^2$ and all $q_1 \in Q_h$. This Petrov-Galerkin scheme may be written as the following system:

$$\mathbf{B}\big((\boldsymbol{u}_1 + \boldsymbol{u}_e, p_1), (\boldsymbol{v}_1, q_1)\big) = \mathbf{F}\big(\boldsymbol{v}_1, q_1\big) \qquad \forall(\boldsymbol{v}_1, q_1) \in \mathbf{V}_h \times Q_h, \tag{18}$$

$$a(\boldsymbol{u}_1 + \boldsymbol{u}_e, \boldsymbol{v}_b)_K - (p_1, \nabla\cdot\boldsymbol{v}_b)_K = (\boldsymbol{f}, \boldsymbol{v}_b)_K \qquad \forall\boldsymbol{v}_b \in H_0^1(K)^2\,,\ \forall\, K \in \mathcal{T}_h\,, \tag{19}$$

where the subindex K stands for integration over K. Equation (19) above may be written in strong form in the following way

$$\boldsymbol{\mathcal{L}}\,\boldsymbol{u}_e \;=\; \boldsymbol{f} - (\sigma\,\boldsymbol{u}_1 + \nabla p_1) \quad \text{in } K. \tag{20}$$

From now on, and just for the derivation of the method, we will suppose that $\boldsymbol{f} \in [V_h]^2$. Now, this differential problem above must be completed with boundary conditions. In order to correct also the residual of the strong equation in the boundary of K, we impose the following boundary condition on $\boldsymbol{u}_e$:

$$\boldsymbol{u}_e \;=\; \boldsymbol{g}_e \quad \text{on } F_i, \quad i = 1,2,3, \tag{21}$$

where $\boldsymbol{g}_e$ will appear as solution of a suitable ODE, with right-hand side depending on $\boldsymbol{f}, \boldsymbol{u}_1$ and p_1, on each sdge F_i (this ODE will be specified, for the basis functions, in §3.1 below). Since this problem is well posed, we can write (18) as follows: Find $(\boldsymbol{u}_1, p_1) \in \mathbf{V}_h \times Q_h$ such that

$$\sum_{K\in\mathcal{T}_h} \left[a(\boldsymbol{u}_1 + \boldsymbol{u}_e^K, \boldsymbol{v}_1)_K - (p_1, \nabla\cdot \boldsymbol{v}_1)_K + (q_1, \nabla\cdot (\boldsymbol{u}_1 + \boldsymbol{u}_e^K))_K \right] = (\boldsymbol{f}, \boldsymbol{v}_1)_\Omega \,, \tag{22}$$

for all $(\boldsymbol{v}_1, q_1) \in \mathbf{V}_h \times Q_h$, where $\boldsymbol{u}_e^K \;:=\; \boldsymbol{u}_e\Big|_K$. Next, in order to give a more practical (and useful in the sequel) formulation, we define, as in (8), an operator $\mathcal{M}_K : \mathbb{P}_1(K)^2 \to H^1(K)^2$ such that

$$\boldsymbol{u}_e^K \;=\; \mathcal{M}_K\,(\boldsymbol{f} - \sigma\,\boldsymbol{u}_1 - \nabla p_1) \quad \forall K \in \mathcal{T}_h\,. \tag{23}$$

Thus, with the characterization (23), the problem (18) leads to the following Petrov-Galerkin Enriched Method (PGEM): Find $(\boldsymbol{u}_1, p_1) \in \mathbf{V}_h \times Q_h$ such that

$$\begin{aligned} &\mathbf{B}_m((\boldsymbol{u}_1, p_1), (\boldsymbol{v}_1, q_1)) \;:= \\ &\sum_{K\in\mathcal{T}_h} [a(\boldsymbol{u}_1 - \mathcal{M}_K\,(\sigma\boldsymbol{u}_1 + \nabla p_1)\,, \boldsymbol{v}_1)_K - (p_1, \nabla\cdot \boldsymbol{v}_1)_K + (q_1, \nabla\cdot(\boldsymbol{u}_1 - \mathcal{M}_K\,(\sigma\boldsymbol{u}_1 + \nabla p_1)))_K] \\ &= (\boldsymbol{f}, \boldsymbol{v}_1)_\Omega - \sum_{K\in\mathcal{T}_h} [\, a(\mathcal{M}_K \boldsymbol{f}, \boldsymbol{v}_1)_K - (q_1, \nabla\cdot(\mathcal{M}_K \boldsymbol{f}))_K\,]\,, \end{aligned} \tag{24}$$

for all $(\boldsymbol{v}_1, q_1) \in \mathbf{V}_h \times Q_h$.

3.1. **The basis functions.** We describe now the way of implementing (24) in terms of its basis functions. Let ψ_1, ψ_2, ψ_3 denote the barycentric coordinates of the element K. We enumerate the sides $F_i, i = 1,2,3$ such that $\psi_i|_{F_i} = 0$ and define as b_K^i the solution of

$$\mathcal{L} b_K^i = \psi_i \quad \text{in } K\,, \tag{25}$$

$$\text{for } j = 1,2,3: \quad \overline{\boldsymbol{\sigma}}_j^i b_K^i - \nu \partial_{\boldsymbol{ss}} b_K^i = \frac{\overline{\boldsymbol{\sigma}}_j^i}{\sigma}\,\psi_i \quad \text{in } F_j\,, \quad b_K^i = 0 \quad \text{on the nodes}\,,$$

where, suggested by [21], we have made the choice

$$\overline{\boldsymbol{\sigma}}_j^i \;=\; \sigma \frac{4|K|^2}{|F_j|^2 |F_i|^2}\,. \tag{26}$$

This local problem may be analytically solved, obtaining the solution

$$b_K^i(x,y) \;=\; \frac{1}{\sigma}\left(\psi_i(x,y) - \frac{\sinh(\alpha_i\psi_i)}{\sinh(\alpha_i)}\right) \quad \text{where} \quad \alpha_i = \sqrt{\frac{4\sigma\,|K|^2}{\nu|F_i|^2}}\,, \tag{27}$$

and hence, we see that, for a linear function $\boldsymbol{g} = (g_1, g_2) = (\sum_{i=1}^3 g_1^i\psi_i, \sum_{j=1}^3 g_2^j\psi_j)$, we have that the operator $\mathcal{M}_K$ defined in (23) is given by

$$\mathcal{M}_K(\boldsymbol{g}) = \left(\sum_{i=1}^3 g_1^i b_K^i, \sum_{j=1}^3 g_2^j b_K^j\right)\,. \tag{28}$$

Hence, an exact expression for the basis functions to be used in the implementation of (24) is available, thus leading to a method which is not of a two level type. Finally, we note that we can exactly formulate the enriched space E_h as the sub-space of $H^1(\mathcal{T}_h)$ which functions are locally linear combinations of the functions b_K^i.

Remark. Let $b_K = \sum_{i=1}^3 b_K^i$, i.e.,

$$b_K(x,y) \;=\; \frac{1}{\sigma}\left(1 - \sum_{i=1}^3 \frac{\sinh(\alpha_i\psi_i)}{\sinh(\alpha_i)}\right)\,. \tag{29}$$

In Figure 1 we depict the function b_K in a patch of equilateral elements for different values of $\alpha_K := \alpha_1 = \alpha_2 = \alpha_3$. In there we can appreciate how this function varies with respect to α_K. This will have a direct impact on the error analysis performed in Section 3.3. We also can compute the mean value of b_K on K. Indeed, from the expression for b_K we obtain

$$\frac{(b_K,1)_K}{|K|} \;=\; \frac{1}{\sigma}\left[1 - 2\sum_{i=1}^3\left(\frac{1}{\alpha_i^2} - \frac{1}{\alpha_i\sinh(\alpha_i)}\right)\right]\,. \tag{30}$$

We further remark that, in the case in which the mesh $\mathcal{T}_h$ is composed by equilateral triangles, then $\overline{\boldsymbol{\sigma}}_j^i = \overline{\boldsymbol{\sigma}}$ for $i,j = 1,2,3$, and then b_K satisfies the following boundary value problem in K:

$$\mathcal{L}\, b_K \;=\; 1 \quad \text{in } K\,, \quad b_K \;=\; g \quad \text{on } \partial K\,, \tag{31}$$

where, for $i = 1,2,3$,

$$\overline{\boldsymbol{\sigma}}\, g \;-\; \nu\partial_{\boldsymbol{ss}} g = \frac{\overline{\boldsymbol{\sigma}}}{\sigma} \quad \text{in } F_i, \quad g \;=\; 0 \quad \text{on the nodes}\,. \tag{32}$$

Finally, using these functions, we may now give a precise definition of the function $\boldsymbol{g}_e$ appearing in (21). Indeed, we have

$$\boldsymbol{g}_e \;=\; \left(\sum_{i=1}^3 (f_i^1 - \sigma u_i^1) b_K^i - b_K\frac{\partial p_1}{\partial x_1}, \sum_{j=1}^3 (f_j^2 - \sigma u_j^2) b_K^j - b_K\frac{\partial p_1}{\partial x_2}\right)\,, \tag{33}$$

where $f_i^k, u_i^k, k = 1, 2, i = 1, 2, 3$, stand for the nodal values of $\boldsymbol{f}$ and $\boldsymbol{u}_1$, respectively. $\square$

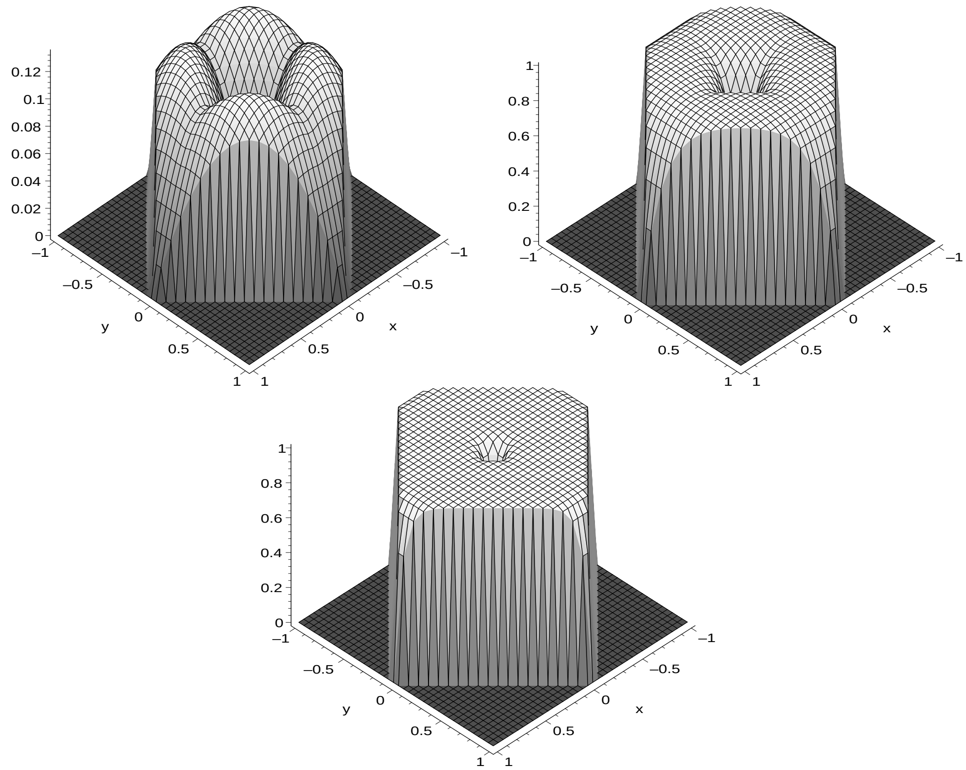

FIGURE 1. Shape of σb_K on a patch of elements with $\alpha_i = 1$, $\alpha_i = 10$ and $\alpha_i = 25$.

3.2. **A link to a stabilized formulation.** We begin by presenting the stabilized finite element method: Find $(\boldsymbol{u}_1, p_1) \in \mathbf{V}_h \times Q_h$ such that

$$\mathbf{B}_\tau((\boldsymbol{u}_1, p_1), (\boldsymbol{v}_1, q_1)) \;=\; \mathbf{F}_\tau(\boldsymbol{v}_1, q_1) \qquad \forall\, (\boldsymbol{v}_1, q_1) \in \mathbf{V}_h \times Q_h\,, \tag{34}$$

where

$$\mathbf{B}_\tau((\boldsymbol{u}_1, p_1), (\boldsymbol{v}_1, q_1)) := \mathbf{B}((\boldsymbol{u}_1, p_1), (\boldsymbol{v}_1, q_1)) - \sum_{K \in \mathcal{T}_h} \tau_K\, (\sigma\, \boldsymbol{u}_1 + \nabla p_1, \sigma\, \boldsymbol{v}_1 - \nabla q_1)_K\,,$$

$$\mathbf{F}_\tau(\boldsymbol{v}_1, q_1) \;:=\; \mathbf{F}(\boldsymbol{v}_1, q_1) - \sum_{K \in \mathcal{T}_h} \tau_K\, (\boldsymbol{f}, \sigma\, \boldsymbol{v}_1 - \nabla q_1)_K\,,$$

and the stabilization parameter is given by

$$\tau_K \;:=\; \frac{1}{\sigma}\left[1 - 2\sum_{i=1}^{3}\left(\frac{1}{\alpha_i^2} - \frac{1}{\alpha_i \sinh(\alpha_i)}\right)\right]. \tag{35}$$

3.2.1. *Derivation.* For completeness of the presentation, we resume the derivation carried out in [8]. We will suppose that the mesh $\mathcal{T}_h$ is made up using equilateral triangles. The first step is to replace in our formulation $\boldsymbol{u}_e$ by

$$\tilde{\boldsymbol{u}}_e := \mathcal{M}_K(\boldsymbol{f} - \sigma\,\overline{\boldsymbol{u}}_1 - \nabla p_1) = b_K\,(\boldsymbol{f} - \sigma\,\overline{\boldsymbol{u}}_1 - \nabla p_1)\,, \tag{36}$$

where, for a function v, $\overline{v}$ denotes its projection onto the $\mathbb{P}_0(K)$ space, i.e.,

$$\overline{v} := \frac{(v,1)_K}{|K|}\,.$$

We further remark that b_K satisfies

$$\|b_K\|_{0,K} \le C\,h_K^3 \quad \text{and} \quad \|b_K\|_{0,\partial K} \le C\,h_K^{5/2}\,, \tag{37}$$

where $C > 0$ is a positive constant depending possibly on σ and ν, but independent of h.

Next, in order to design a stabilized finite element method we integrate by parts and arrive at the following rewriting of (18) (or (24)):

$$\mathbf{B}((\boldsymbol{u}_1,p_1),(\boldsymbol{v}_1,q_1)) + \sum_{K\in\mathcal{T}_h}\Big[(\tilde{\boldsymbol{u}}_e,\sigma\,\boldsymbol{v}_1 - \nabla q_1)_K + (\tilde{\boldsymbol{u}}_e,\nu\partial_{\boldsymbol{n}}\boldsymbol{v}_1 + q_1\mathbf{I}\cdot\boldsymbol{n})_{\partial K}\Big] = \mathbf{F}(\boldsymbol{v}_1,q_1)\,.$$

Next, we neglect the boundary terms (see [8] for a discussion about this matter). Also, using (37) and the approximation properties of the projection (cf. [18]), we obtain

$$\sum_{K\in\mathcal{T}_h}(\tilde{\boldsymbol{u}}_e,\sigma(\boldsymbol{v}_1 - \overline{\boldsymbol{v}}_1))_K \le C\,h_K^3\,\|\boldsymbol{f} - \sigma\,\overline{\boldsymbol{u}}_1 - \nabla p_1\|_{0,K}|\boldsymbol{v}_1|_{1,K}\,,$$

and hence, using (36) and the orthogonality of the projection, the following approximation is justified

$$\sum_{K\in\mathcal{T}_h}(\tilde{\boldsymbol{u}}_e,\sigma\,\boldsymbol{v}_1 - \nabla q_1)_K \approx \sum_{K\in\mathcal{T}_h}\frac{(b_K,1)_K}{|K|}\,(\boldsymbol{f} - \sigma\,\overline{\boldsymbol{u}}_1 - \nabla p_1,\sigma\,\boldsymbol{v}_1 - \nabla q_1)_K\,.$$

Collecting all the previous results, we can present the following stabilized finite element method for (13): Find $(\boldsymbol{u}_1,p_1)\in\mathbf{V}_h\times Q_h$ such that

$$\begin{aligned}\mathbf{B}((\boldsymbol{u}_1,p_1),(\boldsymbol{v}_1,q_1)) - \sum_{K\in\mathcal{T}_h}\frac{(b_K,1)_K}{|K|}\,(\sigma\,\overline{\boldsymbol{u}}_1 + \nabla p_1,\sigma\,\boldsymbol{v}_1 - \nabla q_1)_K \\ = \mathbf{F}(\boldsymbol{v}_1,q_1) - \sum_{K\in\mathcal{T}_h}\frac{(b_K,1)_K}{|K|}\,(\boldsymbol{f},\sigma\,\boldsymbol{v}_1 - \nabla q_1)_K\,,\end{aligned} \tag{38}$$

for all $(\boldsymbol{v}_1,q_1)\in\mathbf{V}_h\times Q_h$. Since we want to present a stabilized finite element method with a classical structure, we replace the added terms in K by $(\sigma\,\boldsymbol{u}_1 + \nabla p_1,\sigma\,\boldsymbol{v}_1 - \nabla q_1)_K$, which introduces a new source of error, but, again, this error is of a smaller size. Thus, using (30) we recover the method (34).

3.3. **Convergence analysis and error estimates.** This section is devoted to the a priori error analysis of the method (34). We will start by giving a technical result concerning the properties of the stabilization parameter τ_K and then we will give a stability result for (34).

Lemma 1. Let $K \in \mathcal{T}_h$, let $\alpha_K = \max\{\alpha_i : i = 1, 2, 3\}$ and $F \in \mathcal{E}(K)$ and $\omega_F = K \cup K'$. Then, the following estimates hold for τ_K:

$$C_1 \min\{1, \alpha_K^2\} \le \sigma\tau_K \le C_2 \min\{1, \alpha_K^2\}, \tag{39}$$

$$\frac{C_1}{1+\alpha_K^2} \le 1 - \sigma\tau_K \le \frac{C_2}{1+\alpha_K^2}, \tag{40}$$

$$|[\![1 - \sigma\tau_K]\!]_F| \le C \min\{1, \alpha_K^2\}, \tag{41}$$

where the (positive) constans C, C_1 and C_2 do not depend on h, σ or ν.

Proof. The results hold using a Taylor series expansion for the function $\sinh(x)$, the definition of τ_K and the mesh regularity. □

Next, let us define the following mesh-dependent norm:

$$\|(\boldsymbol{v}, q)\|_h^2 := \sum_{K\in\mathcal{T}_h} \left[\sigma(1-\sigma\tau_K)\,\|\boldsymbol{v}\|_{0,K}^2 + \nu\,|\boldsymbol{v}|_{1,K}^2 + \tau_K\,\|\nabla q\|_{0,K}^2\right]. \tag{42}$$

Lemma 2. For all $(\boldsymbol{v}, q) \in [H^1(\Omega)]^2 \times H^1(\Omega)$, the bilinear form $\mathbf{B}_\tau$ satisfies

$$\mathbf{B}_\tau((\boldsymbol{v}, q), (\boldsymbol{v}, q)) = \|(\boldsymbol{v}, q)\|_h^2,$$

$$\mathbf{B}_\tau((\boldsymbol{v}, q), (\boldsymbol{w}, r)) \le \|(\boldsymbol{v}, q)\|_h\,\|(\boldsymbol{w}, r)\|_h + \sum_{K\in\mathcal{T}_h} (1-\sigma\tau_K)\,(\nabla q, \boldsymbol{w})_K + \sum_{K\in\mathcal{T}_h} (1-\sigma\tau_K)\,(\nabla\cdot\boldsymbol{v}, r)_K - \sum_{F\in\mathcal{E}_\Omega} [\![\sigma\tau_K]\!]_F\,(\boldsymbol{v}, r)_F,$$

and the discrete problem (34) has a unique solution.

Proof. The first equality follows easily from the definition of $\mathbf{B}_\tau$. The second one is straightfoward from the definition of $\mathbf{B}_\tau$, integration by parts and the Cauchy-Schwarz's inequality. □

The method (34) is not strongly consistent. Hence, we bound the consistency error in the following result.

Lemma 3. Let us suppose that $(\boldsymbol{u},p)\in[H^2(\Omega)\cap H_0^1(\Omega)]^2\times[H^1(\Omega)\cap L_0^2(\Omega)]$. Then, there exists $C>0$ such that

$$\mathbf{B}_\tau((\boldsymbol{u}-\boldsymbol{u}_1,p-p_1),(\boldsymbol{v}_1,q_1)) \;\le\; \sum_{K\in\mathcal{T}_h}\tau_K\nu(\Delta\boldsymbol{u},\sigma\boldsymbol{v}_1)_K + C\,h\,\sqrt{\nu}\,|\boldsymbol{u}|_{2,\Omega}\|(\boldsymbol{v}_1,q_1)\|_h\,.$$

Proof. A simple computation shows that

$$\mathbf{B}_\tau((\boldsymbol{u}-\boldsymbol{u}_1,p-p_1),(\boldsymbol{v}_1,q_1)) = \sum_{K\in\mathcal{T}_h}\tau_K(\nu\Delta\boldsymbol{u},\sigma\boldsymbol{v}_1-\nabla q_1)_K\,,$$

and the result follows from (39), the Cauchy-Schwarz's inequality and the definition of $\|.\|_h$. □

Lemma 4. Let us suppose that $(\boldsymbol{u},p)\in[H^2(\Omega)\cap H_0^1(\Omega)]^2\times[H^1(\Omega)\cap L_0^2(\Omega)]$ is the solution of (13) and that $(\boldsymbol{u}_1,p_1)\in\mathbf{V}_h\times Q_h$ is the solution of (34), and let us denote $(\boldsymbol{e_u},e_p):=(\boldsymbol{u}-\boldsymbol{u}_1,p-p_1)$. Then, there exists $C>0$, independent of h,σ or ν, such that

$$\begin{aligned}\|(\boldsymbol{e_u},e_p)\|_h^2\le C \inf_{(\boldsymbol{v}_1,q_1)\in\mathbf{V}_h\times Q_h}&\Big\{\|(\boldsymbol{u}-\boldsymbol{v}_1,p-q_1)\|_h^2+\sum_{K\in\mathcal{T}_h}\nu h_K^{-2}\,\|\boldsymbol{u}-\boldsymbol{v}_1\|_{0,K}^2+\nu\,h^2\,|\boldsymbol{u}|_{2,\Omega}^2\\ &+\sum_{K\in\mathcal{T}_h}\frac{\min\{1,\alpha_K^2\}}{\sigma}\,(h_K^{-2}\|p-q_1\|_{0,K}^2\;+\;|p-q_1|_{1,K}^2)\Big\}\,.\end{aligned}$$

Proof. Let $(\boldsymbol{v}_1,q_1)\in\mathbf{V}_h\times Q_h$. Then, from Lemmas 2 and 3 there follows

$$\begin{aligned}\|(\boldsymbol{e_u},e_p)\|_h^2 =&\; \mathbf{B}_\tau((\boldsymbol{e_u},e_p),(\boldsymbol{e_u},e_p))\\ =&\; \mathbf{B}_\tau((\boldsymbol{e_u},e_p),(\boldsymbol{u}-\boldsymbol{v}_1,p-q_1))+\mathbf{B}_\tau((\boldsymbol{e_u},e_p),(\boldsymbol{v}_1-\boldsymbol{u}_1,q_1-p_1))\\ \le&\; \|(\boldsymbol{e_u},e_p)\|_h\|(\boldsymbol{u}-\boldsymbol{v}_1,p-q_1)\|_h+\underbrace{\sum_{K\in\mathcal{T}_h}(1-\sigma\tau_K)\,(\nabla e_p,(\boldsymbol{u}-\boldsymbol{v}_1))_K}_{I}\\ &\underbrace{+\sum_{K\in\mathcal{T}_h}(1-\sigma\tau_K)\,(\nabla\cdot\boldsymbol{e_u},(p-q_1))_K-\sum_{F\in\mathcal{E}_\Omega}[\![\sigma\tau_K]\!]_F\,(\boldsymbol{e_u}\cdot\boldsymbol{n},p-q_1)_F}_{II}\\ &\underbrace{\sum_{K\in\mathcal{T}_h}\tau_K(\nu\Delta\boldsymbol{u},\sigma(\boldsymbol{v}_1-\boldsymbol{u}_1))_K}_{III}+C\,h\,\sqrt{\nu}\,|\boldsymbol{u}|_{2,\Omega}\,\|(\boldsymbol{v}_1-\boldsymbol{u}_1,q_1-p_1)\|_h\,.\end{aligned}\tag{43}$$

Now, we proceed term by term. First, using the Cauchy-Schwarz's inequality we arrive at

$$\begin{aligned}
I &= \sum_{K\in\mathcal{T}_h}(1-\sigma\tau_K)(\nabla e_p, \boldsymbol{u}-\boldsymbol{v}_1)_K \\
&\le \sum_{K\in\mathcal{T}_h}(1-\sigma\tau_K)\|\nabla e_p\|_{0,K}\|\boldsymbol{u}-\boldsymbol{v}_1\|_{0,K} \\
&\le C\left\{\sum_{K\in\mathcal{T}_h}\frac{\tau_K^{-1}}{1+\alpha_K^2}\|\boldsymbol{u}-\boldsymbol{v}_1\|_{0,K}^2\right\}^{\frac{1}{2}}\|(\boldsymbol{e_u},e_p)\|_h \\
&\le C\left\{\sum_{K\in\mathcal{T}_h}\sigma\min\{1,\alpha_K^{-2}\}\max\{1,\alpha_K^{-2}\}\|\boldsymbol{u}-\boldsymbol{v}_1\|_{0,K}^2\right\}^{\frac{1}{2}}\|(\boldsymbol{e_u},e_p)\|_h \\
&= C\left\{\sum_{K\in\mathcal{T}_h}\sigma\alpha_K^{-2}\|\boldsymbol{u}-\boldsymbol{v}_1\|_{0,K}^2\right\}^{\frac{1}{2}}\|(\boldsymbol{e_u},e_p)\|_h\,,
\end{aligned} \tag{44}$$

where we have also used that $1-\sigma\tau_K\le 1$, (39), (40) and the definition of the norm $\|\cdot\|_h$. Also, applying (40) we obtain

$$\begin{aligned}
II &= \sum_{K\in\mathcal{T}_h}(1-\sigma\tau_K)(p-q_1,\nabla\cdot\boldsymbol{e_u})_K - \sum_{F\in\mathcal{E}_\Omega}[\![\sigma\tau_K]\!]_F\,(\boldsymbol{e_u}\cdot\boldsymbol{n},p-q_1)_F \\
&\le \sum_{K\in\mathcal{T}_h}\frac{C}{1+\alpha_K^2}\|p-q_1\|_{0,K}|\boldsymbol{e_u}|_{1,K} + \sum_{F\in\mathcal{E}_\Omega}|[\![\sigma\tau_K]\!]_F|\,\|\boldsymbol{e_u}\|_{0,F}\|p-q_1\|_{0,F} \\
&\le C\left\{\sum_{K\in\mathcal{T}_h}\frac{\nu^{-1}}{1+\alpha_K^2}\|p-q_1\|_{0,K}^2\right\}^{\frac{1}{2}}\|(\boldsymbol{e_u},e_p)\|_h + \sum_{F\in\mathcal{E}_\Omega}|[\![\sigma\tau_K]\!]_F|\,\|\boldsymbol{e_u}\|_{0,F}\|p-q_1\|_{0,F}\,.
\end{aligned}$$

Next, using the local trace result (cf. [31]): there exists $C>0$, independent of h, such that, for all $K\in\mathcal{T}_h, F\in\mathcal{E}_K$ and all $v\in H^1(\omega_F)$

$$\|v\|_{0,F}\le C\Big(h_F^{-\frac{1}{2}}\|v\|_{0,\omega_F}+h_F^{\frac{1}{2}}|v|_{1,\omega_F}\Big)\,, \tag{45}$$

the mesh regularity, (40),(41) and the definition of α_K, we obtain

$$
\begin{aligned}
&\sum_{F\in\mathcal{E}_\Omega} |[\![\sigma\tau_K]\!]_F|\, \|\boldsymbol{e_u}\|_{0,F}\|p-q_1\|_{0,F} \\
\leq& C \sum_{F\in\mathcal{E}_\Omega} |[\![\sigma\tau_K]\!]_F|\, \{h_F^{-\frac{1}{2}}\|\boldsymbol{e_u}\|_{0,\omega_F} + h_F^{\frac{1}{2}}|\boldsymbol{e_u}|_{1,\omega_F}\}\{h_F^{-\frac{1}{2}}\|p-q_1\|_{0,\omega_F} + h_F^{\frac{1}{2}}|p-q_1|_{1,\omega_F}\} \\
=& C \sum_{F\in\mathcal{E}_\Omega} |[\![1-\sigma\tau_K]\!]_F|\, \{\|\boldsymbol{e_u}\|_{0,\omega_F} + h_F|\boldsymbol{e_u}|_{1,\omega_F}\}\{h_F^{-1}\|p-q_1\|_{0,\omega_F} + |p-q_1|_{1,\omega_F}\} \\
\leq& C \left\{\sum_{K\in\mathcal{T}_h} (1-\sigma\tau_K)\|\boldsymbol{e_u}\|_{0,K}^2 \,+\, \frac{h_K^2}{1+\alpha_K^2}\,|\boldsymbol{e_u}|_{1,K}^2\right\}^{\frac{1}{2}} \\
&\left\{\sum_{K\in\mathcal{T}_h} \min\{1,\alpha_K^2\}\,(h_K^{-2}\|p-q_1\|_{0,K}^2 \,+\, |p-q_1|_{1,K}^2)\right\}^{\frac{1}{2}} \\
\leq& \left\{\sum_{K\in\mathcal{T}_h} \frac{\min\{1,\alpha_K^2\}}{\sigma}\,(h_K^{-2}\|p-q_1\|_{0,K}^2 \,+\, |p-q_1|_{1,K}^2)\right\}^{\frac{1}{2}} \|(\boldsymbol{e_u},e_p)\|_h\,. \qquad (46)
\end{aligned}
$$

Finally, in an analogous way arrive at the following

$$
\begin{aligned}
III =& \sum_{K\in\mathcal{T}_h} \nu\tau_K(\Delta\boldsymbol{u},\sigma(\boldsymbol{v}_1-\boldsymbol{u}_1))_K = \sum_{K\in\mathcal{T}_h} \nu\tau_K(\Delta\boldsymbol{u},\sigma\boldsymbol{e_u})_K + \sum_{K\in\mathcal{T}_h} \nu\tau_K(\Delta\boldsymbol{u},\sigma(\boldsymbol{u}-\boldsymbol{v}_1))_K \\
\leq& C \left\{\sum_{K\in\mathcal{T}_h} \frac{\nu^2\tau_K^2\sigma}{1-\sigma\tau_K}\,|\boldsymbol{u}|_{2,K}^2\right\}^{\frac{1}{2}} \|(\boldsymbol{e_u},e_p)\|_h + C\sum_{K\in\mathcal{T}_h} \nu\sigma\tau_K|\boldsymbol{u}|_{2,K}\|\boldsymbol{u}-\boldsymbol{v}_1\|_{0,K} \\
\leq& C \left\{\sum_{K\in\mathcal{T}_h} \nu^2\frac{\min\{1,\alpha_K^2\}}{\sigma}(1+\alpha_K^2)\,|\boldsymbol{u}|_{2,K}^2\right\}^{\frac{1}{2}} \|(\boldsymbol{e_u},e_p)\|_h + C\sum_{K\in\mathcal{T}_h} \nu\,|\boldsymbol{u}|_{2,K}\|\boldsymbol{u}-\boldsymbol{v}_1\|_{0,K} \\
\leq& C \left\{\sum_{K\in\mathcal{T}_h} \frac{\nu^2}{\sigma}\min\{1,\alpha_K^2\}\max\{1,\alpha_K^2\}\,|\boldsymbol{u}|_{2,K}^2\right\}^{\frac{1}{2}} \|(\boldsymbol{e_u},e_p)\|_h + C\sum_{K\in\mathcal{T}_h} \nu\,|\boldsymbol{u}|_{2,K}\|\boldsymbol{u}-\boldsymbol{v}_1\|_{0,K} \\
=& C \left\{\sum_{K\in\mathcal{T}_h} \frac{\nu^2\alpha_K^2}{\sigma}\,|\boldsymbol{u}|_{2,K}^2\right\}^{\frac{1}{2}} \|(\boldsymbol{e_u},e_p)\|_h + C\sum_{K\in\mathcal{T}_h} \nu\,|\boldsymbol{u}|_{2,K}\|\boldsymbol{u}-\boldsymbol{v}_1\|_{0,K}\,. \qquad (47)
\end{aligned}
$$

Summing up, from (43)-(47), and the definition of α_K, we obtain

$$
\begin{aligned}
&\|(\boldsymbol{e_u}, e_p)\|_h^2 \le \|(\boldsymbol{e_u}, e_p)\|_h \|(\boldsymbol{u}-\boldsymbol{v}_1, p-q_1)\|_h \\
&+ C \left\{ \sum_{K\in\mathcal{T}_h} \sigma\, \alpha_K^{-2}\, \|\boldsymbol{u}-\boldsymbol{v}_1\|_{0,K}^2 \right\}^{\frac{1}{2}} \|(\boldsymbol{e_u}, e_p)\|_h + C \left\{ \sum_{K\in\mathcal{T}_h} \frac{\nu^{-1}}{1+\alpha_K^2} \|p-q_1\|_{0,K}^2 \right\}^{\frac{1}{2}} \|(\boldsymbol{e_u}, e_p)\|_h \\
&+ C \left\{ \sum_{K\in\mathcal{T}_h} \frac{\min\{1,\alpha_K^2\}}{\sigma} \,(h_K^{-2}\|p-q_1\|_{0,K}^2 + |p-q_1|_{1,K}^2) \right\}^{\frac{1}{2}} \|(\boldsymbol{e_u}, e_p)\|_h \\
&+ C \left\{ \sum_{K\in\mathcal{T}_h} \frac{\nu^2\alpha_K^2}{\sigma} \,|\boldsymbol{u}|_{2,K}^2 \right\}^{\frac{1}{2}} \|(\boldsymbol{e_u}, e_p)\|_h + C \sum_{K\in\mathcal{T}_h} \nu\, |\boldsymbol{u}|_{2,K} \|\boldsymbol{u}-\boldsymbol{v}_1\|_{0,K} \\
&+ Ch\sqrt{\nu}\, |\boldsymbol{u}|_{2,\Omega} \|(\boldsymbol{e_u}, e_p)\|_h + Ch\sqrt{\nu}\, |\boldsymbol{u}|_{2,\Omega} \|(\boldsymbol{u}-\boldsymbol{v}_1, p-q_1)\|_h \\
\le& C \Big\{ \|(\boldsymbol{u}-\boldsymbol{v}_1, p-q_1)\|_h^2 + \sum_{K\in\mathcal{T}_h} \left(\sigma\, \alpha_K^{-2}\, \|\boldsymbol{u}-\boldsymbol{v}_1\|_{0,K}^2 + \nu\, |\boldsymbol{u}|_{2,K} \|\boldsymbol{u}-\boldsymbol{v}_1\|_{0,K} \right) + \nu\, h^2\, |\boldsymbol{u}|_{2,\Omega}^2 \\
&+ \sum_{K\in\mathcal{T}_h} \frac{\nu^{-1}}{1+\alpha_K^2} \|p-q_1\|_{0,K}^2 + \sum_{K\in\mathcal{T}_h} \frac{\min\{1,\alpha_K^2\}}{\sigma} \,(h_K^{-2}\|p-q_1\|_{0,K}^2 + |p-q_1|_{1,K}^2) \Big\} \\
&+ \frac{1}{2} \|(\boldsymbol{e_u}, e_p)\|_h^2 \,,
\end{aligned}
$$

and the result follows using that

$$
\frac{\nu^{-1}}{1+\alpha_K^2} \le C\, \frac{\min\{1,\alpha_K^2\} h_K^{-2}}{\sigma} \,,
$$

and rearranging terms. □

For the proof of the next result we introduce the Clément interpolation operator (cf. [17, 18]) $\mathcal{C}_h : H^1(\Omega) \to V_h$ (if $v \in H_0^1(\Omega)$, then we may define $\mathcal{C}_h(v)$ with values in $V_h \cap H_0^1(\Omega)$), satisfying

$$
\|v - \mathcal{C}_h(v)\|_{0,K} \le C\, \|v\|_{0,\omega_K} \,, \tag{48}
$$

$$
|v - \mathcal{C}_h(v)|_{m,K} \le C\, h_K^{1-m} |v|_{1,\omega_K} \,, \tag{49}
$$

for $m = 0, 1$, with the obvious extension to vector-valued functions.

Lemma 5. Let us suppose that $(\boldsymbol{u},p)\in[H^2(\Omega)\cap H_0^1(\Omega)]^2\times[H^1(\Omega)\cap L_0^2(\Omega)]$ is the solution of (13). Then, there exists $C>0$ such that

$$\|(\boldsymbol{u}-\mathcal{C}_h(\boldsymbol{u}),p-\tilde{p}_h)\|_h^2+\sum_{K\in\mathcal{T}_h}\nu h_K^{-2}\,\|\boldsymbol{u}-\mathcal{C}_h(\boldsymbol{u})\|_{0,K}^2$$

$$+\sum_{K\in\mathcal{T}_h}\frac{\min\{1,\alpha_K^2\}}{\sigma}\,(h_K^{-2}\|p-\tilde{p}_h\|_{0,K}^2\;+\;|p-\tilde{p}_h|_{1,K}^2)\le C\,h^2\nu\,|\boldsymbol{u}|_{2,\Omega}^2+\frac{\min\{1,\alpha_K^2\}}{\sigma}\,|p|_{1,\Omega}^2,$$

where $\tilde{p}_h=\mathcal{C}_h(p)-\frac{(\mathcal{C}_h(p),1)_\Omega}{|\Omega|}\in Q_h$.

Proof. The result follows from the definition of the norm $\|.\|_h$. Indeed, using (48)-(49), (39)-(40) and the mesh regularity we obtain

$$\begin{aligned}\|(\boldsymbol{u}-\mathcal{C}_h(\boldsymbol{u}),p-\tilde{p}_h)\|_h^2&=\sum_{K\in\mathcal{T}_h}\sigma(1-\sigma\tau_K)\,\|\boldsymbol{u}-\mathcal{C}_h(\boldsymbol{u})\|_{0,K}^2+\nu\,|\boldsymbol{u}-\mathcal{C}_h(\boldsymbol{u})|_{1,K}^2+\tau_K\,|p-\tilde{p}_h|_{1,K}^2\\&\le C\sum_{K\in\mathcal{T}_h}\Big[\Big(\frac{\sigma h_K^4}{1+\alpha_K^2}+\nu h_K^2\Big)\,|\boldsymbol{u}|_{2,\omega_K}^2+\frac{\min\{1,\alpha_K^2\}}{\sigma}\,|p|_{1,\omega_K}^2\Big]\\&\le C\Big(\nu\,h^2\,|\boldsymbol{u}|_{2,\Omega}^2+\frac{\min\{1,\alpha_K^2\}}{\sigma}\,|p|_{1,\Omega}^2\Big).\end{aligned}$$

The other terms are bounded in a similar way. □

Finally, using the previous result and the asymptotic behavior of τ_K (cf. Lemma 1) we can prove the following optimal convergence result.

Theorem 6. *Let us suppose that* $(\boldsymbol{u},p)\in[H^2(\Omega)\cap H_0^1(\Omega)]^2\times[H^1(\Omega)\cap L_0^2(\Omega)]$ *is the solution of* (13) *and that* $(\boldsymbol{u}_1,p_1)\in\mathbf{V}_h\times Q_h$ *is the solution of* (34). *Then, there exists* $C>0$ *independent of* h,σ *and* ν, *such that*

$$\|(\boldsymbol{e}_{\boldsymbol{u}},e_p)\|_h\;\le\;C\,(\sqrt{\nu}h\,|\boldsymbol{u}|_{2,\Omega}+\frac{\min\{1,\alpha_K\}}{\sqrt{\sigma}}|p|_{1,\Omega}).$$

Proof. The result follows applying Lemmas 4 and 5 with $\boldsymbol{v}_1=\mathcal{C}_h(\boldsymbol{u})$ and $q_1=\tilde{p}_h$. □

Remark. The estimate from Theorem 6 may written as:

$$\Big[\sum_{K\in\mathcal{T}_h}\frac{\sigma(1-\sigma\tau_K)}{\nu}\|\boldsymbol{e}_{\boldsymbol{u}}\|_{0,K}^2+\frac{\tau_K}{\nu}|e_p|_{1,K}^2\Big]^{\frac{1}{2}}+|\boldsymbol{e}_{\boldsymbol{u}}|_{1,\Omega}\le C\,(h\,|\boldsymbol{u}|_{2,\Omega}+\frac{\min\{1,\alpha_K\}}{\sqrt{\sigma\nu}}|p|_{1,\Omega})\,,$$

which, using (40) leads to

$$\Big[\sum_{K\in\mathcal{T}_h}\frac{\sigma}{\sigma h_K^2+\nu}\,\|\boldsymbol{e}_{\boldsymbol{u}}\|_{0,K}^2\Big]^{\frac{1}{2}}\;+\;|\boldsymbol{e}_{\boldsymbol{u}}|_{1,\Omega}\le C\,(h\,|\boldsymbol{u}|_{2,\Omega}+\frac{\min\{1,\alpha_K\}}{\sqrt{\sigma\nu}}|p|_{1,\Omega})\,,\qquad(50)$$

which may be seen as a robust estimate for the velocity. Now, if $\nu \le \sigma h_K^2$, then (50) provides the following estimate

$$\|\boldsymbol{e}_{\boldsymbol{u}}\|_{0,\Omega} \;\le\; C\, h^2\, (|\boldsymbol{u}|_{2,\Omega} + \frac{1}{\nu}|p|_{1,\Omega})\,, \tag{51}$$

which is an optimal error estimate for $\|\boldsymbol{e}_{\boldsymbol{u}}\|_{0,\Omega}$, which does not need the use of a duality argument. □

In the next result we state an error estimate for the pressure in its natural norm.

Theorem 7. *Under hypothesis of Theorem 6, there exists a positive constant $C > 0$ such that*

$$\|e_p\|_{0,\Omega} \;\le\; C\,\sqrt{\sigma}\max\{1,\sqrt{\nu}\}\left(\sqrt{\nu}h\,|\boldsymbol{u}|_{2,\Omega} + \frac{\min\{1,\alpha_K\}}{\sqrt{\sigma}}|p|_{1,\Omega} + \min\{1,\alpha_K\}\,|\boldsymbol{u}|_{2,\Omega}\right).$$

Proof. From the continuous inf-sup condition (see [26]), there exists $\boldsymbol{w} \in H_0^1(\Omega)^2$ such that $\nabla\cdot\boldsymbol{w} = e_p$ in Ω and $\|\boldsymbol{w}\|_{1,\Omega} \le C\,\|e_p\|_{0,\Omega}$. Let $\boldsymbol{w}_h = \mathcal{C}_h(\boldsymbol{w}) \in \mathbf{V}_h$ be the Clément interpolate of $\boldsymbol{w}$. Then, integrating by parts, (34) (applied to $(\boldsymbol{w}_h, 0)$) and Cauchy-Schwarz's inequality we obtain

$$\begin{aligned}
\|e_p\|_{0,\Omega}^2 &= (\nabla\cdot\boldsymbol{w}, e_p)_\Omega \\
&= (\nabla\cdot(\boldsymbol{w}-\boldsymbol{w}_h), e_p)_\Omega \;+\; (\nabla\cdot\boldsymbol{w}_h, e_p)_\Omega \\
&= -\,(\boldsymbol{w}-\boldsymbol{w}_h, \nabla e_p)_\Omega \;+\; \nu\,(\nabla\boldsymbol{e}_{\boldsymbol{u}}, \nabla\boldsymbol{w}_h)_\Omega \;+\; \sigma\,(\boldsymbol{e}_{\boldsymbol{u}}, \boldsymbol{w}_h)_\Omega \\
&\quad -\sum_{K\in\mathcal{T}_h}\tau_K\,(\sigma\,\boldsymbol{e}_{\boldsymbol{u}} \;+\; \nabla e_p, \sigma\,\boldsymbol{w}_h)_K - \sum_{K\in\mathcal{T}_h}\tau_K(\nu\Delta\boldsymbol{u}, \sigma\boldsymbol{w}_h)_K \\
&= -\,(\boldsymbol{w}-\boldsymbol{w}_h, \nabla e_p)_\Omega \;+\; \nu\,(\nabla\boldsymbol{e}_{\boldsymbol{u}}, \nabla\boldsymbol{w}_h)_\Omega + \sum_{K\in\mathcal{T}_h}((1-\sigma\tau_K)\boldsymbol{e}_{\boldsymbol{u}}, \sigma\boldsymbol{w}_h)_K \\
&\quad -\sum_{K\in\mathcal{T}_h}\tau_K\,(\nabla e_p, \sigma\,\boldsymbol{w}_h)_K - \sum_{K\in\mathcal{T}_h}\tau_K(\nu\Delta\boldsymbol{u}, \sigma\boldsymbol{w}_h)_K \\
&\le C\sum_{K\in\mathcal{T}_h} h_K|\boldsymbol{w}|_{1,\omega_K}\|\nabla e_p\|_{0,K} \;+\; \nu\,|\boldsymbol{e}_{\boldsymbol{u}}|_{1,\Omega}|\boldsymbol{w}_h|_{1,\Omega} + \sum_{K\in\mathcal{T}_h}\sigma(1-\sigma\tau_K)\|\boldsymbol{e}_{\boldsymbol{u}}\|_{0,K}\|\boldsymbol{w}_h\|_{0,K} \\
&\quad + \sum_{K\in\mathcal{T}_h}\sigma\tau_K\,\|\nabla e_p\|_{0,K}\|\boldsymbol{w}_h\|_{0,K} + \sum_{K\in\mathcal{T}_h}\sigma\tau_K\nu|\boldsymbol{u}|_{2,K}\|\boldsymbol{w}_h\|_{0,K}
\end{aligned}$$

$$\leq C \Big[\sum_{K\in\mathcal{T}_h} (\tau_K + \frac{\sigma^2\tau_K^2}{\nu}) \, |e_p|_{1,K}^2 \; + \; \nu|\boldsymbol{e_u}|_{1,K}^2 \; + \; \frac{\sigma^2(1-\sigma\,\tau_K)^2}{\nu} \, \|\boldsymbol{e_u}\|_{0,K}^2 + \sigma^2\tau_K^2\nu \, |\boldsymbol{u}|_{2,K}^2 \Big]^{\frac{1}{2}}$$

$$\Big[\sum_{K\in\mathcal{T}_h} \nu \, |\boldsymbol{w}|_{1,\omega_K}^2 \; + \; \nu \, |\boldsymbol{w}_h|_{1,\Omega}^2 \; + \; \nu \, \|\boldsymbol{w}_h\|_{0,\Omega}^2 \Big]^{\frac{1}{2}}$$

$$\leq C\,\sqrt{\sigma}\,\max\{1,\sqrt{\nu}\}\,\Big[\|(\boldsymbol{e_u}, e_p)\|_h^2 + \sum_{K\in\mathcal{T}_h} \min\{1,\alpha_K^4\} |\boldsymbol{u}|_{2,K}^2 \Big]^{\frac{1}{2}} \Big[|\boldsymbol{w}|_{1,\Omega}^2 \; + \; |\boldsymbol{w}_h|_{1,\Omega}^2 \; + \; \|\boldsymbol{w}_h\|_{0,\Omega}^2 \Big]^{\frac{1}{2}} . \tag{52}$$

Now, using the approximation properties of the Clément interpolant (cf. [18]) we obtain

$$\Big[|\boldsymbol{w}|_{1,\Omega}^2 \; + \; |\boldsymbol{w}_h|_{1,\Omega}^2 \; + \; \|\boldsymbol{w}_h\|_{0,\Omega}^2 \Big]^{\frac{1}{2}} \leq C \, \|\boldsymbol{w}\|_{1,\Omega} \; \leq \; C \, \|e_p\|_{0,\Omega} \, .$$

Hence, dividing in (52) by $\|e_p\|_{0,\Omega}$, using the norm $\|\cdot\|_h$ definition, we have

$$\|p - p_1\|_{0,\Omega} \leq \; C\,\sqrt{\sigma}\max\{1,\sqrt{\nu}\} \left(\|(\boldsymbol{u}-\boldsymbol{u}_1, p - p_1)\|_h + \min\{1,\alpha_K\} \, |\boldsymbol{u}|_{2,\Omega} \right),$$

and the result follows applying Theorem 6. □

Throughout the next lemma we will suppose that the solution of the problem: *Find* $(\boldsymbol{\varphi}, \pi)$ *such that:*

$$\sigma\,\boldsymbol{\varphi} \; - \nu\,\Delta\,\boldsymbol{\varphi} \; - \; \nabla\pi \; = \; \boldsymbol{u} - \boldsymbol{u}_1 \quad , \quad \nabla\cdot\boldsymbol{\varphi} \; = \; 0 \qquad \text{in}\,\Omega\,, \tag{53}$$
$$\boldsymbol{\varphi} \;\; = \;\; \boldsymbol{0} \qquad \text{on}\,\partial\Omega\,,$$

where $(\boldsymbol{u}_1, p_1)$ is the solution of (34), belongs to $[H^2(\Omega)\cap H_0^1(\Omega)]^2 \times [H^1(\Omega)\cap L_0^2(\Omega)]$, and that there exists a constant C, possibly depending on σ and ν, but not on h, such that

$$\|\boldsymbol{\varphi}\|_{2,\Omega} \; + \; \|\pi\|_{1,\Omega} \; \leq \; C \, \|\boldsymbol{u} - \boldsymbol{u}_1\|_{0,\Omega} \, . \tag{54}$$

Theorem 8. *Under the hypothesis of Theorem 6 the following error estimates hold: If* $\nu \leq \sigma h_K^2$*, then there exists* $C > 0$*, independent of* h, σ *and* ν*, such that*

$$\|\boldsymbol{e_u}\|_{0,\Omega} \; \leq \; C\,h^2\,(|\boldsymbol{u}|_{2,\Omega} + \frac{1}{\nu}|p|_{1,\Omega}) \, .$$

If $\sigma h_K^2 < \nu$*, then there exists* $C > 0$*, independent of* h*, but depending on* σ *and* ν*, such that*

$$\|\boldsymbol{e_u}\|_{0,\Omega} \; \leq \; C\,h^2\,(|\boldsymbol{u}|_{2,\Omega} + |p|_{1,\Omega}) \, ,$$

Proof. Since we only need to prove the diffusive-dominated case ($\sigma h_K^2 < \nu$), then we will treat σ and ν as fixed constants. Let $(\boldsymbol{\varphi}_h, \pi_h) := (\mathcal{C}_h(\boldsymbol{\varphi}), \mathcal{C}_h(\pi) - \frac{(\mathcal{C}_h(\pi),1)_\Omega}{|\Omega|}) \in \mathbf{V}_h \times Q_h$. Then, multiplying the first equation in (53) by $\boldsymbol{u} - \boldsymbol{u}_1$ and second by $-(p - p_1)$, from the definition of the bilinear form $\mathbf{B}_\tau$, interpolation inequalities (49)-(48), and Theorems 6 and 7, we obtain

$$
\begin{aligned}
\|\boldsymbol{u}-\boldsymbol{u}_1\|_{0,\Omega}^2 &= \sigma(\boldsymbol{\varphi},\boldsymbol{u}-\boldsymbol{u}_1)_\Omega + \nu(\nabla\boldsymbol{\varphi},\nabla(\boldsymbol{u}-\boldsymbol{u}_1))_\Omega + (\pi,\nabla\cdot(\boldsymbol{u}-\boldsymbol{u}_1))_\Omega - (p-p_1,\nabla\cdot\boldsymbol{\varphi})_\Omega \\
&= \mathbf{B}_\tau((\boldsymbol{u}-\boldsymbol{u}_1,p-p_1),(\boldsymbol{\varphi},\pi)) + \sum_{K\in\mathcal{T}_h}\tau_K\,(\sigma(\boldsymbol{u}-\boldsymbol{u}_1)+\nabla(p-p_1),\sigma\boldsymbol{\varphi}+\nabla\pi)_K \\
&= \mathbf{B}_\tau((\boldsymbol{u}-\boldsymbol{u}_1,p-p_1),(\boldsymbol{\varphi}-\boldsymbol{\varphi}_h,\pi-\pi_h)) + \sum_{K\in\mathcal{T}_h}\tau_K\nu(\Delta\boldsymbol{u},\sigma\boldsymbol{\varphi}_h-\nabla\pi_h)_K \\
&\quad + \sum_{K\in\mathcal{T}_h}\tau_K\,(\sigma(\boldsymbol{u}-\boldsymbol{u}_1)+\nabla(p-p_1),\nu\Delta\boldsymbol{\varphi}+(\boldsymbol{u}-\boldsymbol{u}_1))_K \\
&\le C\left\{\|(\boldsymbol{u}-\boldsymbol{u}_1,p-p_1)\|_h^2+\|\boldsymbol{u}-\boldsymbol{u}_1\|_{0,\Omega}^2+h^2\,|\boldsymbol{u}|_{2,\Omega}^2+\|p-p_1\|_{0,\Omega}^2\right\}^{\frac12} \\
&\quad \left\{\sum_{K\in\mathcal{T}_h}\|\boldsymbol{\varphi}-\boldsymbol{\varphi}_h\|_{0,K}^2+|\boldsymbol{\varphi}-\boldsymbol{\varphi}_h|_{1,K}^2+\|\pi-\pi_h\|_{0,K}^2+\tau_K\|\nabla(\pi-\pi_h)\|_{0,K}^2\right. \\
&\quad \left.+h^2\,\|\varphi_h\|_{0,K}^2+h^2\,|\pi_h|_{1,K}^2+\tau_K\,|\boldsymbol{\varphi}|_{2,K}^2+\tau_K\|\boldsymbol{u}-\boldsymbol{u}_1\|_{0,K}^2\right\}^{\frac12} \\
&\le C\,h^2\,(|\boldsymbol{u}|_{2,\Omega}+\frac{1}{\sqrt{\nu}}\,|p|_{1,\Omega})\left(|\boldsymbol{\varphi}|_{2,\Omega}^2+|\pi|_{1,\Omega}^2+\|\boldsymbol{u}-\boldsymbol{u}_1\|_{0,\Omega}^2\right)^{\frac12},
\end{aligned}
$$

and the result follows applying (54) and dividing by $\|\boldsymbol{u}-\boldsymbol{u}_1\|_{0,\Omega}$. □

4. Numerical validations

4.1. **A problem with an analytical solution.** We first perform a convergence validation. To do this, we set $\Omega=(0,1)\times(0,1)$ and $\boldsymbol{f}$ and the boundary conditions such that the exact solution of (13) is given by

$$
\boldsymbol{u}(x,y)=\left(\frac{\sinh\left(\sqrt{\frac{\sigma}{\nu}}y\right)}{\sinh\left(\sqrt{\frac{\sigma}{\nu}}\right)},0\right)^t,
$$
$$
p(x,y)=(x-0.5)(y-0.5)\,.
$$

In Figures 2-3 we depict the convergence story as $h\to0$ in all the variables for $\sigma=1$ and $\nu=1,10^{-2}$ and 10^{-4}, respectively, where we see that all the variables converge as predicted by the theory.

4.2. **The lid-driven cavity flow.** Next, we address the lid-driven cavity problem, with domain Ω as before, $\boldsymbol{f}=\boldsymbol{0}$, and, in order to test the performance of the method for the large σ case, we perform experiments with $\sigma=1$ and $\sigma=10^4$, both using $\nu=1$. We depict in Figure 4 elevations for the pressure field and in Figure 5 of the horizontal velocity, for an

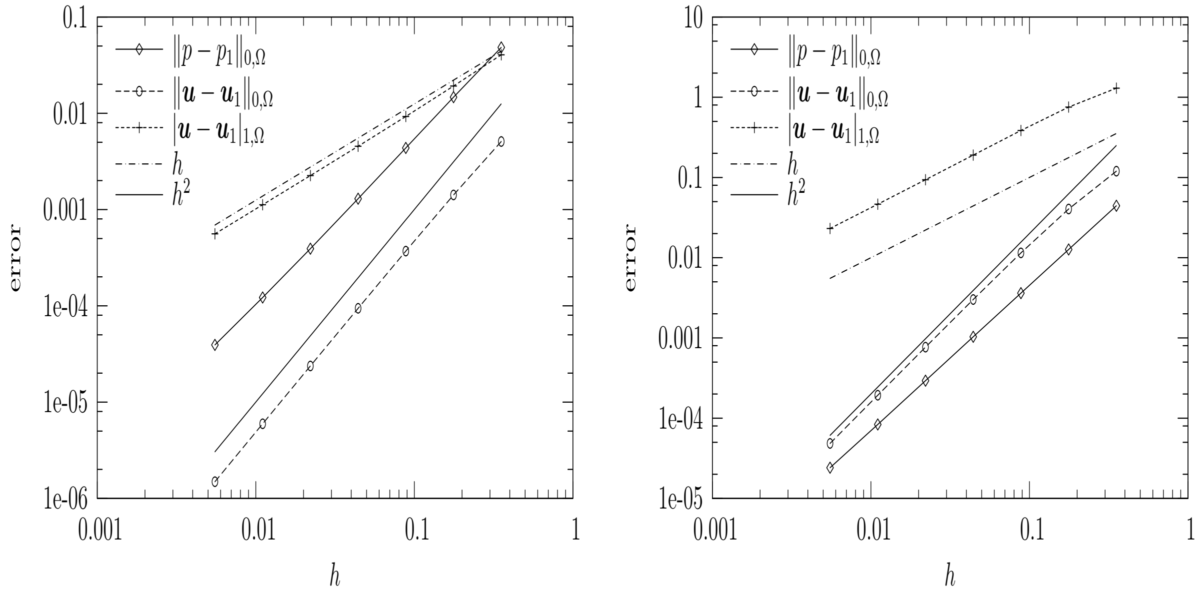

FIGURE 2. Convergence history $\sigma = 1$ and $\nu = 1$ (left) and $\nu = 10^{-2}$ (right).

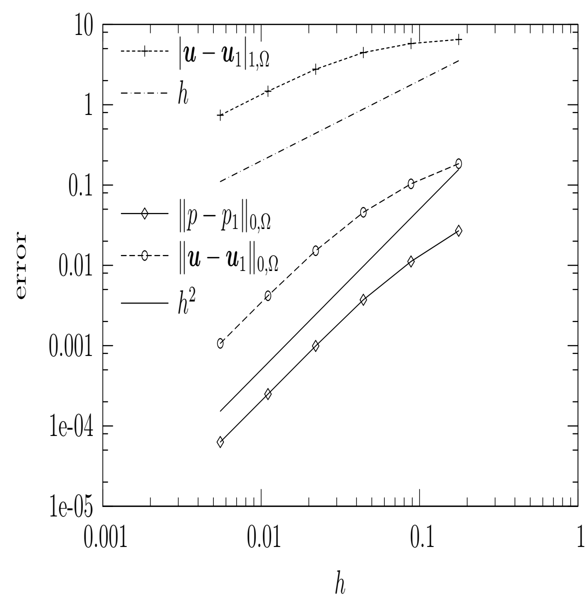

FIGURE 3. Convergence history for $\sigma = 1$ and $\nu = 10^{-4}$.

unstructured (and very close to equilateral) mesh. We observe the absence of oscillations for the pressure in both cases, which shows that the method treats well the inf-sup condition and the presence of a boundary layer for the reaction-dominated regime.

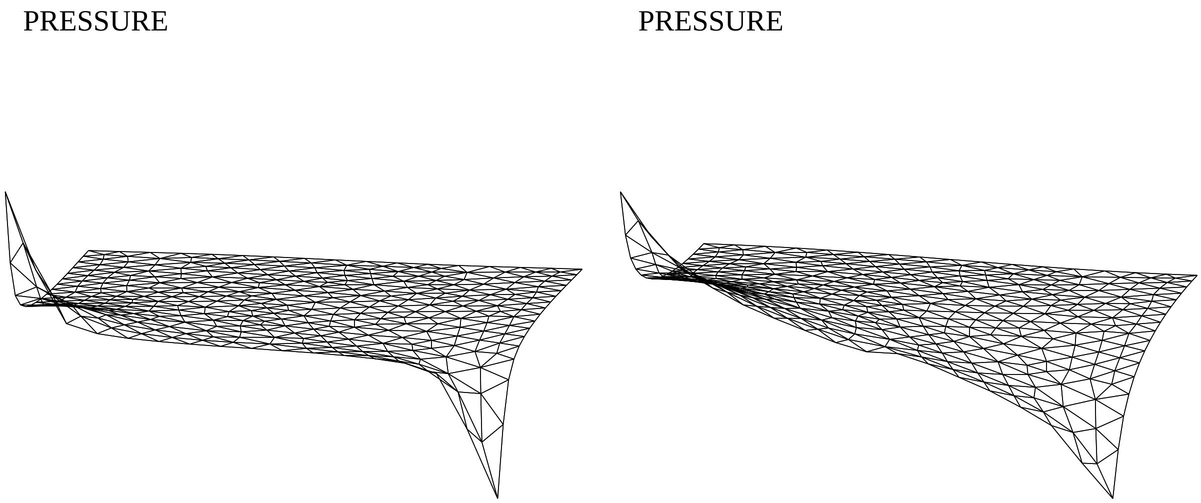

FIGURE 4. Pressure elevation for $\nu = 1$ and $\sigma = 1$ (left) and $\sigma = 10^4$ (right).

Next, we consider a structured mesh and test the method for $\sigma = 10^4$, $\nu = 1$. We see in Figure 6 that a small oscillation appears. This unexpected fact deserves further investigation, but, we also remark that this oscillation may be corrected by changing the definition of α_i as follows:

$$\alpha_i := \sqrt{\frac{8\sigma\,|K|^2}{\nu|F_i|^2}}\,. \tag{55}$$

This fact may be explained as follows, although the method was justified for a regular mesh, we recall that the derivation was performed supposing an equilateral mesh, and the regular mesh we used for this example differs from the equilateral case.

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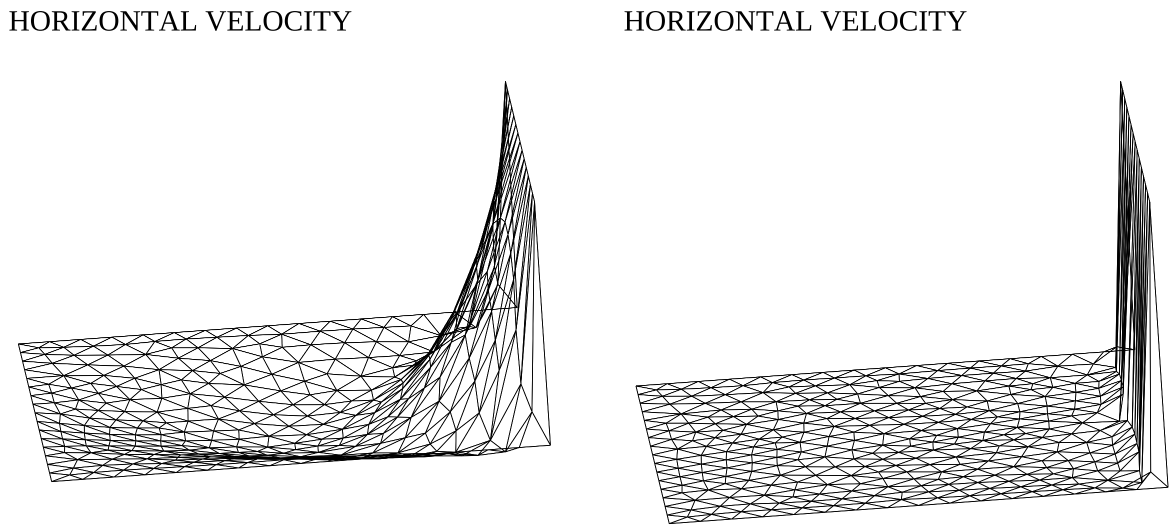

FIGURE 5. Elevation of the horizontal velocity for $\nu = 1$ and $\sigma = 1$ (left) and $\sigma = 10^4$ (right).

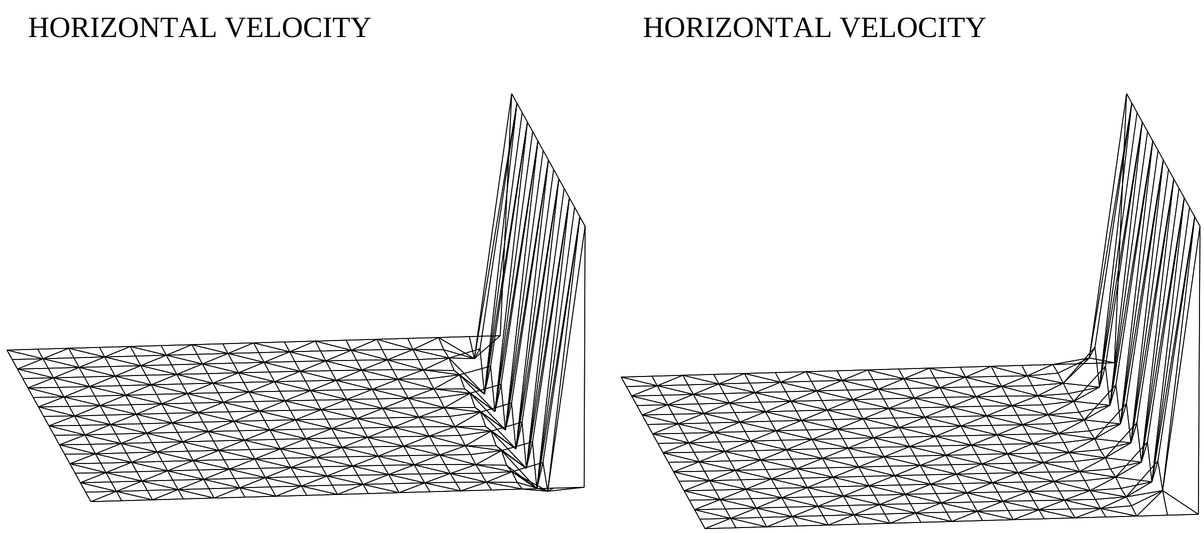

FIGURE 6. Elevation of the horizontal velocity for the standard definition of α_i (left) and (55) (right).

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